1 Introduction

1.1 Prerequisites and Textbooks

The content of this book is sufficiently general and traditional, as to merit a place in the standard curriculum of physics graduate programs. Yet it is hardly to be found in most universities. In fact, it does not even have a single widely accepted name, being called a "many-body theory," "statistical field theory," and the like.

Bits and pieces of this topic can be found in lectures on atomic, condensed matter, nuclear, and high-energy physics, which all may discuss very similar and generic phenomena. A famous Einstein quotation notes that the number of good ideas is so small that the same phenomena pop up repeatedly in many different settings. And indeed, as we will see, one can collect all these pieces together, revealing the logical and historical paths that are common to these settings. Using the examples at hand, we will be demonstrating the surprising unity of physics in its many fields.

Quantum macroscopic phenomena famously include *superfluidity* and *superconductivity*. Their discoveries go back to the low-temperature frontier that was first explored a century ago. Their studies in the 1940s–1970s have resulted in amazing experiments and applications. Since 1990 we have witnessed the discovery of Bose-Einstein condensation in trapped atomic gases and other amazing applications, such as strongly coupled fermionic systems, perhaps with quantum computers to come in the near future.

Past decades have also witnessed remarkable progress in understanding strongly coupled quantum field theories (QFTs). The vacuum we live in—the quantum chromodynamic (QCD) vacuum – is a kind of superconductor, with a gap induced by quark-antiquark pairing. It is also a *dual superconductor* with Bose-condensed color-magnetic monopoles. Experiments on heavy ion collisions reveal the corresponding phase transition to the normal phase, known as the quark-gluon plasma, possessing quite unusual properties of a dual plasma, with a complicated interaction between electrically and magnetically charged quasiparticles.

Yet the major theoretical tools we will discuss are not so new. Feynman's diagrams, invented for quantum electrodynamics (QED) in the 1950s, became the main tool not only of QFTs but also of statistical mechanics (many-body theory). Weakly coupled Bose and Fermi systems were understood first, followed by realistic electron gases and nuclear matter. The Bardeen-Cooper-Schrieffer (BCS) theory explains not only the superconductivity of metals but also a large number of other phenomena as well. Along the way, it has been entirely reformulated, and we will explore it within the framework of renormalization group flow, another unifier of a huge number of phenomena.

Naturally, in the past decades many more discoveries have been made, and new directions have now taken central stage. A big event in the 1990s was cooling of the trapped quantum gases (both Bose and Fermi varieties) to temperatures so low that quantum condensates spontaneously appear. The so-called Feshbach resonance had allowed experimentalists to tune the main interaction parameter—the scattering length a—to any desired value, making it an ideal laboratory for many-body physics.

A much larger experimental effort in the nuclear physics community is the quest to produce and study quark-gluon plasma via high-energy heavy ion collisions. Performed mostly using the two largest devices of modern physics—the Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory and the Large Hadron Collider (LHC) at CERN—they reveal rather unusual properties of such matter. Another objective is to understand the phase transitions and structure of the QCD vacuum state we live in. At the end of the book we will discuss some open issues and current ideas on them.

And last but not least, the exponential growth in computer power made numerical evaluation of the Euclidean path integrals practical in late 1970s. First-principle studies of quantum many-body systems (e.g., liquid helium-II) became possible. This trend swept through QFTs, in particular to lattice gauge theories, and these studies now occupy many of the most powerful supercomputers in the world. These simulations are reproducing the hadronic spectrum and quark-gluon plasma equation of state. Due to these massive computational efforts, these results now complement the experimental effort and even allow researchers to study situations that would be impossible for real experiments.

A word about prerequisites and the use of other sources. It is assumed that the reader is familiar with standard quantum mechanics and statistical mechanics. However, it is *not* assumed that the reader has taken even introductory QFT courses: That is why the early sections contain an "alternative introduction" to the Feynman diagrams. Yet extra knowledge is always helpful: As a good self-education QFT book from the *Nutshell* series, I recommend that by A. Zee [1]. I will mention textbooks on particular subjects, as we proceed.

