Part VI
Example Problems

VI.1 Cloaking
Kurt Bryan and Tanya Leise

1 Imaging

An object is *cloaked* if its presence cannot be detected by an observer using electromagnetic or other forms of imaging. In fact, the observer should not notice that cloaking is even occurring. Cloaking has a long history in science fiction, but recent developments have put the idea on a firmer mathematical and physical basis.

Suppose that an observer seeks to image some bounded region Ω in space. This is to be accomplished by injecting energy into Ω from the outside, then observing the response: that is, the energy that comes out of Ω. An example of this process is radar: the injected energy consists of electromagnetic waves and the observed response consists of the waves reflected by objects in the region. Many other types of energy can be used to form images, for example, acoustic (sonar), mechanical, electrical, or thermal. In each case energy is injected into Ω, response data is collected, and from this information an image may be formed by solving an inverse problem. To cloak an object the relevant physical properties of Ω must be altered so that the energy "flows around" the object, as if the object were not there. The challenge is to do this in a way that is mathematically rigorous and physically implementable. One successful approach to cloaking is based on the idea of *transformation optics*, which we will now discuss in the context of *impedance imaging*.

In impedance imaging, the bounded region $\Omega \subset \mathbb{R}^n$ to be imaged consists of an electrically conductive medium, with $n = 2$ or $n = 3$. Let the vector x represent position in a Cartesian coordinate system, let $u = u(x)$ represent the electrical potential inside Ω, and let $J = J(x)$ represent the electric current flux in Ω. We assume a linear relation $J = -\sigma\nabla u$ (a form of Ohm's law), where for each x the quantity $\sigma = \sigma(x)$ is a symmetric positive-definite $n \times n$ matrix. The matrix σ is called the *conductivity* of Ω and dictates how variations in the potential induce current flow. If $\sigma = \gamma I$ for some scalar function $\gamma(x) > 0$ (I is the $n \times n$ identity matrix), then the conductivity is said to be *isotropic*: there is no preferred direction for current flow. Otherwise, σ is said to be *anisotropic*. In impedance imaging the goal is to recover the internal conductivity of Ω using external measurements.

Specifically, to image a region Ω, the observer applies an electric current flux density g with $\int_{\partial\Omega} g \, \mathrm{d}s = 0$ to the boundary $\partial\Omega$. If charge is conserved inside Ω, then $\nabla \cdot J = 0$ in Ω and the potential u satisfies the boundary-value problem

$$\nabla \cdot \sigma\nabla u = 0 \quad \text{in } \Omega, \tag{1}$$

$$(\sigma\nabla u) \cdot n = g \quad \text{on } \partial\Omega. \tag{2}$$

Here n denotes an outward unit normal vector field on $\partial\Omega$, and we assume that each component of the matrix σ is suitably smooth. The boundary-value problem (1), (2) has a solution u that is uniquely defined up to an arbitrary additive constant. This solution depends on g, of course, but also on the conductivity $\sigma(x)$. By applying different current inputs g and measuring the resulting potential $f = u|_{\partial\Omega}$, an observer builds up information about the conductivity σ. If $\sigma = \gamma I$ is isotropic, then knowledge of the response f for every input g uniquely determines $\gamma(x)$; that is, the observer can "image" an isotropic conductivity with this type of input current/measured voltage data. However, a general anisotropic conductivity cannot be uniquely determined from this type of data, opening the way for cloaking.

2 Cloaking: First Ideas

Suppose a region Ω has isotropic conductivity $\sigma = I$, so $\gamma \equiv 1$. We wish to hide an object with different conductivity inside Ω in such a way that the object is invisible to an observer using impedance imaging. One approach is to remove the conductive material from some subregion $D \subset \Omega$, where $D \cap \partial\Omega = \varnothing$, thereby creating a nonconductive hole for hiding an object. For example, take D to be a ball $B_\rho(p)$ of radius ρ centered at $p \in \Omega$. In this case the boundary-value problem (1), (2) must be amended with the boundary condition $\nabla u \cdot n = 0$ on ∂D, since current cannot flow over ∂D. Unfortunately, this means that the hole D is likely to be visible to the observer, for the region $\Omega \setminus D$ typically yields a different potential response on $\partial\Omega$ than does the region Ω. The difference in input-response mappings for Ω versus $\Omega \setminus B_\rho(p)$ grows like $O(\rho^n)$ with the radius ρ of the hole (measured via an operator norm). The visibility of the hole is therefore proportional to its area or volume. To hide something nontrivial, we need ρ to be large, but the observer can then easily detect it.

