Suppose we can find such functions and that they do not vanish. Then, eliminating G from (32) gives

$$\frac{\Phi_+(t)}{K_+(t)} - \frac{\Phi_-(t)}{K_-(t)} = \frac{g(t)}{K_+(t)}, \quad t \in C,$$

which, again, we can solve using a Cauchy integral and (31).

The problem of finding $K_\pm$ is more delicate. At first sight, we could take the logarithm of (33), giving $\log K_+ - \log K_- = \log G$. This looks similar to (31), but it usually happens that $\log G(t)$ is not continuous for all $t \in C$, which means that we cannot use (30). However, this difficulty can be overcome.

The problem of finding $K_\pm$ such that (33) is satisfied is also the key step in the *Wiener–Hopf technique* (a method for solving linear PDEs with mixed boundary conditions and semi-infinite geometries). In that context, a typical problem would be: factor a given function $L(z)$ as $L(z) = L_+(z)L_-(z)$, where $L_+(z)$ is analytic in an upper half-plane, $\operatorname{Im} z > a$, $L_-(z)$ is analytic in a lower half-plane, $\operatorname{Im} z < b$, and $a < b$ so that the two half-planes overlap. There are also related problems where L is a 2×2 or 3×3 matrix; it is not currently known how to solve such *matrix Wiener–Hopf problems* except in some special cases.

17 Closing Remarks

Complex analysis is a rich, deep, and broad subject with a history going back to Cauchy in the 1820s. Inevitably, we have omitted some important topics, such as approximation theory in the complex plane and analytic number theory. There are numerous fine textbooks, a few of which are listed below. However, do not get the impression that complex analysis is a dead subject; it is not. In this article we have tried to cover the basics, with some indications of where problems and opportunities remain.

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IV.2 Ordinary Differential Equations

James D. Meiss

1 Introduction

Differential equations are near-universal models in applied mathematics. They encapsulate the idea that change occurs incrementally but at rates that may depend upon the state of the system. A system of *ordinary differential equations* (ODEs) prescribes the rate of change of a set of functions, $\boldsymbol{y}(t) = (y_1(t), y_2(t), \dots, y_k(t))$, that depend upon a single variable t, which may be real or complex. The functions y_j are the *dependent* variables of the system, and t is the *independent* variable. (If there is more than one independent variable, then the system becomes a PARTIAL DIFFERENTIAL EQUATION [IV.3] (PDE).) Perhaps the most famous ODE is Newton's second law of motion,

$$m\ddot{\boldsymbol{y}} = \boldsymbol{F}(\boldsymbol{y}, \dot{\boldsymbol{y}}, t),$$

which relates the acceleration of the center of mass $\boldsymbol{y} \in \mathbb{R}^3$ of a body of mass m to an externally applied force $\boldsymbol{F}$. This force commonly depends upon the position of the body, $\boldsymbol{y}$; its velocity, $\dot{\boldsymbol{y}}$ (e.g., electromagnetic or damping forces); and perhaps upon time, t (e.g., time-varying external control). The force may also depend upon positions of other bodies; a prominent example is the n-BODY PROBLEM [VI.16] of gravitation. We will follow the convention of denoting the first derivative by $\dot{\boldsymbol{y}}$ or $\boldsymbol{y}'$, the second by $\ddot{\boldsymbol{y}}$ or $\boldsymbol{y}''$, and, in general, the kth by $\boldsymbol{y}^{(k)}$.

Newton's law is a system of second-order ODEs. More generally, an ODE system is of nth order if it involves the first n derivatives of a k-dimensional vector $\boldsymbol{y}$; formally, therefore, it is a relation of the form

$$G(\boldsymbol{y}, \boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \dots, \boldsymbol{y}^{(n)}; t) = 0. \tag{1}$$

An example is Clairaut's differential equation for a scalar function $y(t)$:

$$-y + t\dot{y} + g(\dot{y}) = 0. \tag{2}$$

This is a first-order ODE since it involves only the first derivative of y. Equations like this are *implicit*, since they can be viewed as implicitly defining the highest derivative as a function of $\boldsymbol{y}$ and its lower derivatives. Clairaut's equation depends explicitly on the independent variable; such ODEs are said to be *nonautonomous*.

Clairaut showed in 1734 that (2) has a particularly simple family of solutions: $y(t) = ct + g(c)$ for any $c \in \mathbb{R}$. While it might not be completely obvious how to find this solution, it is easy to verify that it does solve (2) by simple substitution since, on the proposed solution $\dot{y} = c$, we have $-y+t\dot{y}+g(\dot{y}) = -(ct+g(c))+tc+g(c) \equiv 0$. More generally, a *solution* of the ODE (1) on an interval (a,b) is a function $\boldsymbol{y}(t)$ that makes (1) identically zero for all $t \in (a,b)$. More specifically, a solution may be required to solve an *initial-value problem* (IVP) or a boundary-value problem (see section 5). For the first-order case, the former means finding a function with a given value, $y(t_0) = y_0$, at a given "initial" time t_0. For example, the family of solutions to (2) satisfies the initial condition $y(0) = y_0$ so long as there is a c such that $y_0 = g(c)$, i.e., y_0 is in the range of g. A family $\boldsymbol{y}(t;\boldsymbol{c})$ that satisfies an IVP for a domain of initial values is known as a *general* solution.

Apart from these families of solutions, implicit ODEs can also have *singular* solutions. For example, (2) also has the solution defined parametrically by $(t(s), y(s)) = (-g'(s), g(s) - sg'(s))$. Again, it is easy to verify that this is a solution to (2) by substitution (and implicit differentiation), but it is perhaps not obvious how to find it. Lagrange showed that some singular solutions of an implicit ODE can be found as envelopes of the general solutions, but the general theory was developed later by Cayley and Darboux.

