CHAPTER 1

GEOMETRODYNAMICS IN BRIEF

§1.1. THE PARABLE OF THE APPLE

One day in the year 1666 Newton had gone to the country, and seeing the fall of an apple, as his niece told me, let himself be led into a deep meditation on the cause which thus draws every object along a line whose extension would pass almost through the center of the Earth.

VOLTAIRE (1738)

Once upon a time a student lay in a garden under an apple tree reflecting on the difference between Einstein's and Newton's views about gravity. He was startled by the fall of an apple nearby. As he looked at the apple, he noticed ants beginning to run along its surface (Figure 1.1). His curiosity aroused, he thought to investigate the principles of navigation followed by an ant. With his magnifying glass, he noted one track carefully, and, taking his knife, made a cut in the apple skin one mm above the track and another cut one mm below it. He peeled off the resulting little highway of skin and laid it out on the face of his book. The track ran as straight as a laser beam along this highway. No more economical path could the ant have found to cover the ten cm from start to end of that strip of skin. Any zigs and zags or even any smooth bend in the path on its way along the apple peel from starting point to end point would have increased its length.

"What a beautiful geodesic," the student commented.

His eye fell on two ants starting off from a common point P in slightly different directions. Their routes happened to carry them through the region of the dimple at the top of the apple, one on each side of it. Each ant conscientiously pursued

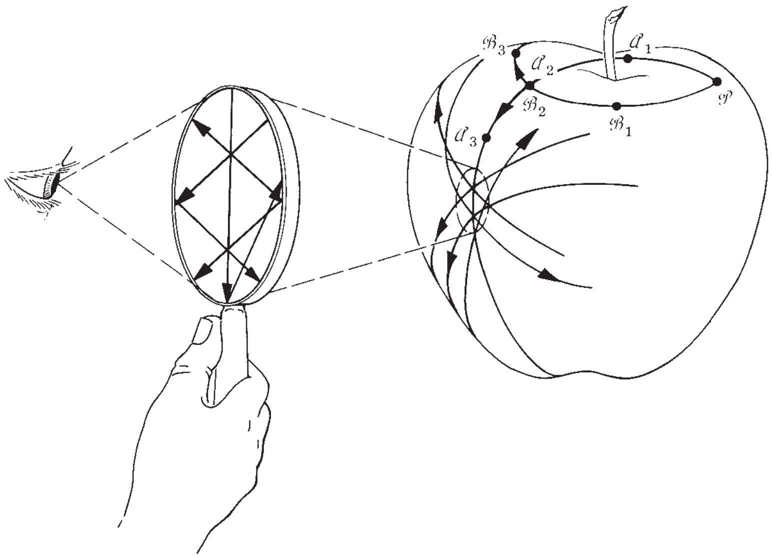


Figure 1.1.
The Riemannian geometry of the spacetime of general relativity is here symbolized by the two-dimensional geometry of the surface of an apple. The geodesic tracks followed by the ants on the apple's surface symbolize the world line followed through spacetime by a free particle. In any sufficiently localized region of spacetime, the geometry can be idealized as flat, as symbolized on the apple's two-dimensional surface by the straight-line course of the tracks viewed in the magnifying glass ("local Lorentz character" of geometry of spacetime). In a region of greater extension, the curvature of the manifold (four-dimensional spacetime in the case of the real physical world; curved two-dimensional geometry in the case of the apple) makes itself felt. Two tracks $\mathcal{A}$ and $\mathcal{B}$, originally diverging from a common point $\mathcal{P}$, later approach, cross, and go off in very different directions. In Newtonian theory this effect is ascribed to gravitation acting at a distance from a center of attraction, symbolized here by the stem of the apple. According to Einstein a particle gets its moving orders locally, from the geometry of spacetime right where it is. Its instructions are simple: to follow the straightest possible track (geodesic). Physics is as simple as it could be locally. Only because spacetime is curved in the large do the tracks cross. Geometrodynamics, in brief, is a double story of the effect of geometry on matter (causing originally divergent geodesics to cross) and the effect of matter on geometry (bending of spacetime initiated by concentration of mass, symbolized by effect of stem on nearby surface of apple).

his geodesic. Each went as straight on his strip of appleskin as he possibly could. Yet because of the curvature of the dimple itself, the two tracks not only crossed but emerged in very different directions.

Einstein's local view of physics contrasted with Newton's "action at a distance"

"What happier illustration of Einstein's geometric theory of gravity could one possibly ask?" murmured the student. "The ants move as if they were attracted by the apple stem. One might have believed in a Newtonian force at a distance. Yet from nowhere does an ant get his moving orders except from the local geometry along his track. This is surely Einstein's concept that all physics takes place by 'local action.' What a difference from Newton's 'action at a distance' view of physics! Now I understand better what this book means."

Physics is simple only when analyzed locally

And so saying, he opened his book and read, "Don't try to describe motion relative to faraway objects. *Physics is simple only when analyzed locally.* And locally the world line that a satellite follows [in spacetime, around the Earth] is already as straight as any world line can be. Forget all this talk about 'deflection' and 'force of gravitation.' I'm inside a spaceship. Or I'm floating outside and near it. Do I feel any 'force of gravitation'? Not at all. Does the spaceship 'feel' such a force? No. Then why talk about it? Recognize that the spaceship and I traverse a region of spacetime free of all force. Acknowledge that the motion through that region is already ideally straight."

The dinner bell was ringing, but still the student sat, musing to himself. "Let me see if I can summarize Einstein's geometric theory of gravity in three ideas: (1) locally, geodesics appear straight; (2) over more extended regions of space and time, geodesics originally receding from each other begin to approach at a rate governed by the curvature of spacetime, and this effect of geometry on matter is what we mean today by that old word 'gravitation'; (3) matter in turn warps geometry. The dimple arises in the apple because the stem is there. I think I see how to put the whole story even more briefly: *Space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve.* In other words, matter here," he said, rising and picking up the apple by its stem, "curves space here. To produce a curvature in space here is to force a curvature in space there," he went on, as he watched a lingering ant busily following its geodesic a finger's breadth away from the apple's stem. "Thus matter here influences matter there. That is Einstein's explanation for 'gravitation.'"

Space tells matter how to move

Matter tells space how to curve

Then the dinner bell was quiet, and he was gone, with book, magnifying glass—and apple.

§1.2. SPACETIME WITH AND WITHOUT COORDINATES

> *Now it came to me: . . . the independence of the gravitational acceleration from the nature of the falling substance, may be expressed as follows: In a gravitational field (of small spatial extension) things behave as they do in a space free of gravitation. . . . This happened in 1908. Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning.*
>
> ALBERT EINSTEIN [in Schilpp (1949), pp. 65–67.]

Nothing is more distressing on first contact with the idea of "curved spacetime" than the fear that every simple means of measurement has lost its power in this unfamiliar context. One thinks of oneself as confronted with the task of measuring the shape of a gigantic and fantastically sculptured iceberg as one stands with a meter stick in a tossing rowboat on the surface of a heaving ocean. Were it the rowboat itself whose shape were to be measured, the procedure would be simple enough. One would draw it up on shore, turn it upside down, and drive tacks in lightly at strategic points here and there on the surface. The measurement of distances from tack to

Problem: how to measure in curved spacetime

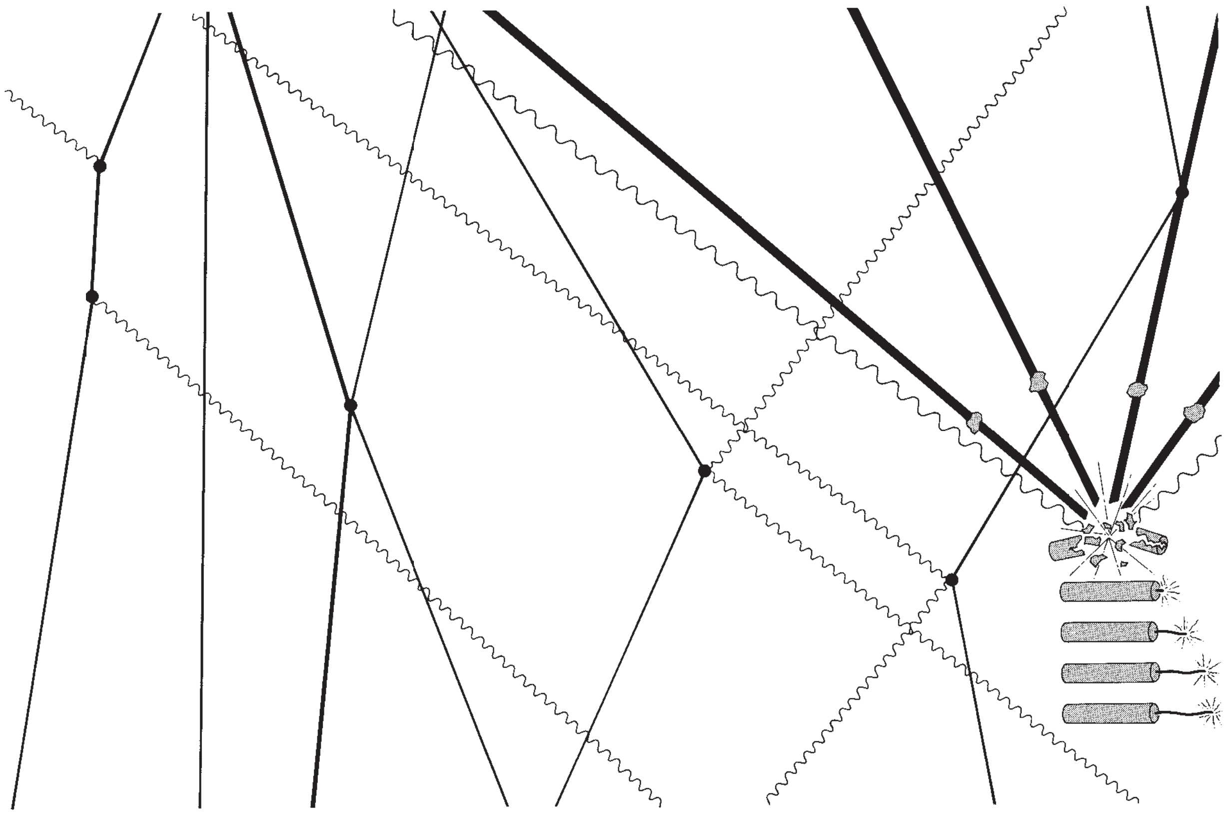


Figure 1.2.
The crossing of straws in a barn full of hay is a symbol for the world lines that fill up spacetime. By their crossings and bends, these world lines mark events with a uniqueness beyond all need of coordinate systems or coordinates. Typical events symbolized in the diagram, from left to right (black dots), are: absorption of a photon; reemission of a photon; collision between a particle and a particle; collision between a photon and a particle; another collision between a photon and a particle; explosion of a firecracker; and collision of a particle from outside with one of the fragments of that firecracker.