The classic books by Feynman on path integrals [2] and statistical mechanics [3] are the foundation of the the early chapters in my book. So the reader is advised to go through the first chapters in the Feynman books. Strangely enough, Feynman's books do not contain Feynman diagrams: Filling this gap will be one of our first tasks. Diagrammatic applications in statistical mechanics were covered in the classic pioneering book by Abrikosov, Gorkov, and Dzyaloshinski [61]. Other recommended books are Fetter and Walecka [5] and Negele and Orland [6]. These are all excellent books. However, they were

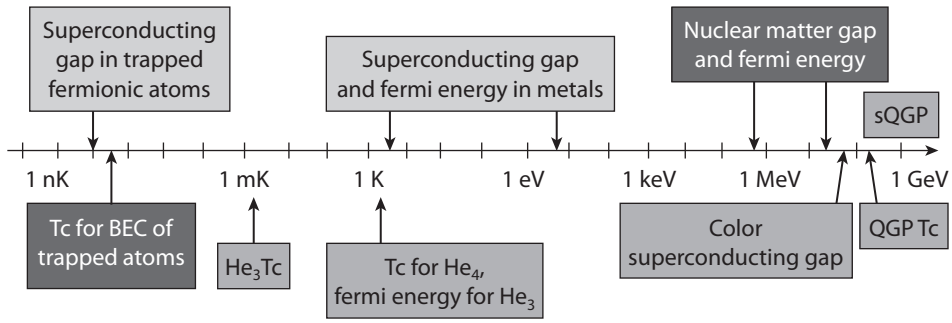

Figure 1.1. The logarithmic temperature scale and certain phenomena to be considered. Each vertical tick is an order of magnitude, and in total the temperature changes from nano-Kelvins to Giga-eV, by about $9+4+9=22$ orders of magnitude. Note that the frontiers of high-energy physics (on the right) and that of low temperature physics (on the left) are at roughly equal distances from room temperature (300 K) in the middle of the plot.

all written for theorists and focus mostly on technical tools rather than on phenomena, so they are not suitable for a first reading. They are also dense, to the extent of being difficult to read, and somewhat dated (e.g., they do not use path integrals, now standard). For a reader interested in a particular application, it would be more useful to look at these books after going through this one.

1.2 Physical Phenomena and Theoretical Tools

Let me start with a map in Figure 1.1, which indicates the location of some of the systems to be discussed in this course on a temperature scale.[1]

A proximity in scale by no means implies similarity of physics. In fact the absolute scale does not matter at all. The first thing physicists do when tackling a problem is introduce the appropriate units to make the problem scaleless. For example, two most extreme cases, the *trapped Fermi atomic gases in the strong coupling regime* at temperatures of the order of nano-Kelvins $T \sim nK$ at one end, and the quark-gluon plasma at the other end, are in fact rather similar. They both are very strongly coupled yet conformal (scale-independent) types of matter, displaying similar kinetic properties, the understanding of which remained challenging until recently.

For example, Fermi systems we will discuss include (in the order to be addressed) the electron gas, liquid ^{3}He, nuclear matter, trapped atomic gases, and quark-gluon plasmas. For all these cases we will see several recurring motives:

1. We use the same set of theoretical tools in all cases, namely, Euclidean path integrals, defined on the Matsubara circle, with the circumference $\beta = \hbar/T$ and with zero temperature corresponding to $\beta \to \infty$.

[1] As in most modern texts, by T I mean the energy, setting the Boltzmann constant $k_B = 1$ and freely combining the Kelvin K and electron-volt eV scales. The standard suffixes are n = nano = 10^{-9}, m = milli = 10^{-3}, k = kilo = 10^3, M = mega = 10^6, G = giga = 10^9.

2. We start by defining the action (Lagrangian). It always consists of a quadratic part (harmonic oscillator $V \sim x^2$ in quantum mechanical examples and the kinetic energy for QFTs) and the more complicated interaction part.
3. We will work out the lowest-order diagrams, with one or two loops, for the total (free) energy.
4. Then we will study the "mass operator" Σ and/or the "polarization operator" Π, correcting the zeroth-order propagators of the theory.
5. We then resum some series of the diagrams, using either geometric series or more sophisticated tools like the renormalization group equation.
6. Because the resummed ("dressed") Green functions possess either shifted or even entirely new poles, we will find that matter supports different propagation modes. Some modify the original particles, making them *quasiparticles*, with the same quantum numbers but new properties. Some are completely new (e.g., *plasmons* in electron gas or *zero sound* in liquid ^{3}He).
7. Furthermore, Bose-Einstein condensation of bosons (or Cooper pairs) of various types effectively destroys particle number conservation. This leads to the so-called *anomalous Green functions* describing the appearance or disappearance of particles. Using such diagrams, known as Gorkov formalism, we will describe multiple superconducting phases and the superfluidity of liquid ^{3}He, with their rich phenomenology.

We will show that superfluid and superconducting phases possess certain topological excitations, the *quantized vortices*, giving rise to complicated phase structures, the *topological matter*, of magnetized (or rotating, or both) matter. Furthermore, the vortices are only one member of the topological soliton family. We will briefly discuss the *skyrmions*, and (more important for this book) *Dirac monopoles* in three dimensions. Topological solitons in four dimensions are called *instantons*;[2] we will show how they lead to fascinating effects in quark-gluon plasmas and the QCD vacuum.