The classical theory of ODEs, originating with Newton in his *Method of Fluxions* in 1671, has as its goal the construction of the general and singular solutions of an ODE in terms of elementary functions. However, in most cases, ODEs do not have such explicit solutions. Indeed most of the well-known SPECIAL FUNCTIONS [IV.7] of mathematics are defined as solutions of differential equations. For example, the BESSEL FUNCTION [III.2] $J_n(x)$ is defined to be the unique solution of the second-order, explicit, nonautonomous, scalar IVP

$$\left.\begin{aligned} x^2y'' + xy' + (x^2 - n^2)y = 0,\\ y(0) = \delta_{n,0}, \qquad y'(0) = \tfrac{1}{2}\delta_{n,1}, \end{aligned}\right\} \tag{3}$$

where $\delta_{i,j}$, the Kronecker delta, is nonzero only when $i = j$ and $\delta_{i,i} = 1$. This equation arises from a number of PDEs through separation of variables. Many of the properties of J_n (e.g., its power-series expansion, asymptotic behavior, etc.) are obtained by direct manipulation of this ODE.

In most applications, the ODE (1) can be written in the explicit form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\boldsymbol{y} = H(\boldsymbol{y}, \dot{\boldsymbol{y}}, \ddot{\boldsymbol{y}}, \dots, \boldsymbol{y}^{(n-1)}; t). \tag{4}$$

Such systems can always be converted into a system of first-order ODEs. For example, if we let $x = (\boldsymbol{y}, \dot{\boldsymbol{y}}, \dots, \boldsymbol{y}^{(n-1)}, t)$ denote a list of $d = nk + 1$ variables, it is then easy to see that (4) can be rewritten as the autonomous first-order system

$$\dot{x} = f(x) \tag{5}$$

for a suitable f. Every coupled set of k, nth-order, explicit ODEs can be written in the form (5).[1] The ODE (5) is a common form in applications, e.g., in population models of ecology or in Hamiltonian dynamics. In general, $x \in M$, where M is a d-dimensional manifold called the *phase space*. For example, the phase space of the planar pendulum is a cylinder with $x = (\theta, p_\theta)$, where θ and p_θ are the angle and angular momentum, respectively.

In general, the function f in (5) gives a velocity vector (an element of the tangent space TM) for each point in the manifold M; thus $f\colon M \to TM$. Such a function is a *vector field*. A solution $\varphi\colon (a,b) \to M$ of (5) is a differentiable curve $x(t) = \varphi(t)$ in M with velocity $f(\varphi(t))$; it is everywhere tangent to f. Given such a curve it is trivial to check to see if it solves (5); by contrast, the construction of solutions is a highly nontrivial task. A *general solution* of (5) has the form $x(t) = \varphi(t;c)$. Here, $c \in \mathbb{R}^d$ is a set of parameters such that, for each $t_0 \in (a,b)$ and each initial condition $x_0 \in M$, the equation $\varphi(t_0;c) = x_0$ can be solved for c.

A *general solution* of (5) is a solution $x(t) = \varphi(t;c)$ that depends upon d parameters, c, such that for any IVP, $x(t_0) = x_0 \in M$ with $t_0 \in (a,b)$, there is a $c \in \mathbb{R}^d$ such that $\varphi(t_0;c) = x_0$. The search for explicit, general solutions of (5) is, in most cases, quixotic.

2 First-Order Differential Equations

Many techniques were developed through the first half of the eighteenth century for obtaining analytical solutions of first-order ODEs. The equations were often motivated by mechanical problems such as the

1. Though (5) is autonomous, the study of nonautonomous equations per se is not without merit. For example, stability of periodic orbits is most fruitfully studied as a nonautonomous linear problem.

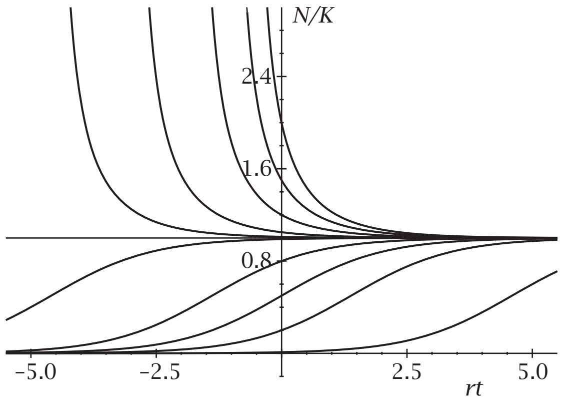

Figure 1 Solutions of the logistic ODE (7). The equilibria $N = 0$ and $N = K$ are the horizontal lines.

isochrone (find a pendulum whose period is independent of amplitude), which was solved by James Bernoulli in 1690, and da Vinci's catenary (find the shape of a suspended cable), which was solved by John Bernoulli in 1691.

During this period a number of methods were devised that can be applied to general categories of systems. In 1691 Leibniz formulated the method of separation of variables: the formal solution of the ODE $\mathrm{d}y/\mathrm{d}x = g(x)h(y)$ has the implicit form

$$\int \frac{\mathrm{d}y}{h(y)} = \int g(x)\,\mathrm{d}x. \tag{6}$$

Any autonomous first-order ODE is separable. For example, for the LOGISTIC POPULATION MODEL [III.19]

$$\dot{N} = rN(1 - N/K), \tag{7}$$

the integrals can be performed and the result solved for N to obtain

$$N(t) = \frac{N_0 K}{N_0 + (K - N_0)\mathrm{e}^{-rt}}. \tag{8}$$

Representative solutions are sketched in figure 1. Note that, whenever $N_0 > 0$, this solution tends, as $t \to \infty$, to K, the "carrying capacity" of the environment. This value and $N = 0$ are the two *equilibria* of (7), since the vector field vanishes at these points.