3 Cloaking via a Transformation

Let us consider the special case in which Ω is the unit disk in $\mathbb{R}^2$. We will show how to make a large nonconductive hole in Ω, say a hole $B_{1/2}(0)$ of radius $\frac{1}{2}$ centered at the origin, essentially undetectable to an observer. We do this by "wrapping" the hole with a carefully designed layer of anisotropic conductor. Let Ω_ρ denote $\Omega \setminus B_\rho(0)$ and let r denote distance from the origin. Choose $\rho \in (0, \frac{1}{2})$ and let ϕ be a smooth invertible mapping from $\overline{\Omega_\rho}$ to $\overline{\Omega_{1/2}}$ with smooth inverse, with the properties that ϕ maps $r = \rho$ to $r = \frac{1}{2}$ and ϕ fixes a neighborhood $\frac{1}{2} < \rho_0 < r \leqslant 1$ of the outer boundary $r = 1$. Such mappings are easily constructed. Let $y = \phi(x)$ and define a function $v(y) = u(\phi^{-1}(y))$ on $\Omega_{1/2}$. The function v satisfies the boundary-value problem

$$\begin{aligned} \nabla \cdot \sigma \nabla v &= 0 \quad \text{in } \Omega_{1/2}, \\ \nabla v \cdot n &= g \quad \text{on } \partial\Omega, \\ (\sigma \nabla v) \cdot n &= 0 \quad \text{on } r = \tfrac{1}{2}, \end{aligned}$$

where σ is the symmetric positive-definite matrix

$$\sigma(y) = \frac{D\phi(x)(D\phi(x))^{\mathrm{T}}}{|\det(D\phi(x))|}$$

and $D\phi$ is the Jacobian of ϕ. Also, because ϕ fixes $r = 1$, we have $v \equiv u$ on the outer boundary.

The quantity σ may be interpreted as a conductivity consisting of an anisotropic shell around the hole, but with $\sigma \equiv I$ near $\partial\Omega$. For any input current g, the potential v on the region $\Omega_{1/2}$ with conductivity σ has the same value on $r = 1$ as the potential u_ρ on Ω_ρ. If $\rho \approx 0$, then $u_\rho - u_0 = O(\rho^2)$, where u_0 is the potential on the region Ω with no hole. The anisotropic conductivity σ thus has the effect of making the hole $B_{1/2}(0)$ appear to be a hole of radius ρ, effectively cloaking the larger hole if $\rho \approx 0$ (see figure 1). In the limit $\rho \to 0^+$, the resulting cloaking conductivity is singular but cloaks perfectly.

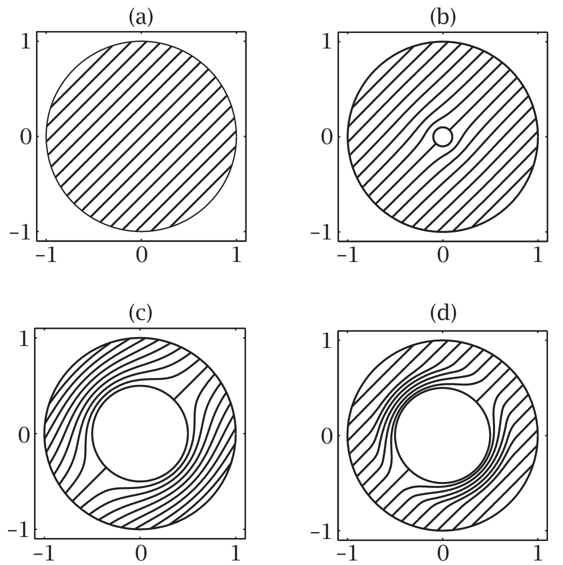

Figure 1 Comparison of flow lines of the current $J = -\sigma\nabla u$ on annuli $\Omega \subset \mathbb{R}^2$, with input flux $g(\theta) = (\sigma\nabla u) \cdot n|_{r=1} = \cos\theta + \sin\theta$ (in polar coordinates). (a) Empty disk. (b) Uncloaked $\Omega_{1/10}$. (c) Uncloaked $\Omega_{1/2}$. (d) Cloaked $\Omega_{1/2}$. The regions in (a)–(c) have isotropic conductivity 1, while the near-cloaked annulus in (d) has conductivity σ given by the transformation described in section 3. For (a), (b), and (c), the resulting boundary potential is $f(\theta) = (\cos\theta + \sin\theta)(1+\rho^2)/(1-\rho^2)$, with $\rho = 0$, $\rho = \frac{1}{10}$, and $\rho = \frac{1}{2}$, respectively, so that these regions will be distinguishable to an observer. In (d), the near-cloaked annulus has anisotropic conductivity σ corresponding to $\rho = \frac{1}{10}$, for which the mapping between f and g is identical to that in (b).