The most complex matter we will discuss near the end of the book is the vacuum of QCD, which has multiple condensates. One is due to quark-antiquark pairing, producing the so-called quark condensate $\langle \bar{q} q \rangle \neq 0$, at the surface of the Dirac sea. It is responsible for a gap in the quark spectrum—an effective quark mass—which is a significant (if not dominant) part of the mass of nucleons (and thus of our own mass). Another condensate is even more unusual: it is a dual superconductor made of bosonic objects with a magnetic charge, the color-magnetic monopoles. It leads to the dual Meissner effect, expelling the electric field into flux tubes and causing "quark confinement." One may ask whether the discussion of these fascinating physical phenomena in this introductory volume is premature. My answer is that, after one goes thorough all the needed theoretical tools and sees how they describe several traditional condensed matter applications, it should not be difficult to understand them.

[2] The solution was found by Polyakov and colleagues in Belavin et al. [179] and called "pseudoparticle," but the name "instanton"—suggested by 't Hooft and meaning "existing for an instant"—became the standard term.

To complete this general introduction, let me praise our tool of choice, on which the whole formalism will be based. It is the *the Feynman path integral* discussed from the beginning in the so-called *Euclidean time formulation*. As we will soon see, (1) it is the simplest and the most natural way to introduce and evaluate the Feynman diagrams; (2) it provides the most natural basis for the semiclassical approximations; and—last but not least—(3) it opens the door to direct first-principle numerical simulations of the path ensembles. We will discuss some computer-based approaches, which range from studies of few-body quantum-mechanical problems and the properties of liquid ^{4}He to numerical simulations of lattice gauge theories, which keep busy a fair fraction of the most powerful supercomputers available.

1.3 Feynman's Path Integrals

The concept is due to Feynman and originated from his studies of QED. The meaning of these path integrals in a quantum mechanical setting is described in his book [2], and in statistical mechanics in [3]. While we will be exploring the main formulas in this book, the reader is encouraged to read a few chapters of the Feynman volumes, where the concept is explained in much more detail.

Let me start by noting that all standard texts on quantum mechanics pay some tribute the so-called *matrix formulation* due to Heisenberg, which operates on abstract states in Hilbert spaces, usually a representation of some closed operator algebra. Then the standard presentations move to the Schrödinger formulation, based on the notion of *wavefunctions*. In stationary (time-independent) problems, the wave functions are the eigenstates of the Hamiltonian

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{1.1}$$

Hence their dependence on time is simple, via[3] the exponential factors e^{-iE_nt}.

The coordinate operator defines another set of states $|x\rangle$. It is rather singular, assuming that a particle is at the point x and nowhere else, so the operator should be treated with care. The stationary wave functions—the main element of the standard textbook Schrödinger formulation of quantum mechanics—are the projection coefficients of one set to another, namely,

$$\psi_n(x) \equiv \langle x|n\rangle. \tag{1.2}$$

Another concept, central for both quantum mechanics and statistical mechanics, is the *density matrix*. Let us discuss it in the simplest setting of single-particle motion in a time-independent one-dimensional potential $V(x)$. First define *the amplitude of the particle's propagation from the point x to the point y, during certain time t*:

$$G(x, y, t) \equiv \langle y|\exp(-i\hat{H}t)|x\rangle = \Sigma_n \psi_n(x)\psi_n^*(y)\exp(-iE_nt). \tag{1.3}$$

[3] Here and in most places below we use units in which $\hbar = 1$.

The first equality in this equation is the definition of the amplitude as a matrix element of the exponent of the Hamiltonian. The second equality is obtained by inserting a complete sum of the projectors over all stationary states

$$\hat{1} = \sum_n |n\rangle\langle n|$$

into the definition. We start with two sets of stationary states $|n\rangle, |n'\rangle$ and use the definition of the wavefunction (1.2). Because in its eigenbasis the Hamiltonian operator has nonzero elements only on the diagonal, the last expression has only a single—not a double—sum over states $n = n'$.

Exercise *First we calculate the amplitude for the free motion of a particle, using the usual plane waves $\psi_p(x) \sim \exp(ipx - iE_p t)$, $E_p = p^2/2m$. The sum over n is thus the integral over momentum p, and the answer is*

$$G(x, y, t) = \sqrt{\frac{m}{2\pi i t}} \exp\left[\frac{\mathrm{im}(x-y)^2}{2t}\right]. \tag{1.4}$$

This expression tells us that a free quantum particle spreads diffusionally, with the distance traveled $|x - y| \sim \sqrt{t}$. (A rather confusing imaginary i in the exponent will soon be gone; see below.) We will later use this expression for normalization of the motion in nonzero potentials. Show that the expression in the exponent of equation (1.4) is none other than the classical action for a straight path going from x to y and taking time t.