The differential equation $N(x, y)y' + M(x, y) = 0$ can be formally rewritten as the vanishing of a differential one-form, $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$. In 1739 Clairaut solved such equations when this one-form is exact, that is (for $\mathbb{R}^2$), when $\partial M/\partial y = \partial N/\partial x$. In 1734 Euler had already developed the more general method of integrating factors: if one can devise a function $F(x, y)$ such that the form $F(M\,\mathrm{d}x + N\,\mathrm{d}y)$ equals the total differential

$$\mathrm{d}H \equiv \frac{\partial H}{\partial x}\,\mathrm{d}x + \frac{\partial H}{\partial y}\,\mathrm{d}y$$

of a function $H(x, y)$, then the solutions to $\mathrm{d}H = 0$ lie on contours of $H(x, y)$. As an example, the integral curves of a Hamiltonian system with one degree of freedom,

$$\dot{x} = -\frac{\partial H}{\partial y}(x, y), \qquad \dot{y} = \frac{\partial H}{\partial x}(x, y), \tag{9}$$

are those curves that are everywhere tangent to the velocity; equivalently, they are orthogonal to the gradient vector $\nabla H \equiv (\partial H/\partial x, \partial H/\partial y)$. Denoting the infinitesimal tangent vector by $(\mathrm{d}x, \mathrm{d}y)$, this requirement becomes the exact one-form $(\mathrm{d}x, \mathrm{d}y) \cdot \nabla H = 0$. Its solutions lie on contours $H(x, y) = E$ with constant "energy." This method gives the phase curves or *trajectories* of the planar system but does not provide the time-dependent functions $x(t)$ and $y(t)$. However, using the constancy of H, the ODE for x, say, becomes $\dot{x} = \partial_y H(x, y(x; E))$, a separable first-order equation whose solution can be obtained *up to the quadrature* (6).

The technique of substitution was also used to solve many special cases (just as for integrating factors, there is no general prescription for finding an appropriate substitution). For example, James Bernoulli's nonautonomous, first-order ODE

$$y' = P(x)y + Q(x)y^n$$

is linearized by the change of variables $z = y^{1-n}$. Similarly, Leibniz showed that the degree-zero, homogeneous equation $y' = G(y/x)$ becomes separable with the substitution $y(x) = xv(x)$.

Discussion of these and other methods can be found in various classic texts, such as Ince (1956).

3 Linear ODEs

In 1743 Euler showed how to solve the nth-order linear constant-coefficient equation

$$\sum_{j=0}^{n} a_j \frac{\mathrm{d}^j y}{\mathrm{d}t^j} = 0 \tag{10}$$

by using the exponential ansatz, $y = \mathrm{e}^{rt}$, to reduce the ODE to the nth-degree *characteristic equation* $p(r) = \sum_{j=0}^{n} a_j r^j = 0$. Each root, r_k, of p provides a solution $y(t) = \mathrm{e}^{r_k t}$. Linearity implies that a superposition of these solutions, $y(t) = \sum_{k=0}^{n} c_k \mathrm{e}^{r_k t}$, is also a solution for any constant coefficients c_k. When $p(r)$ has a root r^* of multiplicity $m > 1$, Euler's reduction-of-order method suggests the further ansatz $y(t) = e^{r^* t}u(t)$. This provides new solutions when u satisfies $u^{(m)} = 0$, which has as its general solution a degree-$(m-1)$ polynomial in t. The general solution therefore becomes a superposition of n linearly independent functions of the form $t^{\ell}e^{r_k t}$. Even when the ODE is real, the roots $r_k = \alpha_k + i\beta_k$ may be complex. In this case, the conjugate root can be used to construct real solutions of the form $t^{\ell}e^{\alpha_k t}\cos(\beta_k t)$ and $t^{\ell}e^{\alpha_k t}\sin(\beta_k t)$ for $\ell = 0, \dots, m-1$. A superposition of these real solutions has n arbitrary real constants c_k, and since the functions are independent, there is a choice of these constants that solves the IVP $y^{(k)}(t_0) = b_k$, $k = 0, \dots, n-1$, for arbitrarily specified values b_k.

More generally, when (5) is linear, it reduces to

$$\dot{x} = Ax \tag{11}$$

for a constant, $n \times n$ matrix A. Formally, the general solution of this system can be written as the matrix exponential: $x(t) = e^{tA}x(0)$. As for more general FUNCTIONS OF MATRICES [II.14], we can view this as defining the symbol e^{tA} as the solution of the ODE. More explicitly, this exponential is defined by the same convergent MacLaurin series as e^{at} for scalar a. If A is semisimple (i.e., if it has a complete eigenvector basis), then A is diagonalized by the matrix P whose columns are eigenvectors: $A = P\Lambda P^{-1}$, where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ is the diagonal matrix of eigenvalues. In this case,

$$e^{tA} = P \operatorname{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t})P^{-1}.$$

More generally, e^{tA} also contains powers of t, generalizing the simpler, scalar situation.

The nonhomogeneous linear system

$$\dot{x} = Ax + g(t)$$

with forcing function $g \in \mathbb{R}^n$ can be solved by Lagrange's method of variation of parameters. The idea is to replace the parameters $x(0)$ in the homogeneous solution by functions $u(t)$. Substitution of $x(t) = e^{tA}u(t)$ into the ODE permits the unknown functions to be isolated and yields the integral form

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}g(s)\,ds.$$

Solution of linear, nonautonomous ODEs, $\dot{x} = A(t)x$, is much more difficult. The source of the difficulty is that $A(t)$ does not generally commute with $A(s)$ when $t \neq s$. Indeed, $e^A e^B \neq e^{A+B}$ unless the matrices do commute (the Baker–Campbell–Hausdorff theorem from Lie theory gives a series expansion for the product). The special case of a time-periodic family of matrices $A(t) = A(t+T)$ can be solved. Floquet showed that the general solution for this case takes the form $x(t) = P(t)e^{tB}x(0)$, where B is a real, constant matrix and $P(t)$ is a periodic matrix with period $2T$. One much-studied example of this form is MATHIEU'S EQUATION [III.21].

More generally, finding a transformation to a set of coordinates in which the effective matrix is constant is called the *reducibility problem*; even for a quasiperiodic dependence on time, this is nontrivial.

4 Singular Points

Consider the linear ODE (10), now allowing the coefficients $a_j(t)$ to be analytic functions of $t \in \mathbb{C}$ so that it is nonautonomous. Cauchy showed that, if the coefficients are analytic in a neighborhood of t_0 and if $a_n(t_0) \neq 0$, this ODE has n independent analytic solutions. The coefficients of the power series of y can be determined from a recursion relation upon substitution of a series for y into the ODE.