Resolution: characterize events by what happens there

tack would record and reveal the shape of the surface. The precision could be made arbitrarily great by making the number of tacks arbitrarily large. It takes more daring to think of driving several score pitons into the towering iceberg. But with all the daring in the world, how is one to drive a nail into spacetime to mark a point? Happily, nature provides its own way to localize a point in spacetime, as Einstein was the first to emphasize. Characterize the point by what happens there! Give a point in spacetime the name "event." Where the event lies is defined as clearly and sharply as where two straws cross each other in a barn full of hay (Figure 1.2). To say that the event marks a collision of such and such a photon with such and such a particle is identification enough. The world lines of that photon and that particle are rooted in the past and stretch out into the future. They have a rich texture of connections with nearby world lines. These nearby world lines in turn are linked in a hundred ways with world lines more remote. How then does one tell the location of an event? Tell first what world lines participate in the event. Next follow each

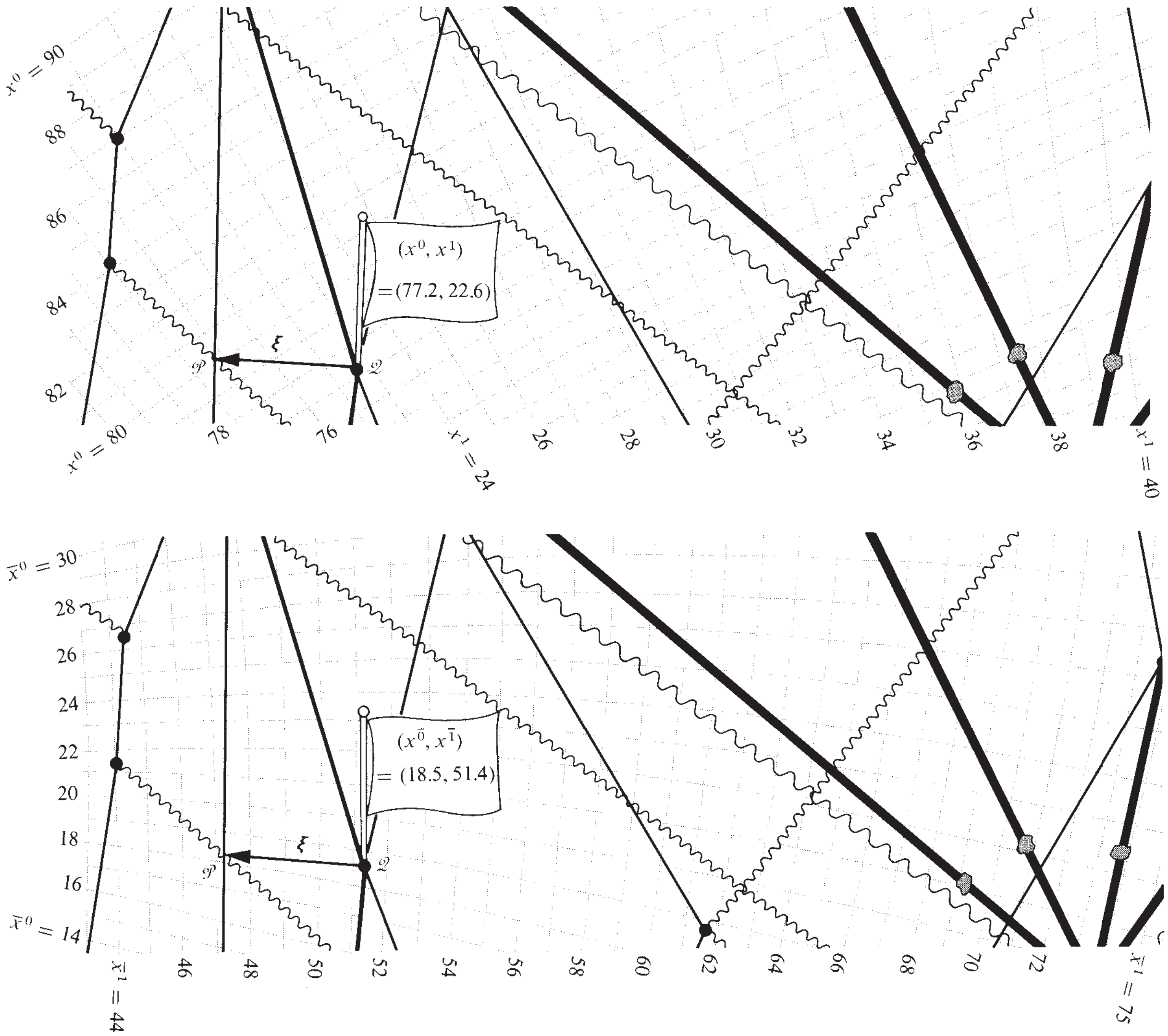


Figure 1.3.

Above: Assigning "telephone numbers" to events by way of a system of coordinates. To say that the coordinate system is "smooth" is to say that events which are almost in the same place have almost the same coordinates. Below: Putting the same set of events into equally good order by way of a different system of coordinates. Picked out specially here are two neighboring events: an event named "$\mathscr{Q}$" with coordinates $(x^0, x^1) = (77.2, 22.6)$ and $(x^{\bar{0}}, x^{\bar{1}}) = (18.5, 51.4)$; and an event named "$\mathscr{P}$" with coordinates $(x^0, x^1) = (79.9, 20.1)$ and $(x^{\bar{0}}, x^{\bar{1}}) = (18.4, 47.1)$. Events $\mathscr{Q}$ and $\mathscr{P}$ are connected by the separation "vector" $\boldsymbol{\xi}$. (Precise definition of a vector in a curved spacetime demands going to the mathematical limit in which the two points have an indefinitely small separation [N-fold reduction of the separation $\mathscr{P} - \mathscr{Q}$], and, in the resultant locally flat space, multiplying the separation up again by the factor N [$\lim N \to \infty$; "tangent space"; "tangent vector"]. Forego here that proper way of stating matters, and forego complete accuracy; hence the quote around the word "vector".) In each coordinate system the separation vector $\boldsymbol{\xi}$ is characterized by "components" (differences in coordinate values between $\mathscr{P}$ and $\mathscr{Q}$):

$$(\xi^0, \xi^1) = (79.9 - 77.2,\ 20.1 - 22.6) = (2.7,\ -2.5),$$
$$(\xi^{\bar{0}}, \xi^{\bar{1}}) = (18.4 - 18.5,\ 47.1 - 51.4) = (-0.1,\ -4.3).$$

See Box 1.1 for further discussion of events, coordinates, and vectors.

of these world lines. Name the additional events that they encounter. These events pick out further world lines. Eventually the whole barn of hay is catalogued. Each event is named. One can find one's way as surely to a given intersection as the city dweller can pick his path to the meeting of St. James Street and Piccadilly. No numbers. No coordinate system. No coordinates.

The name of an event can even be arbitrary

That most streets in Japan have no names, and most houses no numbers, illustrates one's ability to do without coordinates. One can abandon the names of two world lines as a means to identify the event where they intersect. Just as one could name a Japanese house after its senior occupant, so one can and often does attach arbitrary names to specific events in spacetime, as in Box 1.1.

Coordinates provide a convenient naming system

Coordinates, however, are convenient. How else from the great thick catalog of events, randomly listed, can one easily discover that along a certain world line one will first encounter event Trinity, then Baker, then Mike, then Argus—but not the same events in some permuted order?

To order events, introduce coordinates! (See Figure 1.3.) Coordinates are four indexed numbers per event in spacetime; on a sheet of paper, only two. Trinity acquires coordinates

$$(x^0, x^1, x^2, x^3) = (77, 23, 64, 11).$$

Coordinates generally do not measure length

In christening events with coordinates, one demands smoothness but foregoes every thought of mensuration. The four numbers for an event are nothing but an elaborate kind of telephone number. Compare their "telephone" numbers to discover whether two events are neighbors. But do not expect to learn how many meters separate them from the difference in their telephone numbers!

Several coordinate systems can be used at once

Nothing prevents a subscriber from being served by competing telephone systems, nor an event from being catalogued by alternative coordinate systems (Figure 1.3). Box 1.1 illustrates the relationships between one coordinate system and another, as well as the notation used to denote coordinates and their transformations.

Vectors

Choose two events, known to be neighbors by the nearness of their coordinate values in a smooth coordinate system. Draw a little arrow from one event to the other. Such an arrow is called a *vector*. (It is a well-defined concept in flat spacetime, or in curved spacetime in the limit of vanishingly small length; for finite lengths in curved spacetime, it must be refined and made precise, under the new name "tangent vector," on which see Chapter 9.) This vector, like events, can be given a name. But whether named "John" or "Charles" or "Kip," it is a unique, well-defined geometrical object. The name is a convenience, but the vector exists even without it.

Just as a quadruple of coordinates

$$(x^0, x^1, x^2, x^3) = (77, 23, 64, 11)$$

is a particularly useful name for the event "Trinity" (it can be used to identify what other events are nearby), so a quadruple of "components"