The Feynman formulation of quantum mechanics is based on the celebrated path integral representation for this amplitude, namely,

$$G(x, y, t) = \int_{x(0)=x}^{x(t)=y} Dx(t) \exp(-iS[x(t)]). \tag{1.5}$$

From classical mechanics we are familiar with the classical paths that satisfy the Newtonian equations of motion. According to Feynman, however, any quantum system is described by an ensemble of all possible paths, which are not required to satisfy any equations, only the boundary conditions specified as the limits on the integral. By this definition, the path integral should be taken over all paths $x(t)$ with the ends fixed: they all start at point x at time $t = 0$ and end at point y at time t.

Note that the weighted phase factor for a path includes a phase related to its action $S[x(t)]$, not a Hamiltonian. For a particle of mass m moving in a static potential $V(x)$ this factor is, as defined in classical mechanics courses,

$$S = \int dt \left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right], \tag{1.6}$$

with a minus sign for the potential.

The definition of the path integral itself is as follows. Let us split the time interval t into a large number N of small steps, each of a small duration $a = t/N$. The "discretized path" can be defined as the set of all intermediate positions of the particle, $\{x_i\}$, $i = 1, \ldots, N$.

One can evaluate the N-dimensional integrals using x_i as variables. The continuous limit of the path integral is naturally defined as the limit of the discretized path integral under the following condition

$$N \to \infty, \quad a \to 0, \quad t = a\,N = \text{fixed}.$$

Exercise *Derive expression (1.4) for a free particle by evaluating the path integral as N integrals over intermediate positions X_i. In case of difficulty, consult [2], where the evaluation is done in detail.*

In both of his books Feynman shows in detail how one can do path integrals for potentials V containing only linear and quadratic powers of the coordinate. In such cases all integrals over the intermediate coordinates x_i are Gaussian and can be done explicitly. Among such cases are the practically important harmonic oscillator and motion in a magnetic field.

Unfortunately, Gaussian path integrals are the only ones that allow analytic evaluation. Any non-Gaussian path integrals should in general only be calculated either perturbatively (as expansions near Gaussian ones) or numerically. We will discuss actual numerical evaluation of path integrals in the many-body setting in section 12.3.

1.4 Quantum Statistical Mechanics of a Particle

Applications of path integrals in their original formulation is difficult because of the interferences of oscillating weight factors. To get rid of such oscillations, we can analytically continue $G(x, y, t)$ into imaginary (also called Euclidean) time:

$$\tau = it. \tag{1.7}$$

In this case, the paths enter the sum with the new (and nonoscillating) weight $\exp(-S_E[x(\tau)])$, where the *Euclidean action* is defined as

$$S_E = \int dt \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) \right]. \tag{1.8}$$

Note that the relative sign between the kinetic and potential energies has now changed, and the expression looks like a Hamiltonian. This is equivalent to a motion in a flipped $V \to -V$ potential.

The highest weight in the resulting path integral has the "lazy path" $x(\tau) = x_{\min}$, corresponding to the particle sitting at the minimum of $V(x)$ at all times. Because the potential can always be shifted, let us assume that $V(x_{\min}) = 0$ and is positive ($V(x) > 0$) elsewhere. If so, our simplest (lazy) path obviously has zero action.

As emphasized by Feynman, the quantum particle is not just sitting at one location, the bottom of the potential, as a classical particle would do if put there. But the lazy path is just a single path, and the quantum particle is tempted to try many other paths!

The behavior of a quantum particle is a compromise between the *action* (which tries to enforce classical behavior, minimizing the action) and the number of possible paths, or their *entropy* (which counts these possible paths).

Let us now recall the definition of the amplitude, written as a sum over states, and rewrite it in Euclidean time:

$$S(x, y, \tau) = \sum_n \psi_n^*(y)\psi_n(x)e^{-E_n\tau}. \tag{1.9}$$

The oscillations are now changed by decreasing exponentials, so that at large values of τ, the integral is dominated by the lowest (ground) state.

Note that this amplitude satisfies the (analytically continued to τ) Schrödinger equation

$$-\frac{\partial S(x, y, \tau)}{\partial \tau} = \hat{H}_x S(x, y, \tau), \tag{1.10}$$

because both the l.h.s. and the r.h.s. simply add a factor E_n to the sum. Note also that as $\tau \to 0$ in (1.9), the exponent drops out, and the remaining sum becomes (by completeness of the stationary wave function set) just $\delta(x - y)$, which can be used as the initial condition for the equation.