A point at which some of the ratios $a_j(t)/a_n(t)$ are singular is a (fixed) *singular point* of the ODE, and the solution need not be analytic at t_0. There are two distinct cases. A singular point is *regular* if $a_{n-j}(t)/a_n(t)$ has at most a jth-order pole for each $j = 1, \dots, n$. In this case, there is an $r \in \mathbb{C}$ such that there is at least one solution of the form $y(t) = (t-t_0)^r\phi(t)$ with ϕ analytic at t_0. Additional solutions may also have logarithmic singularities. An ODE for which all singular points are regular is called *Fuchsian*.

Most of the SPECIAL FUNCTIONS [IV.7] of mathematical physics are defined as solutions of second-order linear ODEs with regular singular points. Many are special cases of the hypergeometric equation

$$z(1-z)w'' + (\gamma - (\alpha+\beta+1)z)w' - \alpha\beta w = 0 \tag{12}$$

for a complex-valued function $w(z)$. This ODE has regular singular points at $z = 0$, 1, and ∞ (the latter obtained upon transforming the independent variable to $u = 1/z$). For the singular point at 0, following Frobenius, we make the ansatz that the solution has the form of a series:

$$w(z) = z^r \sum_{j=0}^{\infty} c_j z^j.$$

Substitution into (12) yields $c_0 p(r)z^{r-1} + \mathcal{O}(z^r) = 0$, and if this is to vanish with $c_0 \neq 0$, then r must satisfy the *indicial equation* $p(r) = r^2 + (\gamma-1)r = 0$, with roots $r_1 = 0$ and $r_2 = 1-\gamma$. A recursion relation for the c_j, $j > 0$, is obtained from the terms of order z^{r+j-1}.

For $r_1 = 0$, this yields the GAUSS HYPERGEOMETRIC FUNCTION [IV.7 §5]

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{j=0}^{\infty} \frac{\Gamma(\alpha + j)\Gamma(\beta + j)}{j!\Gamma(\gamma + j)} z^j,$$

where the gamma function Γ generalizes the factorial. When $\gamma \notin \mathbb{Z}$, the second solution turns out to be $x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; x)$, which is not analytic at $z = 0$.

When the indicial equation has roots that differ by an integer, a second solution can be found by the method of reduction of order. For example, for the second-order case, suppose $r_1 - r_2 \in \mathbb{N}$ and let $w_1(z) = e^{r_1 t} \phi_1(t)$ be the solution for r_1. Substitution of the ansatz $w(z) = w_1(z) \int v(z)\, dz$ shows that v satisfies a first-order ODE with a regular singular point at 0 whose indicial equation has the negative integer root $r_2 - r_1 - 1$. If the power series for v has $\mathcal{O}(z^{-1})$ terms, then $w(z)$ has logarithmic singularities; ultimately, the second solution has the form $w(z) = z^{r_2} \phi_2(z) + c w_1(z) \ln z$, where ϕ_2 is analytic at 0 and c might be zero. Thus, for example, the second hypergeometric solution for integral γ, where the roots of the indicial equation differ by an integer, has logarithmic singularities.

Near an *irregular* singular point, the solution may have essential singularities. For example, the first-order ODE $z^2 w' = w$ has an irregular singular point at 0; its solution, $w(z) = c e^{-1/z}$, has an essential singularity there. Similarly, Bessel's equation (3) has an irregular singular point at ∞.

Singular points of nonlinear ODEs can be fixed (i.e., determined by singularities of the vector field) or moveable. In the latter case, the position of the singularity depends upon initial conditions. The study of equations whose only movable singularities are poles leads to the theory of PAINLEVÉ TRANSCENDENTS [III.24].

5 Boundary-Value Problems

So far we have considered IVPs for systems of the form (5), that is, when the imposed values occur at one point, $t = t_0$. Another common formulation is that of a boundary-value problem (BVP), where properties of the solution are specified at two distinct points. Such problems commonly occur for ODEs that arise by separation of variables from PDEs. They also occur in control theory, where constraints may be applied at different times.

A classical BVP is the Sturm–Liouville equation:

$$\left.\begin{aligned} -(p(x)y')' + q(x)y &= \lambda r(x) y, \\ \alpha_1 y(a) + \alpha_2 y'(a) &= 0, \\ \beta_1 y(b) + \beta_2 y'(b) &= 0. \end{aligned}\right\} \quad (13)$$

Here, λ is a parameter, e.g., the separation constant for the PDE case, $p \in C^1[a,b]$, $q, r \in C^0[a,b]$, p and the weight function r are assumed to be positive, and $\alpha_1\alpha_2, \beta_1\beta_2 \neq 0$. For example, Bessel's equation (3) takes this form, with $p(x) = -q(x) = x$ and $r(x) = 1/x$, if appropriate boundary conditions are imposed.

The Sturm–Liouville problem has (unique) solutions $y_n(x) \in C^2[a,b]$ only for a discrete set λ_n, $n \in \mathbb{N}$, of values of the separation constant. Moreover, these "eigenfunctions" and their corresponding "eigenvalues" have a number of remarkable properties.

Ordering: $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$.
Oscillation: $y_n(x)$ has $n - 1$ simple zeros in (a, b).
Growth: $\lambda_n \to \infty$ as $n \to \infty$.
Orthogonality: $\int_a^b r(x) y_n(x) y_m(x)\, dx = \delta_{m,n}$.
Completeness: the set y_n is a basis for the space $L^2(a, b)$.

Perhaps the simplest such problem is $y'' = -\lambda y$ with $y(0) = y(1) = 0$. Here, the eigenvalues are $\lambda_n = (n\pi)^2$ and the eigenfunctions are $y_n = \sin(n\pi x)$. The completeness of these functions in $L^2(0, 1)$ is the expression of the convergence of the Fourier sine series. A more interesting problem is the quantum harmonic oscillator, which, when nondimensionalized, is governed by the Schrödinger equation

$$-\psi'' + x^2 \psi = \lambda \psi. \quad (14)$$

Here, λ is related to the energy $E = \frac{1}{2}\lambda\hbar\omega$ for classical frequency ω. This is a Sturm–Liouville problem for $\psi \in L^2(-\infty, \infty)$. The solutions are most easily obtained by the substitution $\psi(x) = e^{-x^2/2} y(x)$ that transforms (14) to the Hermite equation $y'' - 2xy' + (\lambda - 1)y = 0$. This ODE has degree-$(n-1)$ polynomial solutions when $\lambda_n = 2n - 1$, $n \in \mathbb{N}$; otherwise, the wave function ψ is not square integrable. The first five orthonormal eigenstates of (14) are shown in figure 2.