$$(\xi^0, \xi^1, \xi^2, \xi^3) = (1.2, -0.9, 0, 2.1)$$

Box 1.1 MATHEMATICAL NOTATION FOR EVENTS, COORDINATES, AND VECTORS

Events are denoted by capital script, one-letter Latin names such as Sometimes subscripts are used:	$\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, $\mathcal{B}$. $\mathcal{P}_0$, $\mathcal{P}_1$, $\mathcal{B}_6$.
Coordinates of an event $\mathcal{P}$ are denoted by or by or more abstractly by where it is understood that Greek indices can take on any value 0, 1, 2, or 3.	$t(\mathcal{P})$, $x(\mathcal{P})$, $y(\mathcal{P})$, $z(\mathcal{P})$, $x^0(\mathcal{P})$, $x^1(\mathcal{P})$, $x^2(\mathcal{P})$, $x^3(\mathcal{P})$, $x^\mu(\mathcal{P})$ or $x^\alpha(\mathcal{P})$,
Time coordinate (when one of the four is picked to play this role)	$x^0(\mathcal{P})$.
Space coordinates are and are sometimes denoted by It is to be understood that Latin indices take on values 1, 2, or 3.	$x^1(\mathcal{P})$, $x^2(\mathcal{P})$, $x^3(\mathcal{P})$ $x^j(\mathcal{P})$ or $x^k(\mathcal{P})$ or . . . .
Shorthand notation: One soon tires of writing explicitly the functional dependence of the coordinates, $x^\beta(\mathcal{P})$; so one adopts the shorthand notation for the coordinates of the event $\mathcal{P}$, and for the space coordinates. One even begins to think of x^β as representing the event $\mathcal{P}$ itself, but must remind oneself that the values of x^0, x^1, x^2, x^3 depend not only on the choice of $\mathcal{P}$ but also on the *arbitrary* choice of coordinates!	x^β x^j
Other coordinates for the same event $\mathcal{P}$ may be denoted EXAMPLE: In Figure 1.3 $(x^0, x^1) = (77.2, 22.6)$ and $(x^{\bar{0}}, x^{\bar{1}}) = (18.5, 51.4)$ refer to the *same* event. The bars, primes, and hats distinguish one coordinate system from another; by putting them on the indices rather than on the x's, we simplify later notation.	$x^{\bar{\alpha}}(\mathcal{P})$ or just $x^{\bar{\alpha}}$, $x^{\alpha'}(\mathcal{P})$ or just $x^{\alpha'}$, $x^{\hat{\alpha}}(\mathcal{P})$ or just $x^{\hat{\alpha}}$.
Transformation from one coordinate system to another is achieved by the four functions which are denoted more succinctly	$x^{\bar{0}}(x^0, x^1, x^2, x^3)$, $x^{\bar{1}}(x^0, x^1, x^2, x^3)$, $x^{\bar{2}}(x^0, x^1, x^2, x^3)$, $x^{\bar{3}}(x^0, x^1, x^2, x^3)$, $x^{\bar{\alpha}}(x^\beta)$.
Separation vector* (little arrow) reaching from one event $\mathcal{Q}$ to neighboring event $\mathcal{P}$ can be denoted abstractly by It can also be characterized by the coordinate-value differences† between $\mathcal{P}$ and $\mathcal{Q}$ (called "components" of the vector)	$\boldsymbol{u}$ or $\boldsymbol{v}$ or $\boldsymbol{\xi}$, or $\mathcal{P} - \mathcal{Q}$. $\xi^\alpha \equiv x^\alpha(\mathcal{P}) - x^\alpha(\mathcal{Q})$, $\xi^{\bar{\alpha}} \equiv x^{\bar{\alpha}}(\mathcal{P}) - x^{\bar{\alpha}}(\mathcal{Q})$.
Transformation of components of a vector from one coordinate system to another is achieved by partial derivatives of transformation equations since $\xi^{\bar{\alpha}} = x^{\bar{\alpha}}(\mathcal{P}) - x^{\bar{\alpha}}(\mathcal{Q}) = (\partial x^{\bar{\alpha}}/\partial x^\beta)[x^\beta(\mathcal{P}) - x^\beta(\mathcal{Q})]$.†	$\xi^{\bar{\alpha}} = \dfrac{\partial x^{\bar{\alpha}}}{\partial x^\beta}\xi^\beta$,
Einstein summation convention is used here: any index that is repeated in a product is automatically summed on	$\dfrac{\partial x^{\bar{\alpha}}}{\partial x^\beta}\xi^\beta \equiv \sum_{\beta=0}^{3} \dfrac{\partial x^{\bar{\alpha}}}{\partial x^\beta}\xi^\beta$.

*This definition of a vector is valid only in flat spacetime. The refined definition ("tangent vector") in curved spacetime is not spelled out here (see Chapter 9), but flat-geometry ideas apply with good approximation even in a curved geometry, when the two points are sufficiently close.

†These formulas are precisely accurate only when the region of spacetime under consideration is flat and when in addition the coordinates are Lorentzian. Otherwise they are approximate—though they become arbitrarily good when the separation between points and the length of the vector become arbitrarily small.

is a convenient name for the vector "John" that reaches from

$$(x^0, x^1, x^2, x^3) = (77, 23, 64, 11)$$

to

$$(x^0, x^1, x^2, x^3) = (78.2, 22.1, 64.0, 13.1).$$

How to work with the components of a vector is explored in Box 1.1.

Coordinate singularities normally unavoidable

There are many ways in which a coordinate system can be imperfect. Figure 1.4 illustrates a coordinate singularity. For another example of a coordinate singularity, run the eye over the surface of a globe to the North Pole. Note the many meridians that meet there ("collapse of cells of egg crates to zero content"). Can't one do better? Find a single coordinate system that will cover the globe without singularity? A theorem says no. Two is the minimum number of "coordinate patches" required to cover the two-sphere without singularity (Figure 1.5). This circumstance emphasizes anew that points and events are primary, whereas coordinates are a mere bookkeeping device.

Continuity of spacetime

Figures 1.2 and 1.3 show only a few world lines and events. A more detailed diagram would show a maze of world lines and of light rays and the intersections between them. From such a picture, one can in imagination step to the idealized limit: an infinitely dense collection of light rays and of world lines of infinitesimal test particles. With this idealized physical limit, the mathematical concept of a continuous four-dimensional "manifold" (four-dimensional space with certain smoothness properties) has a one-to-one correspondence; and in this limit continuous, differentiable (i.e., smooth) coordinate systems operate. The mathematics then supplies a tool to reason about the physics.

The mathematics of manifolds applied to the physics of spacetime

Dimensionality of spacetime

A simple countdown reveals the dimensionality of the manifold. Take a point $\mathscr{P}$ in an n-dimensional manifold. Its neighborhood is an n-dimensional ball (i.e., the interior of a sphere whose surface has $n - 1$ dimensions). Choose this ball so that its boundary is a smooth manifold. The dimensionality of this manifold is $(n - 1)$. In this $(n - 1)$-dimensional manifold, pick a point $\mathscr{Q}$. Its neighborhood is an $(n - 1)$-dimensional ball. Choose this ball so that . . . , and so on. Eventually one comes by this construction to a manifold that is two-dimensional but is not yet known to be two-dimensional (two-sphere). In this two-dimensional manifold, pick a point $\mathscr{M}$. Its neighborhood is a two-dimensional ball ("disc"). Choose this disc so that its boundary is a smooth manifold (circle). In this manifold, pick a point $\mathscr{N}$. Its neighborhood is a one-dimensional ball, but is not yet known to be one-dimensional ("line segment"). The boundaries of this object are two points. This circumstance tells that the intervening manifold is one-dimensional; therefore the previous manifold was two-dimensional; and so on. The dimensionality of the original manifold is equal to the number of points employed in the construction. For spacetime, the dimensionality is 4.

This kind of mathematical reasoning about dimensionality makes good sense at the everyday scale of distances, at atomic distances (10^{-8} cm), at nuclear dimensions (10^{-13} cm), and even at lengths smaller by several powers of ten, if one judges by the concord between prediction and observation in quantum electrodynamics at high

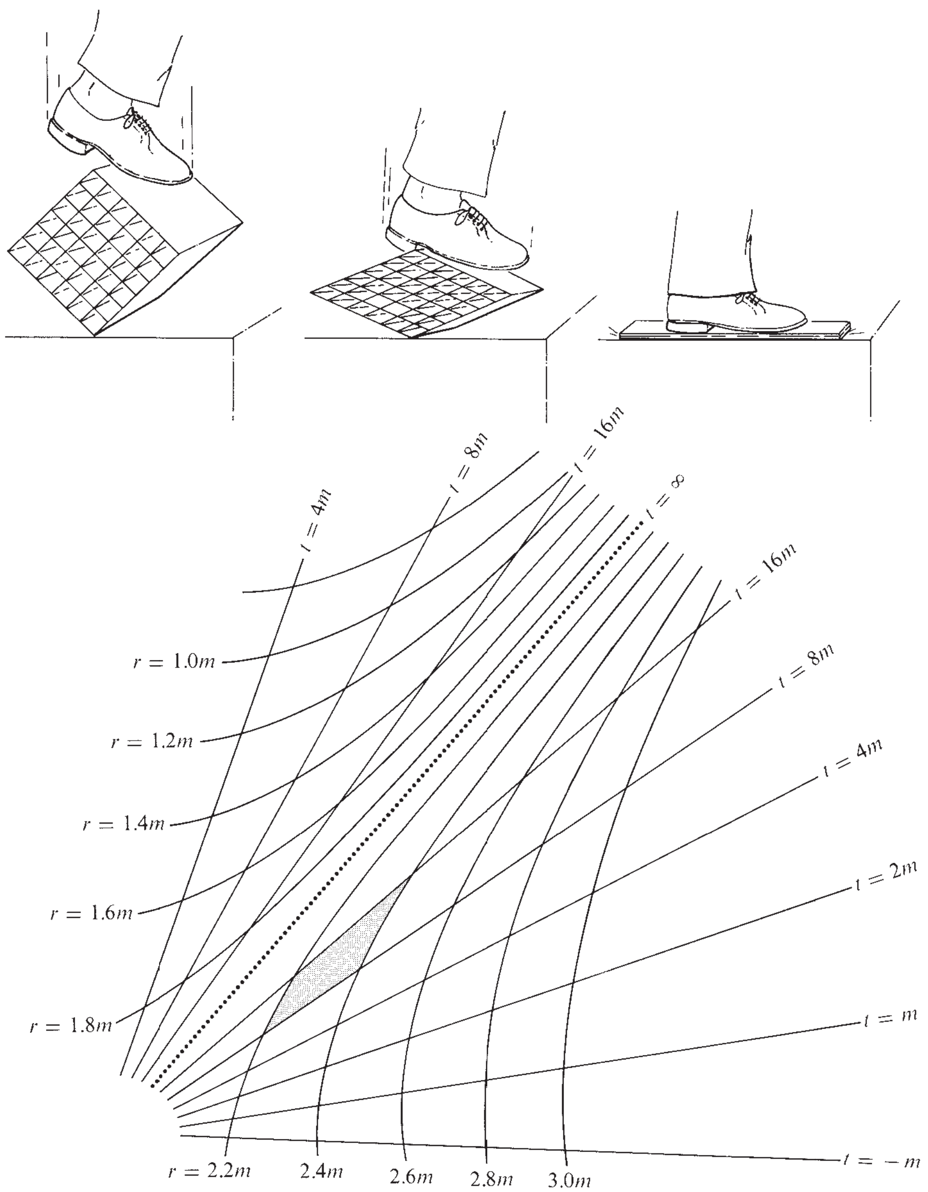


Figure 1.4.

How a mere coordinate singularity arises. Above: A coordinate system becomes *singular* when the "cells in the egg crate" are squashed to zero volume. Below: An example showing such a singularity in the Schwarzschild coordinates r, t often used to describe the geometry around a black hole (Chapter 31). For simplicity the angular coordinates θ, ϕ have been suppressed. The singularity shows itself in two ways. First, all the points along the dotted line, while quite distinct one from another, are designated by the same pair of (r, t) values; namely, $r = 2m$, $t = \infty$. The coordinates provide no way to distinguish these points. Second, the "cells in the egg crate," of which one is shown grey in the diagram, collapse to zero content at the dotted line. In summary, there is nothing strange about the geometry at the dotted line; all the singularity lies in the coordinate system ("poor system of telephone numbers"). No confusion should be permitted to arise from the accidental circumstance that the t coordinate attains an infinite value on the dotted line. No such infinity would occur if t were replaced by the new coordinate $\bar{t}$, defined by

$$(t/2m) = \tan(\bar{t}/2m).$$

When $t = \infty$, the new coordinate $\bar{t}$ is $\bar{t} = \pi m$. The r, $\bar{t}$ coordinates still provide no way to distinguish the points along the dotted line. They still give "cells in the egg crate" collapsed to zero content along the dotted line.