Next we define the set of all periodic paths by putting $x = y$. For these paths the Euclidean time interval can be closed into a circle. Let us call its circumference β, $\tau \in [0, \beta]$ related to the temperature via[4]

$$\beta = \frac{\hbar}{T} \tag{1.11}$$

The path integral over the periodic paths thus defined is the quantity

$$\rho(x, T) = \sum |\psi_n(x)|^2 exp(-E_n/T), \tag{1.12}$$

which is nothing other than the probability of finding a quantum particle at point x in a heat bath with temperature T.

Furthermore, let us integrate over the initial (=final) value x, using the normalization condition of the states $\int dx|\psi_n(x)|^2 = 1$. What is obtained is in fact the basic quantity of statistical mechanics, the *statistical sum*

$$Z = \sum_n e^{-E_n/T} = \int Dx(\tau)e^{-S_E}, \tag{1.13}$$

for which we have now found a path integral definition.

Indeed, the r.h.s. is the path integral over all periodic paths, with the duration parameter $\tau \in [0, \beta]$ defined on the so-called Matsubara circle with the circumference β as defined in (1.11). This expression for Z in terms of the path integral opens many new application in quantum statistical mechanics. This fact has been realized in several key papers from the 1950s, such as [7–9].

[4] Unlike in the previous expressions, where $\hbar = 1$, here I remind the reader about the presence ot this term. In the classical limit of $\hbar \to 0$, the circumference $\beta \to 0$.

Example: (from [3]): The transition amplitude for a quantum particle in a harmonic oscillator potential

$$V = \frac{m^2\Omega^2}{2}x^2 \tag{1.14}$$

has the following form:

$$G_{\rm osc}(x,\, y,\, \tau) = \left(\frac{m\Omega}{2\pi\hbar \sinh(\Omega\tau)}\right)^{1/2} \exp\left[-\left(\frac{m\Omega}{2\hbar \sinh(\Omega\tau)}\right)((x^2+y^2)\cosh(\Omega\tau) - 2xy)\right]. \tag{1.15}$$

Exercise *While it can be derived from multiple Gaussian integrals, show instead that equation* (1.15) *satisfies both equation* (1.10) *and the natural initial condition at* $\tau \to 0$.

Exercise *For* $x = y$ *the density matrix is the probability of finding the particle at* x *at temperature* T. *Show that the expression in the exponent in (1.15) for the harmonic oscillator is the action of a classical path for inverted potential* $-V$, *rolling toward the maximum* $x = 0$, *starting and ending at* x *and taking time* $\beta = \hbar/T$. *Such paths, called "fluctons," can be defined for other potentials and even QFTs. They are a basis for multidimensional semiclassical approximations. The reader can find more on this topic in [10].*

The diagonal elements of the amplitude S are the probability of finding a particle at point x: so setting $y = x$ and $\tau = \beta$, we find that the harmonic oscillator at *any* temperature has a Gaussian distribution of particles

$$P(x) = \sqrt{\frac{m\Omega}{2\pi\hbar \sinh(\hbar\Omega\beta)}} \exp\left(-\frac{x^2}{2\langle x^2\rangle}\right), \tag{1.16}$$

where the (temperature-dependent) width is given by

$$\langle x^2\rangle = \frac{1}{2m\Omega}\coth\left(\frac{\Omega}{2T}\right). \tag{1.17}$$

This expression, which we will meet again later in the book, has two important limits. At small $T \to 0$, the width corresponds to the quantum mechanical ground-state wavefunction $\psi_0(x)$ of the oscillator. In the opposite limit of high $T \to \infty$, it corresponds to the classical thermal result $\langle x^2\rangle = \frac{T}{m\Omega^2}$. Let us rewrite this expression once again to elucidate its physical nature. Since for harmonic oscillators the total energy is just twice the potential energy, which is related to mean $\langle x^2\rangle$, we also have an expression for the mean energy of the oscillator at temperature T. Check that it can be put into the following "physical" form:

$$\langle E\rangle = \Omega\left(\frac{1}{2} + \frac{1}{e^{\Omega/T}-1}\right). \tag{1.18}$$

Now we can see the meaning of the two terms in parentheses: they are the energies corresponding to (T-independent) zero-point quantum oscillations (familiar from quantum mechanics courses) plus the energy of the thermal excitation (familiar from statistical mechanics courses). Note that we automatically get the correct Planck

(or Bose) distribution from the transition amplitude in Euclidean time: we will see similar "miracles" many more times below.

Let me now outline the rather straightforward transition, from quantum mechanics to the QFT context, which we will need later. The path integral over $(1+0)$-dimensional time-space, over all particle paths $x(\tau)$, is directly generalized to the path integral over the *field histories* in $(1+d)$ time-space dimensions. So the functional integrals change their integration variable

$$x(\tau) \to \phi(\tau, \vec{x}), \tag{1.19}$$

preserving its physical meaning. The weight is still given by the Euclidean action of the corresponding QFT. Most importantly, the same Euclidean time setting on a thermal or Matsubara circle can still be used for the partition function.