6 Equilibria and Stability

Apart from the solution of linear PDEs, linear systems of ODEs find their primary application in the study of the stability of equilibria of the nonlinear system (5). A point x^* is an equilibrium if $f(x^*) = 0$. If $f \in C^1$, then

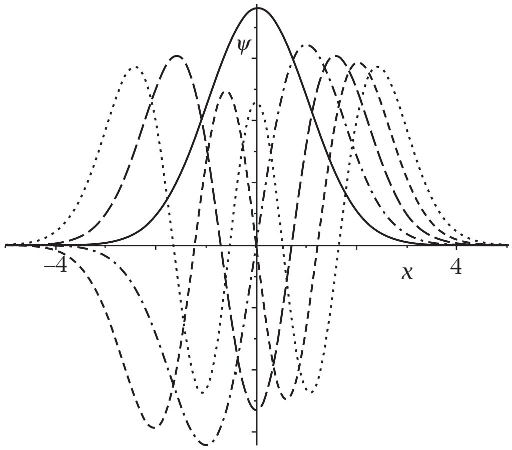

Figure 2 The first five eigenstates of the Sturm–Liouville problem (14).

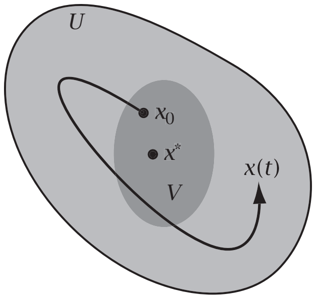

Figure 3 A Lyapunov-stable equilibrium x^*.

the dynamics of a nearby point $x(t) = x^* + \delta x(t)$ may be approximated by $\delta\dot{x} = \mathrm{D}f(x^*)\delta x$, where $(\mathrm{D}f)_{ij} = \partial f_i/\partial x_j$ is the Jacobian matrix of the vector field.

An equilibrium is *Lyapunov stable* if, for *every* neighborhood U, there is a neighborhood $V \subset U$ such that if $x(0) \in V$ then $x(t) \in U$ for all $t > 0$ (see figure 3). For the case in which $A = \mathrm{D}f(x^*)$ is a hyperbolic matrix (its spectrum does not intersect the imaginary axis), the stability of x^* can be decided by the eigenvalues of A. Indeed, the Hartman–Grobman theorem states that in this case there is a neighborhood U of x^* such that there is a coordinate change (a homeomorphism) that takes the dynamics of (5) in U to that of (11). In this case we say that the two dynamical systems are topologically conjugate on U.

An equilibrium is stable if all of the eigenvalues of A are in the left half of the complex plane, $\mathrm{Re}(\lambda) < 0$. Indeed, in this case it is *asymptotically* stable: there is a neighborhood U such that every solution that starts in U remains in U and converges to x^* as $t \to \infty$. In this case, x^* is a stable *node*. When there are eigenvalues with both positive and negative real parts, then x^* is a *saddle*. The case of complex eigenvalues deserves special mention, since the solution of the linear system then involves trigonometric functions and there are solutions in $\mathbb{R}^d$ that are infinite spirals. This is not necessarily the case when nonlinear terms are added, however: the homeomorphism that conjugates the system in U may unwrap the spirals.

As an example consider the damped Duffing oscillator[2]

$$\dot{x} = y, \qquad \dot{y} = -\mu y + x(1 - x^2), \tag{15}$$

with the phase portrait shown in figure 4 when $\mu = \frac{1}{2}$. There are three equilibria: $(0,0)$ and $(\pm 1, 0)$. The Jacobian at the origin is

$$\mathrm{D}f(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix},$$

with eigenvalues $\lambda_{1,2} = -\frac{1}{4}(1 \pm \sqrt{17})$. Since these are real and of opposite signs, the origin is a saddle. By contrast, the Jacobian of the other fixed points is

$$\mathrm{D}f(\pm 1,0) = \begin{pmatrix} 0 & 1 \\ -2 & -\frac{1}{2} \end{pmatrix},$$

with the complex eigenvalues $\lambda_{1,2} = -\frac{1}{4}(1 \pm \mathrm{i}\sqrt{31})$. Since the real parts are negative, these points are both attracting foci. They are still foci in the nonlinear system, as illustrated in figure 4, since trajectories that approach them cross the line $y = 0$ infinitely many times. Apart from the saddle and its *stable manifold* (the dotted curve in the figure), every other trajectory is asymptotic to one of the foci; these are *attractors* whose basins of attraction are separated by the stable manifold of the saddle.

The stability of a nonhyperbolic equilibrium (when A has eigenvalues on the imaginary axis) is delicate and depends in detail on the nonlinear terms, i.e., the $\mathcal{O}(\delta x^2)$ terms in the expansion of f about x^*. For example, the system

$$\dot{x} = -y + ax(x^2 + y^2), \qquad \dot{y} = x + ay(x^2 + y^2) \tag{16}$$

has only one equilibrium, $(0,0)$. The Jacobian at the origin has eigenvalues $\lambda = \pm\mathrm{i}$; its dynamics are that of a *center*. Nevertheless, the dynamics of (16) near $(0,0)$ depend upon the value of a. This can be easily seen by transforming to polar coordinates using $(x, y) =$

2. George Duffing studied the periodically forced version of (15) in 1918.

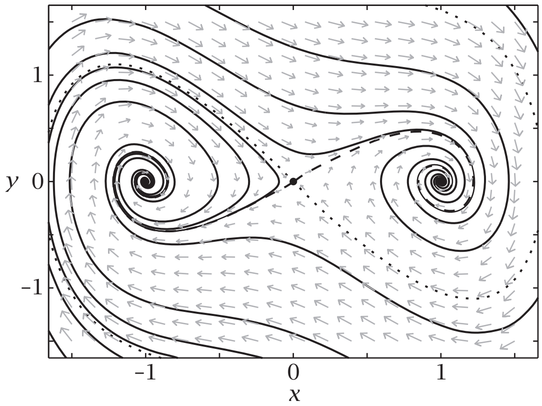

Figure 4 The phase portrait of (15) for $\mu = \frac{1}{2}$. Arrows depict the vector field, and dots depict the three equilibria. The unstable (dashed) and stable (dotted) manifolds of the saddle are shown.