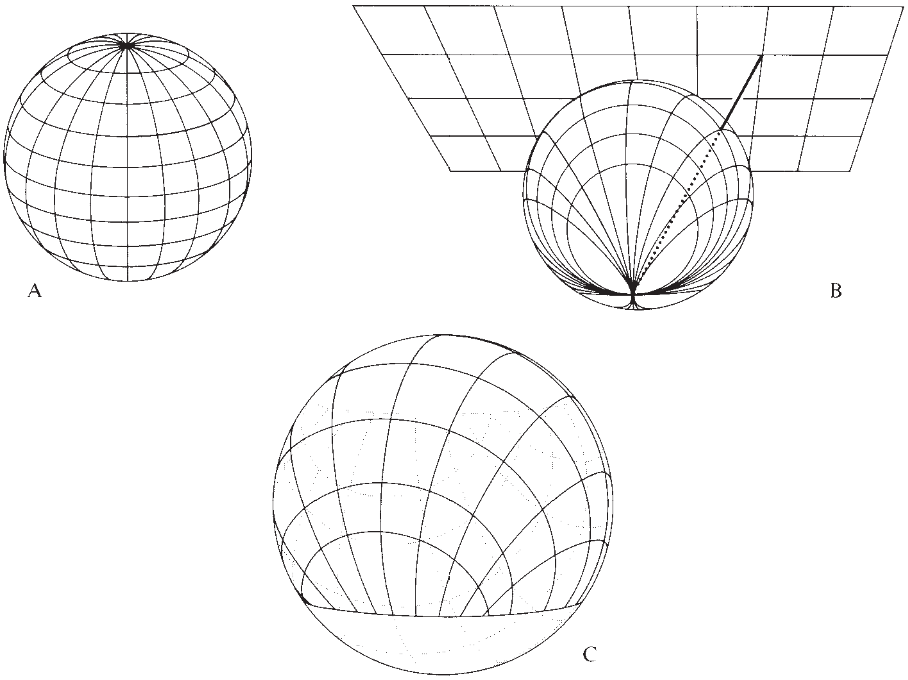


Figure 1.5.
Singularities in familiar coordinates on the two-sphere can be eliminated by covering the sphere with two overlapping coordinate patches. A. Spherical polar coordinates, singular at the North and South Poles, and discontinuous at the international date line. B. Projection of the Euclidean coordinates of the Euclidean two-plane, tangent at the North Pole, onto the sphere via a line running to the South Pole; coordinate singularity at the South Pole. C. Coverage of two-sphere by two overlapping coordinate patches. One, constructed as in B, covers without singularity the northern hemisphere and also the southern tropics down to the Tropic of Capricorn. The other (grey) also covers without singularity all of the tropics and the southern hemisphere besides.

energies (corresponding de Broglie wavelength 10^{-16} cm). Moreover, classical general relativity thinks of the spacetime manifold as a deterministic structure, completely well-defined down to arbitrarily small distances. Not so quantum general relativity or "quantum geometrodynamics." It predicts violent fluctuations in the geometry at distances on the order of the Planck length,

Breakdown in smoothness of spacetime at Planck length

$$\begin{aligned} L^* &= (\hbar G/c^3)^{1/2} \\ &= [(1.054 \times 10^{-27}\text{ g cm}^2/\text{sec})(6.670 \times 10^{-8}\text{ cm}^3/\text{g sec}^2)]^{1/2} \times \\ &\qquad\qquad \times (2.998 \times 10^{10}\text{ cm/sec})^{-3/2} \qquad (1.1) \\ &= 1.616 \times 10^{-33}\text{ cm}. \end{aligned}$$

No one has found any way to escape this prediction. As nearly as one can estimate, these fluctuations give space at small distances a "multiply connected" or "foamlike" character. This lack of smoothness may well deprive even the concept of dimensionality itself of any meaning at the Planck scale of distances. The further exploration of this issue takes one to the frontiers of Einstein's theory (Chapter 44).

If spacetime at small distances is far from the mathematical model of a continuous manifold, is there not also at larger distances a wide gap between the mathematical idealization and the physical reality? The infinitely dense collection of light rays and of world lines of infinitesimal test particles that are to define all the points of the manifold: they surely are beyond practical realization. Nobody has ever found a particle that moves on timelike world lines (finite rest mass) lighter than an electron. A collection of electrons, even if endowed with zero density of charge (e^+ and e^- world lines present in equal numbers) will have a density of mass. This density will curve the very manifold under study. Investigation in infinite detail means unlimited density, and unlimited disturbance of the geometry.

Difficulty in defining geometry even at classical distances?

However, to demand investigatability in infinite detail in the sense just described is as out of place in general relativity as it would be in electrodynamics or gas dynamics. Electrodynamics speaks of the strength of the electric and magnetic field at each point in space and at each moment of time. To measure those fields, it is willing to contemplate infinitesimal test particles scattered everywhere as densely as one pleases. However, the test particles do not have to be there at all to give the field reality. The field has everywhere a clear-cut value and goes about its deterministic dynamic evolution willy-nilly and continuously, infinitesimal test particles or no infinitesimal test particles. Similarly with the geometry of space.

No; one must accept geometry at classical distances as meaningful

In conclusion, when one deals with spacetime in the context of classical physics, one accepts (1) the notion of "infinitesimal test particle" and (2) the idealization that the totality of identifiable events forms a four-dimensional continuous manifold. Only at the end of this book will a look be taken at some of the limitations placed by the quantum principle on one's way of speaking about and analyzing spacetime.

§1.3. WEIGHTLESSNESS

"Gravity is a great mystery. Drop a stone. See it fall. Hear it hit. No one understands why." What a misleading statement! Mystery about fall? What else should the stone do except fall? To fall is normal. The abnormality is an object standing in the way of the stone. If one wishes to pursue a "mystery," do not follow the track of the falling stone. Look instead at the impact, and ask what was the force that pushed the stone away from its natural "world line," (i.e., its natural track through spacetime). That could lead to an interesting issue of solid-state physics, but that is not the topic of concern here. Fall is. Free fall is synonymous with weightlessness: absence of any force to drive the object away from its normal track through spacetime. Travel aboard a freely falling elevator to experience weightlessness. Or travel aboard a spaceship also falling straight toward the Earth. Or, more happily, travel aboard a spaceship in that state of steady fall toward the Earth that marks a circular orbit. In each case one is following a natural track through spacetime.

Free fall is the natural state of motion

The traveler has one chemical composition, the spaceship another; yet they travel together, the traveler weightless in his moving home. Objects of such different nuclear constitution as aluminum and gold fall with accelerations that agree to better than one part in 10^{11}, according to Roll, Krotkov, and Dicke (1964), one of the most important null experiments in all physics (see Figure 1.6). Individual molecules fall in step, too, with macroscopic objects [Estermann, Simpson, and Stern (1938)]; and so do individual neutrons [Dabbs, Harvey, Paya, and Horstmann (1965)], individual

All objects fall with the same acceleration

(*continued on page 16*)

Figure 1.6.
Principle of the Roll-Krotkov-Dicke experiment, which showed that the gravitational accelerations of gold and aluminum are equal to 1 part in 10^{11} or better (Princeton, 1964). In the upper lefthand corner, equal masses of gold and aluminum hang from a supporting bar. This bar in turn is supported at its midpoint. If both objects fall toward the sun with the same acceleration of $g = 0.59$ cm/sec^2, the bar does not turn. If the Au mass receives a higher acceleration, $g + \delta g$, then the gold end of the bar starts to turn toward the sun in the Earth-fixed frame. Twelve hours later the sun is on the other side, pulling the other way. The alternating torque lends itself to recognition against a background of noise because of its precise 24-hour period. Unhappily, any substantial mass nearby, such as an experimenter, located at M, will produce a torque that swamps the effect sought. Therefore the actual arrangement was as shown in the body of the figure. One gold weight and two aluminum weights were supported at the three corners of a horizontal equilateral triangle, 6 cm on a side (three-fold axis of symmetry, giving zero response to all the simplest nonuniformities in the gravitational field). Also, the observers performed all operations remotely to eliminate their own gravitational effects*. To detect a rotation of the torsion balance as small as $\sim 10^{-9}$ rad without disturbing the balance, Roll, Krotkov, and Dicke reflected a very weak light beam from the optically flat back face of the quartz triangle. The image of the source slit fell on a wire of about the same size as the slit image. The light transmitted past the wire fell on a photomultiplier. A separate oscillator circuit drove the wire back and forth across the image at 3,000 hertz. When the image was centered perfectly, only even harmonics of the oscillation frequency appeared in the light intensity. However, when the image was displaced slightly to one side, the fundamental frequency appeared in the light intensity. The electrical output of the photomultiplier then contained a 3,000-hertz component. The magnitude and sign of this component were determined automatically. Equally automatically a proportional D.C. voltage was applied to the electrodes shown in the diagram. It restored the torsion balance to its zero position. The D.C. voltage required to restore the balance to its zero position was recorded as a measure of the torque acting on the pendulum. This torque was Fourier-analyzed over a period of many days. The magnitude of the Fourier component of 24-hour period indicated a ratio $\delta g/g = (0.96 \pm 1.04) \times 10^{-11}$. Aluminum and gold thus fall with the same acceleration, despite their important differences summarized in the table.