Let us jump a bit ahead and outline three major applications of the Euclidean path integrals. They can be evaluated by

1. perturbative series,
2. semiclassical approximation, and
3. direct numerical methods.

In this book we will discuss (1) in significant detail, presentation of (2) will be mostly missed (see [10] if interested), and (3) will be covered for a few basic applications.

1.5 A Few Gaussian Integrals

Two basic one-dimensional Gaussian integrals (defined for $A > 0$) are:

$$I_1 = \int_{-\infty}^{\infty} dx e^{-Ax^2} = \sqrt{\frac{\pi}{A}} \tag{1.20}$$

$$I_2 = \int_{-\infty}^{\infty} dx e^{-Ax^2} x^2 = \frac{\sqrt{\pi}}{2} A^{-3/2}. \tag{1.21}$$

Note that the second can be obtained from the first by a derivative over A, and their ratio gives the r.m.s. average $\langle x^2 \rangle = 1/(2A)$.

Let us generalize the integral to several dimensions, with coordinates x_i, $i = 1, \ldots, N$. A generic integral has some quadratic form A_{ij}

$$I_3 = \int_{-\infty}^{\infty} \Pi_{n=1}^{n=N} dx^n e^{-\sum_{i,j} A_{ij} x^i x^j}. \tag{1.22}$$

Without loss of generality, the form can be taken to be symmetric and can be diagonalized. If all eigenvalues of A are positive, $\lambda_n > 0$, the integral is well defined. Since it simply factorizes, the result is the following product of I_1 integrals, so

$$I_3 = \frac{\pi^{N/2}}{\Pi_{n=1}^{n=N} \sqrt{\lambda_n}} = \frac{\pi^{N/2}}{\sqrt{\det A}}. \tag{1.23}$$

Using this result, we can easily get an expression to be used below for a *correlator* of two coordinates:

$$\langle x_i x_j \rangle = \frac{\int_{-\infty}^{\infty} \left(\Pi_{n=1}^{n=N} dx^n\right) x_i x_j e^{-\sum_{i,j} A_{ij} x^i x^j}}{\int_{-\infty}^{\infty} \left(\Pi_{n=1}^{n=N} dx^n\right) e^{-\sum_{i,j} A_{ij} x^i x^j}}. \tag{1.24}$$

In the diagonal basis of the form, $A_{nj} y_j = \lambda_n y_n$, this integral factorizes and is simply given by the ratio of two basic integrals:

$$\langle y_i y_j \rangle = \delta_{ij} \frac{1}{2\lambda_i} \tag{1.25}$$

In general, coordinates x_i can be defined as a superposition of the diagonal ones $x_i = \sum_j C_{ij} y_j$ with some coefficients C_{ij}. Therefore, a correlator of two general coordinates can be written as

$$\langle x_i x_{i'} \rangle = \frac{1}{2} \sum_{ii'j} \frac{C_{ij} C_{i'j}}{\lambda_j} = \frac{1}{2} \left(A^{-1}\right)_{ii'} \tag{1.26}$$

the matrix element of the matrix A^{-1} *inverse* to the original matrix.

This is the essence of the perturbative construction: The propagators representing correlations of two quantum paths/fields are nothing other than the inverse of the relevant quadratic form in the action. The Euclidean time helps to avoid unwanted zeros in this inversion.

1.6 Another Take on the Path Integral for the Harmonic Oscillator

As explained above, the path integral for the harmonic oscillator is defined using the (Euclidean) action

$$S = \int d\tau \left(\frac{m\dot{x}^2}{2} + \frac{m\Omega^2 x^2}{2} \right), \tag{1.27}$$

where $\dot{x} = dx/d\tau$. We can recognize a certain Gaussian form, since $S \sim x^2$. Feynman originally used the discretization $x(\tau) \to x(\tau_n = n\beta/N)$ and approximated the derivative by a difference

$$\dot{x}(\tau) \to \frac{x(\tau_{n+1}) - x(\tau_n)}{a},$$

in which case we have a nondiagonal matrix.

Let us now try another approach. The differential operator of kinetic energy is simple enough to be diagonalized. It acts as written on both paths, but using periodicity in time, we can integrate the kinetic term in parts, rewriting the relevant differential operator as the one acting on one of the paths $x(\tau)$:

$$\hat{A}_{\text{osc}} = -m\frac{d^2}{d\tau^2} + m\Omega^2, \tag{1.28}$$

so that now the exponent contains $\int d\tau x(\tau) \hat{A}_{\text{osc}} x(\tau)$. Note the minus sign of the second derivative, which comes from integration by parts.