$(r\cos\theta, r\sin\theta)$:

$$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y}) = ar^3,$$
$$\dot{\theta} = \frac{1}{r^2}(x\dot{y} - y\dot{x}) = 1.$$

Thus if $a < 0$, the origin is a global attractor: every trajectory limits to the origin as $t \to \infty$. If $a > 0$, the origin is a repellor. The study of nonhyperbolic equilibria is the first step in BIFURCATION THEORY [IV.21].

7 Existence and Uniqueness

Before one attempts to find solutions to an ODE, it is important to know whether solutions exist, and if they exist whether there is more than one solution to a given IVP. There are two types of problems that can occur.

The first is that the velocity f may be unbounded on M; in this case, a solution might exist but only over a finite interval of time. For example, the system $\dot{x} = x^2$ for $x \in \mathbb{R}$ has the general solution $x(t) = x_0/(1 - tx_0)$. Note that $|x| \to \infty$ as $t \to 1/x_0$, even though f is a "nice" function: it is smooth, and moreover, it is analytic. The problem is, however, that as $|x|$ increases, the velocity increases even more rapidly, leading to infinite speed in finite time. The existence theorem deals with this problem by being local; it guarantees existence only on a compact interval.

The second problem is that f may not be smooth enough to guarantee a unique solution. One might expect that it is sufficient that f be continuous. However, the simple system $\dot{x} = \sqrt{|x|}$ for $x \in \mathbb{R}$ has infinitely many solutions that satisfy the initial condition $x(0) = 0$. The obvious solution is $x(t) \equiv 0$, but $x(t) = \frac{1}{4}\operatorname{sgn}(t)t^2$ is also a solution. Moreover, any function $x(t)$ that is zero up to an arbitrary time $t_0 > 0$ and then connects to the parabola $\frac{1}{4}(t - t_0)^2$ also solves the IVP. Elimination of this problem requires assuming that f is *more* than continuous; it must be at least Lipschitz. A function $f\colon M \to \mathbb{R}^d$ is *Lipschitz* on $M \subset \mathbb{R}^m$ if there is a constant K such that for all $x, y \in M$, $\|f(x) - f(y)\| \leqslant K\|x - y\|$.

With this concept, we can state a theorem of existence and uniqueness. Let $B_r(x)$ denote the closed ball of radius r about x.

Theorem 1 (Picard–Lindelöf). *Suppose that for $x_0 \in \mathbb{R}^d$ there exists $b > 0$ such that $f\colon B_b(x_0) \to \mathbb{R}^d$ is Lipschitz. Then the IVP (5) with $x(t_0) = x_0$ has a unique solution $x\colon [t_0 - a, t_0 + a] \to B_b(x_0)$ with $a = b/V$, where $V = \max_{x \in B_b(x_0)} \|f(x)\|$.*

This theorem can be proved iteratively (e.g., by Picard iteration), but the most elegant proof uses the contraction mapping theorem.

8 Flows

When the vector field of (5) satisfies the conditions of the Picard–Lindelöf theorem, the solution is necessarily a C^1 function of time. It is also a Lipschitz function of the initial condition. Suppose now that $f \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and is locally Lipschitz. Though the theorem guarantees the existence only on a (perhaps small) interval $t \in [t_0 - a, t_0 + a]$, this solution can be uniquely extended to a maximal open interval $J = (\alpha, \beta)$ such that the solution is unbounded as t approaches α or β when they are finite. As noted in section 7, unbounded solutions may arise even for "nice" vector fields; however, if f is bounded or globally Lipschitz, then $J = \mathbb{R}$.

If $\varphi_t(x_0)$ denotes the maximally extended solution, then $\varphi\colon J \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies a number of conditions:

- $\varphi \in C^1$,
- $\varphi_0(x) = x$, and
- $\varphi_t \circ \varphi_s = \varphi_{t+s}$ whenever t, s, and $t + s \in J$.

The last condition encapsulates the idea of autonomy: flowing from the point $\varphi_s(x)$ for a time t is the same as flowing for time $t + s$ from x; the origin of time is a matter of convention.

For example, (8), the solution of the logistic ODE, gives such a function if it is rewritten as $\varphi_t(N_0) = N(t)$.

In this case (for r and K positive), $J = \mathbb{R}$ if $0 \leqslant N_0 \leqslant K$, and $J = (\alpha, \infty)$ with

$$\alpha = \frac{1}{r} \ln\left(1 - \frac{K}{N_0}\right) < 0$$

if $N_0 > K$. Indeed, it is apparent from figure 1 that solutions with initial conditions above the carrying capacity K grow rapidly for decreasing t; the theory implies that $\varphi_t(N_0) \to \infty$ as $t \downarrow \alpha$.

More generally, any function satisfying these conditions is called a *flow*. The flow is *complete* if $J = \mathbb{R}$, and it is a *semiflow* if α is finite but β is still ∞. It is not hard to see that every flow is the solution of a differential equation (5) for some C^0 vector field. Flows form one fundamental part of the theory of DYNAMICAL SYSTEMS [IV.20].

9 Phase-Plane Analysis

A system of two differential equations

$$\left.\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y) \end{aligned}\right\} \qquad (17)$$

can be qualitatively analyzed by considering a few simple properties of P and Q. The goals of such an analysis include determining the asymptotic behavior for $t \to \pm\infty$ and the stability of any equilibria or periodic orbits.

The *nullclines* are the sets

$$N_{\mathrm{h}} = \{x, y : Q(x, y) = 0\},$$
$$N_{\mathrm{v}} = \{x, y : P(x, y) = 0\}.$$

Typically these are curves on which the instantaneous motion is horizontal or vertical, respectively. The set of equilibria is precisely the intersection of the nullclines, $E = N_{\mathrm{h}} \cap N_{\mathrm{v}}$. The web of nullclines divides the phase plane into sectors in which the velocity vector lies in one of the four quadrants.