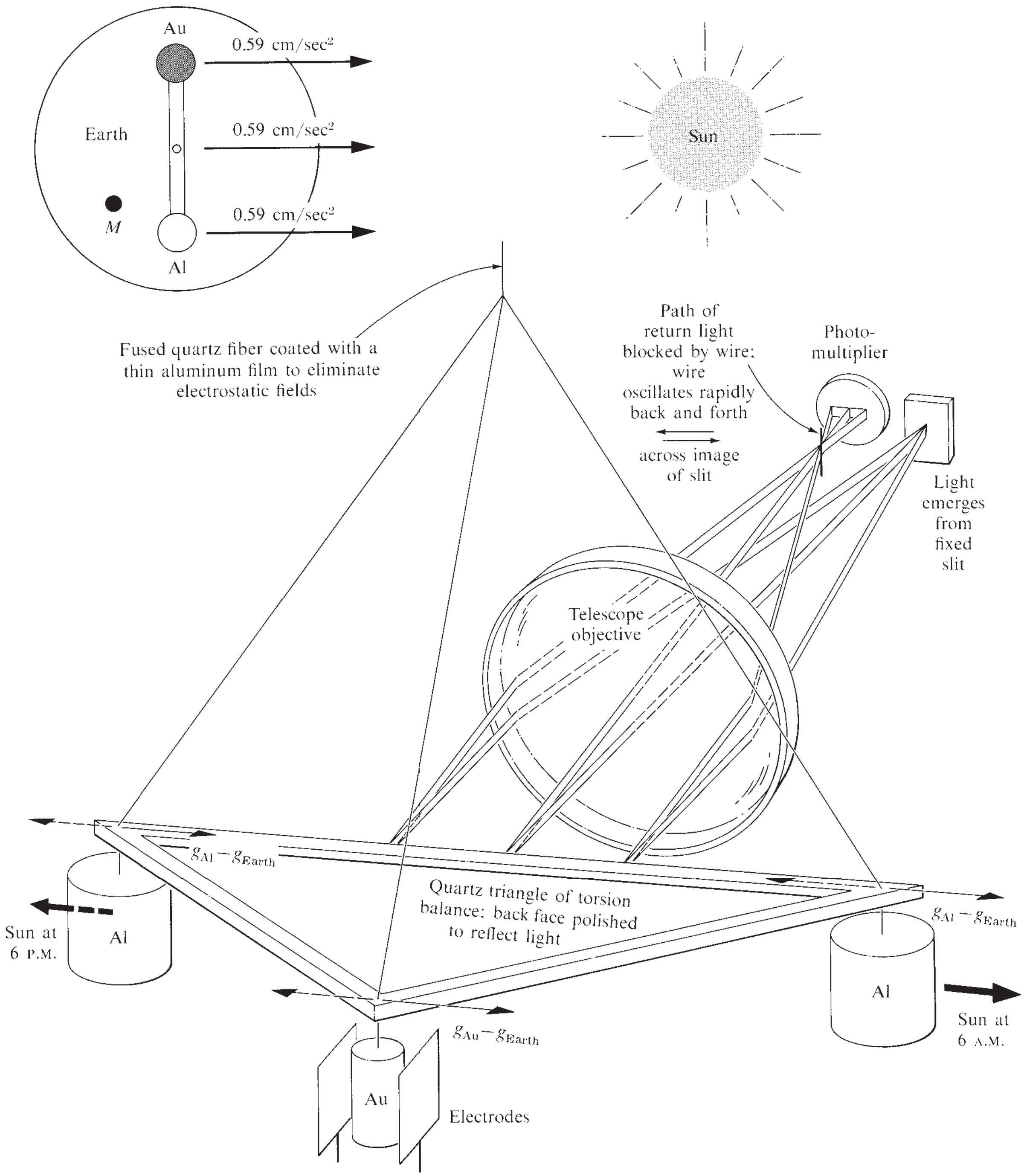
Ratios	*Al*	*Au*
$\dfrac{\text{Number of neutrons}}{\text{Number of protons}}$	1.08	1.5
$\dfrac{\text{Mass of kinetic energy of K-electron}}{\text{Rest mass of electron}}$	0.005	0.16
$\dfrac{\text{Electrostatic mass-energy of nucleus}}{\text{Mass of atom}}$	0.001	0.004

The theoretical implications of this experiment will be discussed in greater detail in Chapters 16 and 38.

Braginsky and Panov (1971) at Moscow University performed an experiment identical in principle to that of Dicke-Roll-Krotkov, but with a modified experimental set-up. Comparing the accelerations of platinum and aluminum rather than of gold and aluminum, they say that

$$\delta g/g \lesssim 1 \times 10^{-12}.$$

*Other perturbations had to be, and were, guarded against. (1) A bit of iron on the torsion balance as big as 10^{-3} cm on a side would have contributed, in the Earth's magnetic field, a torque a hundred times greater than the measured torque. (2) The unequal pressure of radiation on the two sides of a mass would have produced an unacceptably large perturbation if the temperature difference between these two sides had exceeded 10^{-4} °K. (3) Gas evolution from one side of a mass would have propelled it like a rocket. If the rate of evolution were as great as 10^{-8} g/day, the calculated force would have been $\sim 10^{-7}$ g cm/sec^2, enough to affect the measurements. (4) The rotation was measured with respect to the pier that supported the equipment. As a guarantee that this pier did not itself rotate, it was anchored to bed rock. (5) Electrostatic forces were eliminated; otherwise they would have perturbed the balance.


Au
0.59 cm/sec²
Earth
0.59 cm/sec²
M
0.59 cm/sec²
Al
Sun
Fused quartz fiber coated with a thin aluminum film to eliminate electrostatic fields
Path of return light blocked by wire; wire oscillates rapidly back and forth across image of slit
Photo-multiplier
Light emerges from fixed slit
Telescope objective
$g_{Al} - g_{Earth}$
Quartz triangle of torsion balance; back face polished to reflect light
$g_{Al} - g_{Earth}$
Sun at 6 P.M.
Al
Al
Sun at 6 A.M.
$g_{Au} - g_{Earth}$
Au
Electrodes

electrons [Witteborn and Fairbank (1967)] and individual mu mesons [Beall (1970)]. What is more, not one of these objects has to see out into space to know how to move.

Contemplate the interior of a spaceship, and a key, penny, nut, and pea by accident or design set free inside. Shielded from all view of the world outside by the walls of the vessel, each object stays at rest relative to the vessel. Or it moves through the room in a straight line with uniform velocity. That is the lesson which experience shouts out.

Forego talk of acceleration! That, paradoxically, is the lesson of the circumstance that "all objects fall with the same acceleration." Whose fault were those accelerations, after all? They came from allowing a groundbased observer into the act. The

Box 1.2 MATERIALS OF THE MOST DIVERSE COMPOSITION FALL WITH THE SAME ACCELERATION ("STANDARD WORLD LINE")

Aristotle: "the downward movement of a mass of gold or lead, or of any other body endowed with weight, is quicker in proportion to its size."

Pre-Galilean literature: metal and wood weights fall at the same rate.

Galileo: (1) "the variation of speed in air between balls of gold, lead, copper, porphyry, and other heavy materials is so slight that in a fall of 100 cubits [about 46 meters] a ball of gold would surely not outstrip one of copper by as much as four fingers. Having observed this, I came to the conclusion that in a medium totally void of resistance all bodies would fall with the same speed." (2) later experiments of greater precision "diluting gravity" and finding same time of descent for different objects along an inclined plane.

Newton: inclined plane replaced by arc of pendulum bob; "time of fall" for bodies of different composition determined by comparing time of oscillation of pendulum bobs of the two materials. Ultimate limit of precision in such experiments limited by problem of determining effective length of each pendulum: (acceleration) = $(2\pi/\text{period})^2$(length).

Lorand von Eötvös, Budapest, 1889 and 1922: compared on the rotating earth the vertical defined by a plumb bob of one material with the vertical defined by a plumb bob of other material. The two hanging masses, by the two unbroken threads that support them, were drawn along identical world lines through spacetime (middle of the laboratory of Eötvös!). If cut free, would they also follow identical tracks through spacetime ("normal world line of test mass")? If so, the acceleration that draws the actual world line from the normal free-fall world line will have a standard value, $\boldsymbol{a}$. The experiment of Eötvös did not try to test agreement on the magnitude of $\boldsymbol{a}$ between the two masses. Doing so would have required (1) cutting the threads and (2) following the fall of the two masses. Eötvös renounced this approach in favor of a static observation that he could make with greater precision, comparing the *direction* of $\boldsymbol{a}$ for the two masses. The direction of the supporting thread, so his argument ran, reveals the direction in which the mass is being dragged away from its normal world line of "free fall" or "weightlessness." This acceleration is the vectorial resultant of (1) an acceleration of magnitude $\boldsymbol{g}$, directed outward against so-called gravity, and (2) an acceleration directed toward the axis of rotation of the earth, of magnitude $\omega^2 R \sin\theta$ (ω, angular vepush of the ground under his feet was driving him away from a natural world line. Through that flaw in his arrangements, he became responsible for all those accelerations. Put him in space and strap rockets to his legs. No difference!* Again the responsibility for what he sees is his. Once more he notes that "all objects fall with

*"No difference" spelled out amounts to Einstein's (1911) principle of the local equivalence between a "gravitational field" and an acceleration: "*We arrive at a very satisfactory interpretation of this law of experience, if we assume that the systems* K *and* K′ *are physically exactly equivalent, that is, if we assume that we may just as well regard the system* K *as being in a space free from gravitational fields, if we then regard* K *as uniformly accelerated. This assumption of exact physical equivalence makes it impossible for us to speak of the absolute acceleration of the system of reference, just as the usual theory of relativity forbids us to talk of the absolute velocity of a system; and it makes the equal falling of all bodies in a gravitational field seem a matter of course.*"

locity; R, radius of earth; θ, polar angle measured from North Pole to location of experiment). This centripetal acceleration has a vertical component $-\omega^2 R \sin^2\theta$ too small to come into discussion. The important component is $\omega^2 R \sin\theta\cos\theta$, directed northward and parallel to the surface of the earth. It deflects the thread by the angle

$$\frac{\text{horizontal acceleration}}{\text{vertical acceleration}} = \frac{\omega^2 R \sin\theta\cos\theta}{g} = \frac{3.4\ \text{cm/sec}^2}{980\ \text{cm/sec}^2}\sin\theta\cos\theta = 1.7\times10^{-3}\ \text{radian at}\ \theta = 45^\circ$$

from the straight line connecting the center of the earth to the point of support. A difference, δg, of one part in 10^8 between g for the two hanging substances would produce a difference in angle of hang of plumb bobs equal to 1.7×10^{-11} radian at Budapest ($\theta = 42.5^\circ$). Eötvös reported $\delta g/g$ less than a few parts in 10^9.

Roll, Krotkov, and Dicke, Princeton, 1964: employed as fiducial acceleration, not the 1.7 cm/sec^2 steady horizontal acceleration, produced by the earth's rotation at $\theta = 45^\circ$, but the daily alternating 0.59 cm/sec^2 produced by the sun's attraction. Reported $|g(\text{Au}) - g(\text{Al})|/g$ less than 1×10^{-11}. See Figure 1.6.

Braginsky and Panov, Moscow, 1971: like Roll, Krotkov, and Dicke, employed Sun's attraction as fiducial acceleration. Reported $|g(\text{Pt}) - g(\text{Al})|/g$ less than 1×10^{-12}.

Beall, 1970: particles that are deflected less by the Earth's or the sun's gravitational field than a photon would be, effectively travel faster than light. If they are charged or have other electromagnetic structure, they would then emit Čerenkov radiation, and reduce their velocity below threshold in less than a micron of travel. The threshold is at energies around 10^3 mc^2. Ultrarelativistic particles in cosmic-ray showers are not easily identified, but observations of 10^{13} eV muons show that muons are not "too light" by as much as 5×10^{-5}. Conversely, a particle P bound more strongly than photons by gravity will transfer the momentum needed to make pair production $\gamma \rightarrow \mathrm{P} + \overline{\mathrm{P}}$ occur within a submicron decay length. The existence of photons with energies above 10^{13} eV shows that $e^\pm$ are not "too heavy" by 5 parts in 10^9, $\mu^\pm$ not by 2 in 10^4, Λ, Ξ^-, Ω^- not by a few per cent.