This operator plays the role of the quadratic form in section 1.5. But unlike the matrix form there, we now have a differential operator. It acts in the Hilbert space of functions, which has infinitely many dimensions rather than a finite number of them. Yet the same approach can in fact be used: the matrix or the differential operator may and should be diagonalized!

The eigenfunctions and eigenvalues of this particular operator are in fact easy to find: they are Fourier harmonics

$$\psi_n = \exp(i\omega_n \tau), \quad \lambda_n = \frac{m}{2}(\omega_n^2 + \Omega^2), \tag{1.29}$$

which are discrete because of the periodicity in τ:

$$\omega_n = \frac{2\pi n}{\beta} = 2\pi T n. \tag{1.30}$$

Note that the integer n runs over all integers, from $-\infty$ to ∞, including zero. The decomposition of the paths, periodic in τ with the period $\beta = \hbar/T$, is thus just what is called the Fourier decomposition.

Following the same logic as in section 1.5 on Gaussian integrals, we arrive at the following expression for the correlator of coordinates at different times:

$$\langle x(\tau)x(\tau')\rangle = \frac{T}{m}\sum_n \frac{\exp[i\omega_n(\tau-\tau')]}{\omega_n^2 + \Omega^2}. \tag{1.31}$$

We can recognize the coefficients C_{ij} from equation (1.26) in the numerator, related the original and the diagonal coordinates, as the Fourier harmonics, with the inverse eigenvalues in the denominator. The only differences from the formula we had before is that the sum over n is now infinite and the harmonics are complex, so one of them appears complex conjugated.[5] However, these differences do not matter: what is important is that by acting with the operator on this expression, we get the eigenvalue in the numerator, canceling the denominator. The resulting sum is

$$\sum_n \exp[i\omega_n(\tau-\tau')] \sim \delta(\tau-\tau'), \tag{1.32}$$

because Fourier harmonics form a complete set on periodic functions. So the expression above *is* indeed the inverse of the differential operator.

This correlator plays a very important role in perturbation theory and has a special name: the *propagator*. Note that it is a function of the time difference, because the action has no explicit time dependence anywhere, so all time moments are equal. Thus we can set $\tau' = 0$ and call the propagator a function of one variable, $G(\tau)$. The calculation we just made,

$$m\left(-\frac{d^2}{d\tau^2} + \Omega^2\right) G(\tau) = T\sum_n \frac{-(i\omega_n)^2 + \Omega^2}{\omega_n^2 + \Omega^2} e^{i\omega_n \tau} = \delta(\tau), \tag{1.33}$$

shows that $G(\tau)$ is nothing other than the *Green function* of classical equations of motion. Indeed, the operator acting on one of the arguments of the Green function returns the delta function, or unit operator. The mathematical meaning of the Green function is just

[5] It is the same for the quadratic form if we had not assumed there that the C_{ij} are real.

the *inversion* of the corresponding differential operator, a direct generalization of the inverse matrix in section 1.5.

For clarity of the presentation, let me add that in this case the Fourier transform corresponds to the multidimensional Gaussian integral as follows. The functions we integrate over, the analog of the original coordinates x_n, are the set of all periodic functions $f(\tau)$. These can be decomposed into the other set, the diagonal coordinates y_n, with the coefficients being the Fourier coefficients $\tilde{f}_n$. The relation between the two sets is given by

$$\tilde{f}_n = \int d\tau f(\tau) e^{i\frac{2\pi}{\beta}n\tau} \tag{1.34}$$

with the inverse relation being

$$f(\tau) = \frac{1}{\beta}\sum_n \tilde{f}_n e^{-i\frac{2\pi}{\beta}n\tau}. \tag{1.35}$$

You should not be intimidated by the fact that one of the formulas has an integral and the other a sum (although an infinite one): both representations—in time or in frequencies—provide mathematically identical descriptions of all paths we need to sum over in the path integral.

At this point it is also useful to discuss the limit of large $\beta \to \infty$ or vanishingly small temperature $T \to 0$. Physically, this is the case when one wants to go from statistical mechanics back to quantum mechanics or QFTs at zero temperature. In this case the periodicity condition is no longer restrictive, and we can use Fourier transforms with *continuous* frequencies. The formulas for the direct and the inverse transforms in this case are

$$\tilde{f}(\omega) = \int d\tau f(\tau) e^{i\omega\tau}, \quad f(\tau) = \int \frac{d\omega}{2\pi}\tilde{f}(\omega) e^{-i\omega t}, \tag{1.36}$$

and the expression for the propagator would have an integral over continuous ω instead of the discrete sum over n.