For example, the Lotka–Volterra system

$$\left.\begin{aligned} \dot{x} &= bx(1 - x - 2y), \\ \dot{y} &= cy(1 - 2x - y) \end{aligned}\right\} \qquad (18)$$

can be thought of as a model of competition between two species with normalized populations $x \geqslant 0$ and $y \geqslant 0$. The species have per capita birth rates b and c, respectively, when their populations are small, but these decrease if either or both of x and y grows because of competition for the same resource. In the absence of competition, the environment has a carrying capacity of one population unit. The nullclines are pairs of lines $N_{\mathrm{h}} = \{y = 0\} \cup \{y = 1 - 2x\}$ and

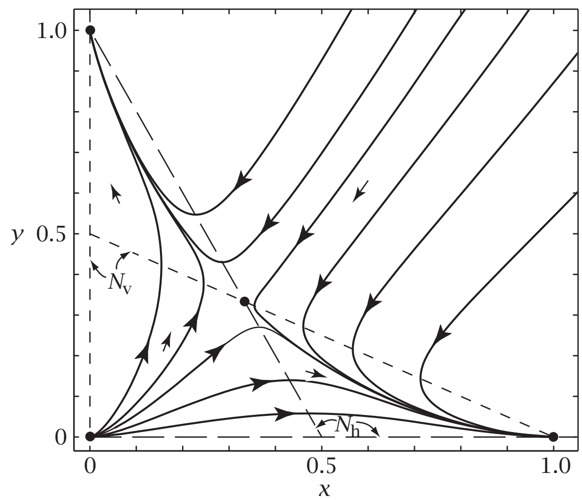

Figure 5 The phase portrait for (18) for $2c = 3b > 0$. Representative velocity vectors are shown in each sector defined by the nullclines, as are several numerically generated trajectories.

$N_{\mathrm{v}} = \{x = 0\} \cup \{y = \frac{1}{2}(1 - x)\}$. Consequently, there are four equilibria: $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(\frac{1}{3}, \frac{1}{3})$. The nullclines divide the biologically relevant domain into four regions within which the velocity lies in one of the four quadrants, as shown in figure 5. In particular, when both x and y are large (e.g., bigger than the carrying capacity), the velocity must be in the third quadrant since both $\dot{x} < 0$ and $\dot{y} < 0$. Since a component of the velocity can reverse only upon crossing a nullcline (and in this case does reverse), the remainder of the qualitative behavior is then determined.

From this simple observation one can conclude that the origin is a *source*, i.e., every nearby trajectory approaches the origin as $t \to -\infty$. By contrast, the two equilibria on the axes are *sinks* since all nearby trajectories approach them as $t \to +\infty$. The remaining equilibrium is a *saddle* since there are approaching and diverging solutions nearby. Moreover, every trajectory in the positive quadrant is bounded, and almost all trajectories asymptotically approach one of the two sinks. The only exceptions are a pair of trajectories that are on the stable manifold of the saddle. Details that are not determined by this analysis include the timescale over which this behavior occurs and the curvature of the solution curves, which depends upon the ratio b/c. This model demonstrates the ecological phenomenon of competitive exclusion; typically, only one species survives.

10 Limit Cycles

If the simplest solutions of ODE systems are equilibria, periodic orbits form the second class. A solution $\Gamma = \{x(t) : 0 \leqslant t < T\}$ of an autonomous ODE is periodic with (minimal) period T if Γ is a simple closed loop in the phase space. Indeed, uniqueness of solutions implies that, if $x(T) = x(0)$, then $x(nT) = x(0)$ for all $n \in \mathbb{Z}$.

A one-dimensional autonomous ODE (5) cannot have any periodic solutions. Indeed, every solution of such a system is a monotone function of t. Periodic trajectories are common in two dimensions. For example, each planar Hamiltonian system (9) has periodic trajectories on every closed nondegenerate ($\nabla H \neq 0$) contour $H(x, y) = E$. These periodic trajectories are not isolated. An isolated periodic orbit is called a limit cycle. More generally, a *limit cycle* is a periodic orbit that is the forward (ω) or backward (α) limit of another trajectory.

The van der Pol oscillator,

$$\dot{x} = y, \qquad \dot{y} = -x + 2\mu y - x^2 y, \tag{19}$$

was introduced in 1922 as a model of a nonlinear circuit with a triode tube. Here, x represents the current through the circuit, and y represents the voltage drop across an inductor. The parameter μ corresponds to the "negative" resistance of the triode passing a small current. This system has a unique periodic solution when $\mu > 0$ (see figure 6). The creation of this limit cycle at $\mu = 0$ follows from the HOPF BIFURCATION [IV.21 §2] theorem. Its uniqueness is a consequence of a more general theorem due to Liénard.

Planar vector fields can therefore have equilibria and periodic orbits. Are there more complicated trajectories, e.g., quasiperiodic or chaotic orbits? The negation of this speculation is contained in the theorem proposed by Poincaré and proved later by Bendixson: the set of limit points of any bounded trajectory in the plane can contain only equilibria and periodic orbits. There is therefore *no* CHAOS *[II.3] in two dimensions*! From the point of view of finding periodic trajectories, this theorem implies the following.