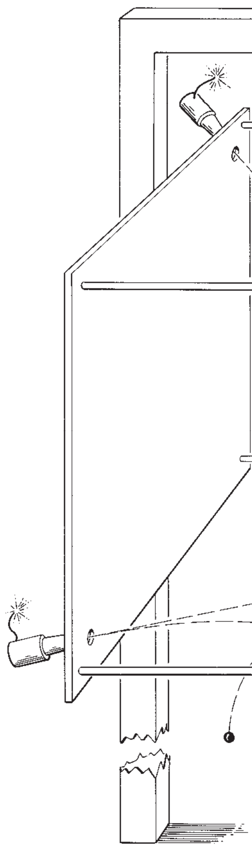


Figure 1.7.
"Weightlessness" as test for a local inertial frame of reference ("Lorentz frame"). Each spring-driven cannon succeeds in driving its projectile, a steel ball bearing, through the aligned holes in the sheets of lucite, and into the woven-mesh pocket, when the frame of reference is free of rotation and in free fall ("normal world line through spacetime"). A cannon would fail (curved and ricocheting trajectory at bottom of drawing) if the frame were hanging as indicated when the cannon went off ("frame drawn away by pull of rope from its normal world line through spacetime"). Harold Waage at Princeton has constructed such a model for an inertial reference frame with lucite sheets about 1 m square. The "fuses" symbolizing time delay were replaced by electric relays. Penetration fails if the frame (1) rotates, (2) accelerates, or (3) does any combination of the two. It is difficult to cite any easily realizable device that more fully illustrates the meaning of the term "local Lorentz frame."

Eliminate the acceleration by use of a local inertial frame

the same acceleration." Physics looks as complicated to the jet-driven observer as it does to the man on the ground. Rule out both observers to make physics look simple. Instead, travel aboard the freely moving spaceship. Nothing could be more natural than what one sees: every free object moves in a straight line with uniform velocity. This is the way to do physics! Work in a very special coordinate system: a coordinate frame in which one is weightless; *a local inertial frame of reference.* Or calculate how things look in such a frame. Or—if one is constrained to a ground-based frame of reference—use a particle moving so fast, and a path length so limited, that the ideal, freely falling frame of reference and the actual ground-based frame get out of alignment by an amount negligible on the scale of the experiment. [Given a 1,500-m linear accelerator, and a 1 GeV electron, time of flight $\simeq$ (1.5 $\times$ 10^5 cm)/

$(3 \times 10^{10}$ cm/sec$) = 0.5 \times 10^{-5}$ sec; fall in this time $\sim\frac{1}{2}gt^2 = (490$ cm/sec$^2)(0.5 \times 10^{-5}$ sec$)^2 \simeq 10^{-8}$ cm.]

In analyzing physics in a local inertial frame of reference, or following an ant on his little section of apple skin, one wins simplicity by foregoing every reference to what is far away. Physics is simple only when viewed locally: that is Einstein's great lesson.

Newton spoke differently: "Absolute space, in its own nature, without relation to anything external, remains always similar and immovable." But how does one give meaning to Newton's absolute space, find its cornerstones, mark out its straight lines? In the real world of gravitation, no particle ever follows one of Newton's straight lines. His ideal geometry is beyond observation. "A comet going past the sun is deviated from an ideal straight line." No. There is no pavement on which to mark out that line. The "ideal straight line" is a myth. It never happened, and it never will.

Newton's absolute space is unobservable, nonexistent

> "It required a severe struggle [for Newton] to arrive at the concept of independent and absolute space, indispensible for the development of theory. . . . Newton's decision was, in the contemporary state of science, the only possible one, and particularly the only fruitful one. But the subsequent development of the problems, proceeding in a roundabout way which no one could then possibly foresee, has shown that the resistance of Leibniz and Huygens, intuitively well-founded but supported by inadequate arguments, was actually justified. . . . It has required no less strenuous exertions subsequently to overcome this concept [of absolute space]"
>
> [A. EINSTEIN (1954)].

What is direct and simple and meaningful, according to Einstein, is the geometry in every local inertial reference frame. There every particle moves in a straight line with uniform velocity. *Define* the local inertial frame so that this simplicity occurs for the first few particles (Figure 1.7). In the frame thus defined, every other free particle is observed also to move in a straight line with uniform velocity. Collision and disintegration processes follow the laws of conservation of momentum and energy of special relativity. That all these miracles come about, as attested by tens of thousands of observations in elementary particle physics, is witness to the inner workings of the machinery of the world. The message is easy to summarize: (1) physics is always and everywhere locally Lorentzian; i.e., locally the laws of special relativity are valid; (2) this simplicity shows most clearly in a local Lorentz frame of reference ("inertial frame of reference"; Figure 1.7); and (3) to test for a local Lorentz frame, test for weightlessness!

But Einstein's local inertial frames exist, are simple

In local inertial frames, physics is Lorentzian

§1.4. LOCAL LORENTZ GEOMETRY, WITH AND WITHOUT COORDINATES

On the surface of an apple within the space of a thumbprint, the geometry is Euclidean (Figure 1.1; the view in the magnifying glass). In spacetime, within a limited region, the geometry is Lorentzian. On the apple the distances between point and point accord with the theorems of Euclid. In spacetime the intervals ("proper distance," "proper time") between event and event satisfy the corresponding theorems of Lorentz-Minkowski geometry (Box 1.3). These theorems lend themselves

Local Lorentz geometry is the spacetime analog of local Euclidean geometry.

(*continued on page 23*)

Box 1.3 LOCAL LORENTZ GEOMETRY AND LOCAL EUCLIDEAN GEOMETRY: WITH AND WITHOUT COORDINATES

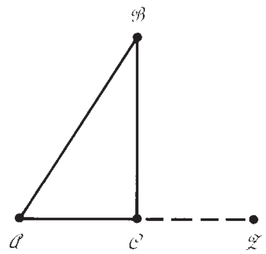
I. Local Euclidean Geometry

What does it mean to say that the geometry of a tiny thumbprint on the apple is Euclidean?

A. *Coordinate-free language* (Euclid):
Given a line $\mathscr{A}\mathscr{C}$. Extend it by an equal distance $\mathscr{C}\mathscr{Z}$. Let $\mathscr{B}$ be a point not on $\mathscr{A}\mathscr{Z}$ but equidistant from $\mathscr{A}$ and $\mathscr{Z}$. Then

$$s_{\mathscr{A}\mathscr{B}}{}^2 = s_{\mathscr{A}\mathscr{C}}{}^2 + s_{\mathscr{B}\mathscr{C}}{}^2.$$

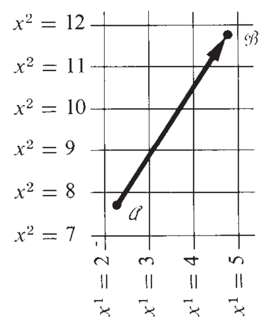
(Theorem of Pythagoras; also other theorems of Euclidean geometry.)



B. *Language of coordinates* (Descartes):
From any point $\mathscr{A}$ to any other point $\mathscr{B}$ there is a distance s given in suitable (Euclidean) coordinates by

$$s_{\mathscr{A}\mathscr{B}}{}^2 = [x^1(\mathscr{B}) - x^1(\mathscr{A})]^2 + [x^2(\mathscr{B}) - x^2(\mathscr{A})]^2.$$

If one succeeds in finding any coordinate system where this is true for all points $\mathscr{A}$ and $\mathscr{B}$ in the thumbprint, then one is guaranteed that (i) this coordinate system is locally Euclidean, and (ii) the geometry of the apple's surface is locally Euclidean.

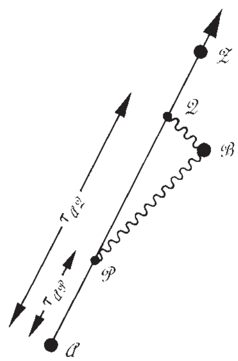


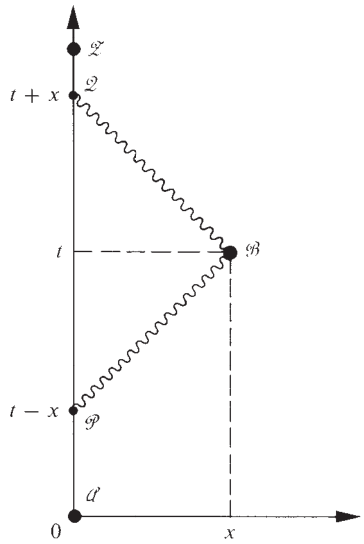
II. Local Lorentz Geometry

What does it mean to say that the geometry of a sufficiently limited region of spacetime in the real physical world is Lorentzian?

A. *Coordinate-free language* (Robb 1936):
Let $\mathscr{A}\mathscr{Z}$ be the world line of a free particle. Let $\mathscr{B}$ be an event not on this world line. Let a light ray from $\mathscr{B}$ strike $\mathscr{A}\mathscr{Z}$ at the event $\mathscr{Q}$. Let a light ray take off from such an earlier event $\mathscr{P}$ along $\mathscr{A}\mathscr{Z}$ that it reaches $\mathscr{B}$. Then the proper distance $s_{\mathscr{A}\mathscr{B}}$ (spacelike separation) or proper time $\tau_{\mathscr{A}\mathscr{B}}$ (timelike separation) is given by

$$s_{\mathscr{A}\mathscr{B}}{}^2 \equiv -\tau_{\mathscr{A}\mathscr{B}}{}^2 = -\tau_{\mathscr{A}\mathscr{Q}}\tau_{\mathscr{A}\mathscr{P}}.$$



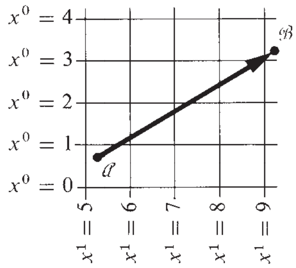


Proof of above criterion for local Lorentz geometry, using coordinate methods in the local Lorentz frame where particle remains at rest:

$$\begin{aligned}\tau_{\mathcal{A}\mathcal{B}}{}^2 &= t^2 - x^2 = (t - x)(t + x)\\ &= \tau_{\mathcal{A}\mathcal{P}}\tau_{\mathcal{A}\mathcal{Q}}.\end{aligned}$$

B. *Language of coordinates* (Lorentz, Poincaré, Minkowski, Einstein):
From any event $\mathcal{A}$ to any other nearby event $\mathcal{B}$, there is a proper distance $s_{\mathcal{A}\mathcal{B}}$ or proper time $\tau_{\mathcal{A}\mathcal{B}}$ given in suitable (local Lorentz) coordinates by

$$\begin{aligned}s_{\mathcal{A}\mathcal{B}}{}^2 = -\tau_{\mathcal{A}\mathcal{B}}{}^2 = &-[x^0(\mathcal{B}) - x^0(\mathcal{A})]^2\\ &+ [x^1(\mathcal{B}) - x^1(\mathcal{A})]^2\\ &+ [x^2(\mathcal{B}) - x^2(\mathcal{A})]^2\\ &+ [x^3(\mathcal{B}) - x^3(\mathcal{A})]^2.\end{aligned}$$



If one succeeds in finding any coordinate system where this is locally true for all neighboring events $\mathcal{A}$ and $\mathcal{B}$, then one is guaranteed that (i) this coordinate system is locally Lorentzian, and (ii) the geometry of spacetime is locally Lorentzian.