4 Generalizations

The transformation optics approach to cloaking is not limited to impedance imaging. Many forms of imaging with physics governed by a suitable partial differential equation may be amenable to cloaking. The key is to define a suitable mapping ϕ from the region containing something to be hidden to a region that looks "empty." Under the appropriate change of variable, one obtains a new partial differential equation that describes how to cloak the region, and the coefficients in this new partial differential equation may possess a reasonable physical interpretation (e.g., an anisotropic conductivity). The transformation thus prescribes the required physical properties the medium must possess in order to direct the flow of energy around an obstacle in order to cloak it. The construction of materials with the required properties is an active and very challenging area of research.

Further Reading

Bryan, K., and T. Leise. 2010. Impedance imaging, inverse problems, and Harry Potter's cloak. *SIAM Review* 52:359–77.

Greenleaf, A., Y. Kurylev, M. Lassas, and G. Uhlmann. 2009. Cloaking devices, electromagnetic wormholes, and transformation optics. *SIAM Review* 51:3–33.

Kohn, R. V., H. Shen, M. S. Vogelius, and M. I. Weinstein. 2008. Cloaking via change of variables in electric impedance tomography. *Inverse Problems* 24:1–21.

VI.2 Bubbles
Andrea Prosperetti

Ordinarily, the term "bubble" designates a mass of gas and/or vapor enclosed in a different medium, most often a liquid. Soap bubbles and the bubbles in a boiling pot are very familiar examples, but bubbles occur in many, and very diverse, situations of great importance in science and technology: air entrainment in breaking waves, with implications for the acidification of the oceans; water oxygenation, with consequences for, for example, marine life and water purification; volcanic eruptions; vapor generation, e.g., for heat transfer, power generation, and distillation; cavitation and cavitation damage, e.g., in hydraulic machinery and on ship propellers; medicine, e.g., in decompression sickness (the "bends"), kidney stone fragmentation (lithotripsy), plaque removal in dentistry, blood flow visualization, and cancer treatment; beverage carbonation; curing of concrete; bread making; and many others. There is therefore a very extensive literature on this subject across a wide variety of fields.

It is often the case that liquid masses in an immiscible liquid are also referred to as bubbles, rather than, more properly, "drops." This is more than a semantic issue as the most characteristic—and, indeed, defining—feature of bubbles is their large compressibility. When a bubble contains predominantly vapor (i.e., a gas below its critical point), condensation and evaporation are strong contributors to volume changes, which can be so extreme as to lead to the complete disappearance of the bubble or, conversely, to its explosive growth. Bubbles containing predominantly an incondensible gas are usually less compressible but still far more so than the surrounding medium. Some unexpectedly large effects are associated with this compressibility because, in a sense, a bubble represents a singularity for the host liquid.

To appreciate this feature one can look at the simplest mathematical model, in which the bubble is spherical with a time-dependent radius $R(t)$ and is surrounded by an infinite expanse of incompressible liquid. The dynamics of the bubble volume is governed by the so-called Rayleigh–Plesset equation, which, after neglecting surface tension and viscous effects, takes the form

$$R(t)\frac{\mathrm{d}U}{\mathrm{d}t} + \tfrac{3}{2}U^2 = \frac{1}{\rho}(p_i - p_\infty). \tag{1}$$

Here, $U = \mathrm{d}R/\mathrm{d}t$, ρ is the liquid density, and p_i, p_∞ are the bubble's internal pressure and the ambient pressure (i.e., the pressure far from the bubble). If these two pressures can be approximated as constants, this equation has an energy first integral, which, for a vanishing initial velocity, is given by

$$U(t) = \pm\sqrt{\frac{2(p_i - p_\infty)}{3\rho}\left[1 - \frac{R^3(0)}{R^3(t)}\right]}, \tag{2}$$

with the upper sign for growth ($p_i > p_\infty$, $R(0) \leqslant R(t)$) and the lower sign for collapse ($p_i < p_\infty$, $R(0) \geqslant R(t)$). In this latter case, by the time that $R(t)$ has become much smaller than $R(0)$, this expression would predict $U \propto -R^{-3/2}$, which diverges as $R(t) \to 0$. Of course, many physical effects that are neglected in this simple model (particularly the ultimate increase of p_i but also liquid compressibility, loss of sphericity, viscosity, and others) prevent an actual divergence from occurring, but, nevertheless, this feature is qualitatively robust and responsible for the unexpected violence of many bubble phenomena.

The approximation $p_i \simeq$ const. is reasonable for much of the lifetime of a bubble that contains mostly a low-density vapor. A frequently used model for an incondensible gas bubble assumes a polytropic pressure–volume relation of the form $p_i \times (\text{volume})^\kappa =$ const., with κ a number between 1 (for an isothermal bubble) and the ratio of the gas specific heats (for an adiabatic bubble). The internal pressure p_i in the simple model (1) is then replaced by $p_i = p_{i0}(R_0/R)^{3\kappa}$ for some reference values R_0 and p_{i0}.

The interaction of bubbles with pressure disturbances, such as sound, is often of interest, e.g., in underwater sound propagation, medical ultrasonics,