Theorem 2 (Poincaré–Bendixson). *Suppose that $A \subset \mathbb{R}^2$ is bounded and positively invariant and that φ is a complete semiflow in A. Then, if A contains no equilibria, it must contain a periodic orbit.*

For example, consider the system

$$\dot{x} = y, \qquad \dot{y} = -x + yh(r),$$

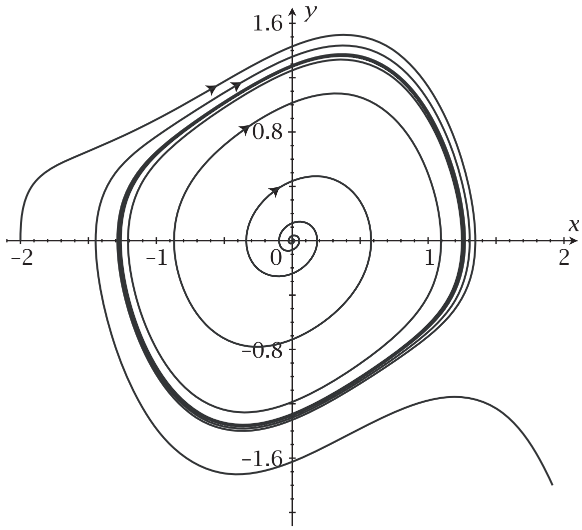

Figure 6 The phase portrait of the van der Pol oscillator (19) for $\mu = 0.2$.

where $r = \sqrt{x^2 + y^2}$, and let A be the annulus $\{(x, y) : a < r < b\}$. Thus A contains no equilibria for any $0 < a < b$. Converting to polar coordinates gives, for the radial equation,

$$\dot{r} = \frac{y^2}{r} h(r).$$

Now suppose that there exist $0 < a < b$ such that $h(b) < 0 < h(a)$. On the circle $r = a$ we then have $\dot{r} \geqslant 0$, implying that trajectories cannot leave A through its inner boundary. Similarly, trajectories cannot leave through $r = b$ because $\dot{r} \leqslant 0$ on this circle. By the Poincaré–Bendixson theorem, therefore, there is a periodic orbit in A.

11 Heteroclinic Orbits

Suppose that φ is a complete C^{r+1} flow that has a saddle equilibrium at x^*. The k-generalized eigenvectors of $\mathrm{D}f(x^*)$ corresponding to the stable eigenvalues, $\mathrm{Re}(\lambda_i) < 0$, define a k-dimensional tangent plane E^{s} at x^*. This linear plane can be extended to form a set of trajectories of the nonlinear flow whose forward evolution converges to x^*:

$$W^{\mathrm{s}}(x^*) = \Big\{x \in M \setminus \{x^*\} : \lim_{t \to \infty} \varphi_t(x) = x^*\Big\}.$$

The STABLE MANIFOLD THEOREM [IV.20] implies that this set is a k-dimensional, C^r, immersed manifold that is tangent to E^{s} at x^*. Similarly, a saddle has an unstable manifold

$$W^{\mathrm{u}}(x^*) = \Big\{x \in M \setminus \{x^*\}: \lim_{t\to-\infty} \varphi_t(x) = x^*\Big\}$$

that is tangent to the $(n-k)$-dimensional plane spanned by the unstable eigenvectors of x^*. It is important to note that this set is defined by its backward asymptotic behavior and not by the idea that it escapes from x^*. These concepts can also be generalized to hyperbolic invariant sets.

Poincaré realized that intersections of stable and unstable manifolds can give rise to complicated orbits. He called an orbit Γ *homoclinic* if $\Gamma \in W^{\mathrm{u}}(x^*) \cap W^{\mathrm{s}}(x^*)$. Similarly, an orbit is *heteroclinic* if $\Gamma \in W^{\mathrm{u}}(a) \cap W^{\mathrm{s}}(b)$ for distinct saddles a and b.

Planar Hamiltonian systems often have homoclinic or heteroclinic orbits. For example, the conservative Duffing oscillator, (15) with $\mu = 0$, has Hamiltonian $H(x,y) = \frac{1}{2}(y^2 - x^2 + \frac{1}{2}x^4)$. This function has a critical level set $H = 0$ that is a figure-eight intersecting the saddle equilibrium at $(0,0)$. Since energy is conserved, trajectories remain on each level set; in particular, every trajectory on the figure-eight is biasymptotic to the origin (these are homoclinic trajectories). For this case, the stable and unstable manifolds coincide, and we say that there is a *homoclinic connection*. This set is also called a *separatrix* since it separates motion that encircles each center from that enclosing both centers. Such a homoclinic connection is fragile; for example, it is destroyed whenever $\mu \neq 0$ in (15). More generally, a homoclinic BIFURCATION [IV.21] corresponds to the creation/destruction of a homoclinic orbit from a periodic one.

If, however, the intersection of $W^{\mathrm{u}}(a)$ with $W^{\mathrm{s}}(b)$ is transverse, it cannot be destroyed by a small perturbation. A transversal intersection of two submanifolds is one for which the union of their tangent spaces at an intersection point spans TM:

$$T_x W^{\mathrm{u}}(a) \oplus T_x W^{\mathrm{u}}(b) = T_x M.$$

Note that for this to be the case, we must have $\dim(W^{\mathrm{u}}) + \dim(W^{\mathrm{s}}) \geqslant \dim(M)$. Every such intersection point lies on a heteroclinic orbit that is structurally stable. Poincaré realized that in certain cases the existence of such a transversal heteroclinic orbit implies infinite complexity. This idea was formalized by Steve Smale in his construction of the *Smale horseshoe*. The existence of a transversal heteroclinic orbit implies a chaotic invariant set.

12 Other Techniques and Concepts

Differential equations often have discrete or continuous SYMMETRIES [IV.22], and these are useful in constructing new solutions and reducing the order of the system. Given sufficiently many symmetries and invariants, a system of ODEs can be effectively solved, that is, it is integrable.

One often finds that no analytical method leads to explicit solutions of an ODE. In this case, NUMERICAL SOLUTION [IV.12] techniques are invaluable.

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IV.3 Partial Differential Equations

Lawrence C. Evans

1 Overview

This article is an extremely rapid survey of the modern theory of partial differential equations (PDEs). Sources of PDEs are legion: mathematical physics, geometry, probability theory, continuum mechanics, optimization theory, etc. Indeed, *most of the fundamental laws of the physical sciences are partial differential equations* and most papers published in applied mathematics concern PDEs.

The following discussion is consequently very broad but also very shallow, and it will certainly be inadequate for any given PDE the reader may care about. The goal is rather to highlight some of the many key insights and unifying principles across the entire subject.

1.1 Confronting PDEs

Among the greatest accomplishments of the physical and other sciences are the discoveries of fundamental laws, which are usually PDEs. The great problems for mathematicians, both pure and applied, are then to understand the solutions of these equations,