III. Statements of Fact

The geometry of an apple's surface is locally Euclidean everywhere. The geometry of spacetime is locally Lorentzian everywhere.

Box 1.3 (*continued*)

IV. Local Geometry in the Language of Modern Mathematics

A. *The metric for any manifold:*
At each point on the apple, at each event of spacetime, indeed, at each point of any "Riemannian manifold," there exists a geometrical object called the *metric tensor* $\boldsymbol{g}$. It is a machine with two input slots for the insertion of two vectors:

$$\begin{array}{c} \text{slot 1} \quad \text{slot 2} \\ \downarrow \qquad\quad \downarrow \\ \boldsymbol{g}(\qquad , \qquad). \end{array}$$

If one inserts the same vector $\boldsymbol{u}$ into both slots, one gets out the square of the length of $\boldsymbol{u}$:

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{u}) = \boldsymbol{u}^2.$$

If one inserts two different vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$ (it matters not in which order!), one gets out a number called the "scalar product of $\boldsymbol{u}$ on $\boldsymbol{v}$" and denoted $\boldsymbol{u} \cdot \boldsymbol{v}$:

$$\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}.$$

The metric is a linear machine:

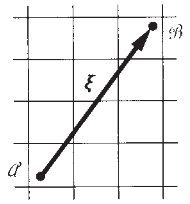
$$\begin{aligned} \boldsymbol{g}(2\boldsymbol{u} + 3\boldsymbol{w}, \boldsymbol{v}) &= 2\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) + 3\boldsymbol{g}(\boldsymbol{w}, \boldsymbol{v}), \\ \boldsymbol{g}(\boldsymbol{u}, a\boldsymbol{v} + b\boldsymbol{w}) &= a\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) + b\boldsymbol{g}(\boldsymbol{u}, \boldsymbol{w}). \end{aligned}$$

Consequently, in a given (arbitrary) coordinate system, its operation on two vectors can be written in terms of their components as a bilinear expression:

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{u}, \boldsymbol{v}) &= g_{\alpha\beta} u^\alpha v^\beta \\ &\qquad \text{(implied summation on } \alpha, \beta) \\ &= g_{11}u^1v^1 + g_{12}u^1v^2 + g_{21}u^2v^1 + \cdots . \end{aligned}$$

The quantities $g_{\alpha\beta} = g_{\beta\alpha}$ (α and β running from 0 to 3 in spacetime, from 1 to 2 on the apple) are called the "components of $\boldsymbol{g}$ in the given coordinate system."

B. *Components of the metric in local Lorentz and local Euclidean frames:*
To connect the metric with our previous descriptions of the local geometry, introduce local Euclidean coordinates (on apple) or local Lorentz coordinates (in spacetime).



Let $\boldsymbol{\xi}$ be the separation vector reaching from $\mathscr{A}$ to $\mathscr{B}$. Its components in the local Euclidean (Lorentz) coordinates are

$$\xi^\alpha = x^\alpha(\mathscr{B}) - x^\alpha(\mathscr{A})$$

(cf. Box 1.1). Then the squared length of $\boldsymbol{u}_{\mathscr{A}\mathscr{B}}$, which is the same as the squared distance from $\mathscr{A}$ to $\mathscr{B}$, must be (cf. I.B. and II.B. above)

$$\begin{aligned} \boldsymbol{\xi} \cdot \boldsymbol{\xi} &= \boldsymbol{g}(\boldsymbol{\xi}, \boldsymbol{\xi}) = g_{\alpha\beta}\xi^\alpha\xi^\beta \\ &= s_{\mathscr{A}\mathscr{B}}{}^2 = (\xi^1)^2 + (\xi^2)^2 \text{ on apple} \\ &= -(\xi^0)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 \\ &\qquad\qquad \text{in spacetime.} \end{aligned}$$

Consequently, the components of the metric are

$$\begin{aligned} &g_{11} = g_{22} = 1,\ g_{12} = g_{21} = 0; \\ &\text{i.e., } g_{\alpha\beta} = \delta_{\alpha\beta} \qquad \text{on apple, in local Euclidean coordinates;} \\ &g_{00} = -1,\ g_{0k} = 0,\ g_{jk} = \delta_{jk} \\ &\qquad \text{in spacetime, in local Lorentz coordinates.} \end{aligned}$$

These special components of the metric in local Lorentz coordinates are written here and hereafter as $g_{\hat{\alpha}\hat{\beta}}$ or $\eta_{\alpha\beta}$, by analogy with the Kronecker delta $\delta_{\alpha\beta}$. In matrix notation:

$$\|g_{\hat{\alpha}\hat{\beta}}\| = \|\eta_{\alpha\beta}\| = \begin{array}{c|c} & \beta \longrightarrow \\ & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \hline \alpha \downarrow \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left\| \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

to empirical test in the appropriate, very special coordinate systems: Euclidean coordinates in Euclidean geometry; the natural generalization of Euclidean coordinates (local Lorentz coordinates; local inertial frame) in the local Lorentz geometry of physics. However, the theorems rise above all coordinate systems in their content. They refer to intervals or distances. Those distances no more call on coordinates for their definition in our day than they did in the time of Euclid. Points in the great pile of hay that is spacetime; and distances between these points: that is geometry! State them in the coordinate-free language or in the language of coordinates: they are the same (Box 1.3).

§1.5. TIME

Time is defined so that motion looks simple.

Time is awake when all things sleep.
Time stands straight when all things fall.
Time shuts in all and will not be shut.
Is, was, and shall be are Time's children.
O Reasoning, be witness, be stable.

VYASA, the *Mahabarata* (*ca.* A.D. 400)

Relative to a local Lorentz frame, a free particle "moves in a straight line with uniform velocity." What "straight" means is clear enough in the model inertial reference frame illustrated in Figure 1.7. But where does the "uniform velocity" come in? Or where does "velocity" show itself? There is not even one clock in the drawing!

A more fully developed model of a Lorentz reference frame will have not only holes, as in Fig. 1.7, but also clock-activated shutters over each hole. The projectile can reach its target only if it (1) travels through the correct region in space and (2) gets through that hole in the correct interval of time ("window in time"). How then is time defined? Time is defined so that motion looks simple!

The time coordinate of a local Lorentz frame is so defined that motion looks simple

No standard of time is more widely used than the day, the time from one high noon to the next. Take that as standard, however, and one will find every good clock or watch clashing with it, for a simple reason. The Earth spins on its axis and also revolves in orbit about the sun. The motion of the sun across the sky arises from neither effect alone, but from the two in combination, different in magnitude though they are. The fast angular velocity of the Earth on its axis (roughly 366.25 complete turns per year) is wonderfully uniform. Not so the apparent angular velocity of the sun about the center of the Earth (one turn per year). It is greater than average by 2 per cent when the Earth in its orbit (eccentricity 0.017) has come 1 per cent closer than average to the sun (Kepler's law) and lower by 2 per cent when the Earth is 1 per cent further than average from the sun. In the first case, the momentary rate of rotation of the sun across the sky, expressed in turns per year, is approximately

$$366.25 - (1 + 0.02);$$

in the other,

$$366.25 - (1 - 0.02).$$

Taking the "mean solar day" to contain $24 \times 3{,}600 = 86{,}400$ standard seconds, one sees that, when the Earth is 1 per cent closer to (or further from) the sun than average, then the number of standard seconds from one high noon to the next is greater (or less) than normal by

$$\frac{0.02 \text{ (drop in turns per year)}}{365.25 \text{ (turns per year on average)}} 86{,}400 \text{ sec} \sim 4.7 \text{ sec}.$$

This is the bookkeeping on time from noon to noon. No standard of time that varies so much from one month to another is acceptable. If adopted, it would make the speed of light vary from month to month!

This lack of uniformity, once recognized (and it was already recognized by the ancients), forces one to abandon the solar day as the standard of time; that day does not make motion look simple. Turn to a new standard that eliminates the motion of the Earth around the sun and concentrates on the spin of the Earth about its axis: the sidereal day, the time between one arrival of a star at the zenith and the next arrival of that star at the zenith. Good! Or good, so long as one's precision of measurement does not allow one to see changes in the intrinsic angular velocity of the Earth. What clock was so bold as first to challenge the spin of the Earth for accuracy? The machinery of the heavens.

Halley (1693) and later others, including Kant (1754), suspected something was amiss from apparent discrepancies between the paths of totality in eclipses of the sun, as predicted by Newtonian gravitation theory using the standard of time then current, and the location of the sites where ancient Greeks and Romans actually recorded an eclipse on the day in question. The moon casts a moving shadow in space. On the day of a solar eclipse, that shadow paints onto the disk of the spinning Earth a black brush stroke, often thousands of kilometers in length, but of width generally much less than a hundred kilometers. He who spins the globe upon the table and wants to make the shadow fall rightly on it must calculate back meticulously to determine two key items: (1) where the moon is relative to Earth and sun at each moment on the ancient day in question; and (2) how much angle the Earth has turned through from then until now. Take the eclipse of Jan. 14, A.D. 484, as an example (Figure 1.8), and assume the same angular velocity for the Earth in the intervening fifteen centuries as the Earth had in 1900 (astronomical reference point). One comes out wrong. The Earth has to be set back by 30° (or the moon moved from its computed position, or some combination of the two effects) to make the Athens observer fall under the black brush. To catch up those 30° (or less, if part of the effect is due to a slow change in the angular momentum of the moon), the Earth had to turn faster in the past than it does today. Assigning most of the discrepancy to terrestrial spin-down (rate of spin-down compatible with modern atomic-clock evidence), and assuming a uniform rate of slowing from then to now