Completing this somewhat lengthy discussion of the propagator $G(\tau)$ for a harmonic oscillator, let us find its explicit form. It can be done either by performing the summation in (1.31), which I leave as an exercise, or by solving the equation for the Green function.

Let me do it by using some educated guesses. The first is to realize that at the point $\tau = 0$ (same as $\tau = \beta$ on the circle), the function $G(\tau)$ must have a jump in the first derivative: The second derivative will then produce $\delta(\tau)$, as required by the equation. The second consideration is that, starting from the opposite point on the circle, $\tau_{\text{opposite}} = \beta/2$, the function should be smooth and symmetric in both directions. The third is that the solutions of the equations of motion without the r.h.s. are just exponents of the type $\exp(\pm\Omega\tau)$. Therefore the needed symmetric function must be $\cosh(\Omega(\tau - \beta/2))$ times some constant C. The constant can be found from the derivative jump required or from the fact that at $\tau = 0$ the function is just the average of the coordinate fluctuation squared $\langle x(0)^2\rangle$ (see l.h.s of (1.31)), which we already know from (1.17). Thus the explicit form we are looking for is

$$G(\tau) = \frac{\cosh(\Omega(\tau - \beta/2))}{2m\Omega\sinh(\Omega\beta/2)}. \tag{1.37}$$

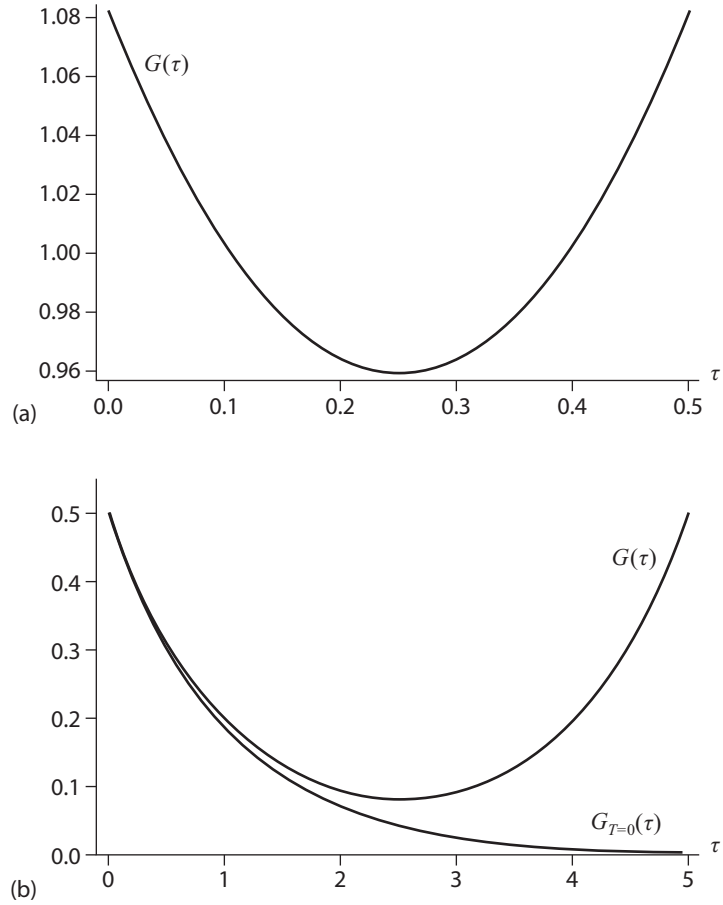

Figure 1.2. The correlator of coordinates $G(\tau)$ for harmonic oscillator, for $m = 1$ and two values of $\beta = \hbar/T =$ (a) $1/\Omega$, (b) $5/\Omega$. Panel (b) also shows the limit $T = 0$ (1.38).

For the small (zero) temperature T limit, the circumference of the circle goes to zero, and point $\tau = 0$ can be considered as the middle of the $[-\infty, \infty]$ line. In this case the propagator takes a very simple form:

$$G(T \to 0, \tau) = \frac{1}{2m\Omega} e^{-\Omega|\tau|}. \tag{1.38}$$

Note the modulus of the time in the exponent: this function is indeed singular at the origin and decreases in both directions, $\tau \to \pm\infty$.

It will be important later to get some feeling for which T values are "low" or "high": thus I have plotted the propagator $G(\tau)$ in Figure 1.2. Panel a for the case $T/\hbar\Omega = 1$ can in fact be considered a high T case, as $G(\tau)$ depends on τ weakly, deviating from 1 by only several percentage points. Panel b corresponds to the case $T/\hbar\Omega = 0.2$: it can be considered a low T case, as $G(\tau)$ follows the $T = 0$ line, falling significantly—by an order of magnitude—until the midpoint.