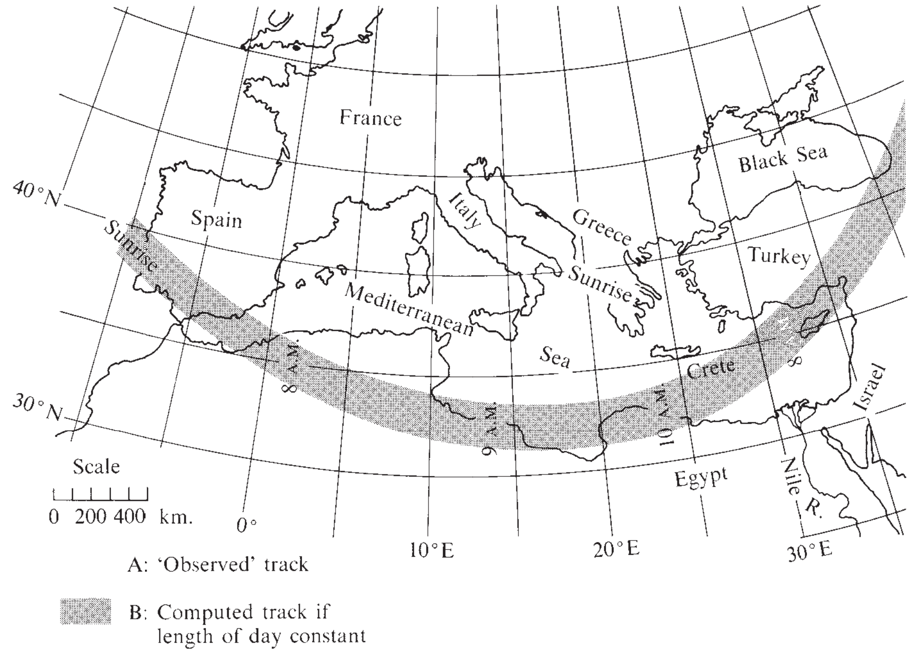


Figure 1.8.
Calculated path of totality for the eclipse of January 14, A.D. 484 (left; calculation based on no spin-down of Earth relative to its 1900 angular velocity) contrasted with the same path as set ahead enough to put the center of totality (at sunrise) at Athens [displacement very close to 30°; actual figure of deceleration adopted in calculations, 32.75 arc sec/(century)^2]. This is "undoubtedly the most reliable of all ancient European eclipses," according to Dr. F. R. Stephenson, of the Department of Geophysics and Planetary Physics of the University of Newcastle upon Tyne, who most kindly prepared this diagram especially for this book. He has also sent a passage from the original Greek biography of Proclus of Athens (died at Athens A.D. 485) by Marinus of Naples, reading, "Nor were there portents wanting in the year which preceded his death; for example, such a great eclipse of the Sun that night seemed to fall by day. For a profound darkness arose so that stars even appeared in the sky. This happened in the eastern sky when the Sun dwelt in Capricorn" [from Westermann and Boissonade (1878)].

Does this 30° for this eclipse, together with corresponding amounts for other eclipses, represent the "right" correction? "Right" is no easy word. From one total eclipse of the sun in the Mediterranean area to another is normally many years. The various provinces of the Greek and Roman worlds were far from having a uniform level of peace and settled life, and even farther from having a uniform standard of what it is to observe an eclipse and put it down for posterity. If the scores of records of the past are unhappily fragmentary, even more unhappy has been the willingness of a few uncritical "investigators" in recent times to rush in and identify this and that historical event with this and that calculated eclipse. Fortunately, by now a great literature is available on the secular deceleration of the Earth's rotation, in the highest tradition of critical scholarship, both astronomical and historical. In addition to the books of O. Neugebauer (1959) and Munk and MacDonald (1960), the paper of Curott (1966), and items cited by these workers, the following are key items. (For direction to them, we thank Professor Otto Neugebauer—no relation to the other Neugebauer cited below!) For the ancient records, and for calculations of the tracks of ancient eclipses, F. K. Ginzel (1882, 1883, 1884); for an atlas of calculated eclipse tracks, Oppolzer (1887) and Ginzel (1899); and for a critical analysis of the evidence. P. V. Neugebauer (1927, 1929, and 1930). This particular eclipse was chosen rather than any other because of the great reliability of the historical record of it.

(angular velocity correction proportional to first power of elapsed time: angle correction itself proportional to square of elapsed time), one estimates from a correction of

$$30° \text{ or } 2 \text{ hours} \qquad 1{,}500 \text{ years ago}$$

the following corrections for intermediate times:

$$30°/10^2, \text{ or } 1.2 \text{ min} \qquad 150 \text{ years ago,}$$

$$30°/10^4, \text{ or } 0.8 \text{ sec} \qquad 15 \text{ years ago.}$$

Thus one sees the downfall of the Earth as a standard of time and its replacement by the orbital motions of the heavenly bodies as a better standard: a standard that does more to "make motion look simple." Astronomical time is itself in turn today being supplanted by atomic time as a standard of reference (see Box 1.4, "Time Today").

Good clocks make spacetime trajectories of free particles look straight

Look at a bad clock for a good view of how time is defined. Let t be time on a "good" clock (time coordinate of a local inertial frame); it makes the tracks of free particles through the local region of spacetime look straight. Let $T(t)$ be the reading of the "bad" clock; it makes the world lines of free particles through the local region of spacetime look curved (Figure 1.9). The old value of the acceleration, translated into the new ("bad") time, becomes

$$0 = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dT}{dt}\frac{dx}{dT}\right) = \frac{d^2T}{dt^2}\frac{dx}{dT} + \left(\frac{dT}{dt}\right)^2\frac{d^2x}{dT^2}.$$

To explain the apparent accelerations of the particles, the user of the new time introduces a force that one knows to be fictitious:

$$F_x = m\frac{d^2x}{dT^2} = -m\frac{\left(\dfrac{dx}{dT}\right)\left(\dfrac{d^2T}{dt^2}\right)}{\left(\dfrac{dT}{dt}\right)^2}. \tag{1.2}$$

It is clear from this example of a "bad" time that Newton thought of a "good" time when he set up the principle that "Time flows uniformly" ($d^2T/dt^2 = 0$). Time is defined to make motion look simple!

The principle of uniformity, taken by itself, leaves free the scale of the time variable. The quantity $T = at + b$ satisfies the requirement as well as t itself. The history of timekeeping discloses many choices of the unit and origin of time. Each one required some human action to give it sanction, from the fiat of a Pharaoh to the communique of a committee. In this book the amount of time it takes light to travel one centimeter is decreed to be the unit of time. Spacelike intervals and timelike intervals are measured in terms of one and the same geometric unit: the centimeter. Any other decision would complicate in analysis what is simple in nature. No other choice would live up to Minkowski's words, "Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."

Our choice of unit for measuring time: *the geometrodynamic centimeter.*

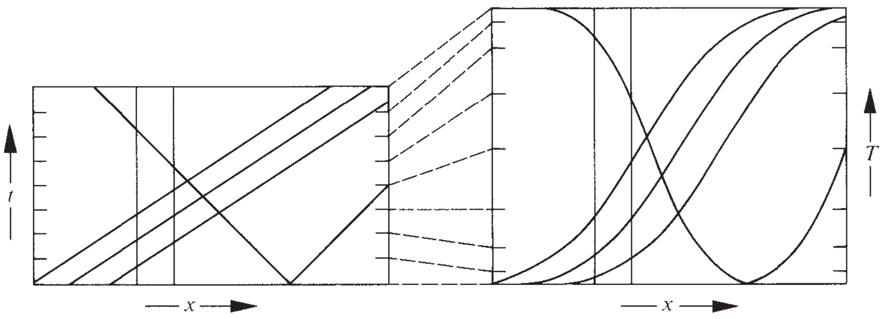


Figure 1.9.
Good clock (left) vs. bad clock (right) as seen in the maps they give of the same free particles moving through the same region of spacetime. The world lines as depicted at the right give the impression that a force is at work. The good definition of time eliminates such fictitious forces. The dashed lines connect corresponding instants on the two time scales.

One can measure time more accurately today than distance. Is that an argument against taking the elementary unit to be the centimeter? No, provided that this definition of the centimeter is accepted: *the geometrodynamic standard centimeter is the fraction*

$$1/(9.460546 \times 10^{17}) \tag{1.3}$$

of the interval between the two "effective equinoxes" that bound the tropical year 1900.0. The tropical year 1900.0 has already been recognized internationally as the fiducial interval by reason of its definiteness and the precision with which it is known. Standards committees have *defined the ephemeris second* so that 31,556,925.974 sec make up that standard interval. Were the speed of light known with perfect precision, the standards committees could have given in the same breath the number of centimeters in the standard interval. But it isn't; it is known to only six decimals. Moreover, the *international centimeter* is defined in terms of the orange-red wavelength of Kr^{86} to only nine decimals (16,507.6373 wavelengths). Yet the standard second is given to 11 decimals. We match the standard second by arbitrarily defining the geometrodynamic standard centimeter so that

$$9.4605460000 \times 10^{17}$$

such centimeters are contained in the standard tropical year 1900.0. The speed of light then becomes exactly

$$\frac{9.4605460000 \times 10^{17}}{31{,}556{,}925.974} \text{ geometrodynamic cm/sec.} \tag{1.4}$$

This is compatible with the speed of light, as known in 1967, in units of "international cm/sec":

$$29{,}979{,}300{,}000 \pm 30{,}000 \text{ international cm/sec.}$$

Box 1.4 TIME TODAY

Prior to 1956 the second was defined as the fraction 1/86,400 of the mean solar day.

From 1956 to 1967 the "second" meant the ephemeris second, defined as the fraction 1/(31,556,925.9747) of the tropical year 00h00m00s December 31, 1899.

Since 1967 the standard second has been the SI (Système International) second, defined as 9,192,631,770 periods of the unperturbed microwave transition between the two hyperfine levels of the ground state of Cs^{133}.

Like the foregoing evolution of the unit for the time *interval,* the evolution of a time *coordinate* has been marked by several stages.

Universal time, UTO, is based on the count of days as they actually occurred historically; in other words, on the actual spin of the earth on its axis; historically, on mean solar time (solar position as corrected by the "equation of time"; i.e., the faster travel of the earth when near the sun than when far from the sun) as determined at Greenwich Observatory.

UT1, the "navigator's time scale," is the same time as corrected for the wobble of the earth on its axis ($\Delta t \sim 0.05$ sec).

UT2 is UT1 as corrected for the periodic fluctuations of unknown origin with periods of one-half year and one year ($\Delta t \sim 0.05$ sec; measured to 3 ms in one day).

Ephemeris Time, ET (as defined by the theory of gravitation and by astronomical observations and calculations), is essentially determined by the orbital motion of the earth around the sun. "Measurement uncertainties limit the realization of accurate ephemeris time to about 0.05 sec for a nine-year average."

Coordinated Universal Time (UTC) is broadcast on stations such as WWV. It was adopted internationally in February 1971 to become effective January 1, 1972. The clock rate is controlled by atomic clocks to be as uniform as possible for one year (atomic time is measured to ~0.1 microsec in 1 min, with diffusion rates of 0.1 microsec per day for ensembles of clocks), but is changed by the infrequent addition or deletion of a second—called a "leap second"—so that UTC never differs more than 0.7 sec from the navigator's time scale, UT1.

THE TIMES
Wednesday
June 21 1972

Time suspended for a second

Time will stand still throughout the world for one second at midnight, June 30. All radio time signals will insert a "leap second" to bring Greenwich Mean Time into line with the earth's loss of three thousandths of a second a day.

The signal from the Royal Greenwich Observatory to Broadcasting House at midnight GMT (1 am BST July 1) will be six short pips marking the seconds 55 to 60 inclusive, followed by a lengthened signal at the following second to mark the new minute.

The foregoing account is abstracted from J. A. Barnes (1971). The following is extracted from a table (not official at time of receipt), kindly supplied by the Time and Frequency Division of the U.S. National Bureau of Standards in Boulder, Colorado.

Timekeeping capabilities of some familiar clocks are as follows:

Tuning fork wrist watch (1960),
1 min/mo.

Quartz crystal clock (1921–1930),
1 μsec/day,
1 sec/yr.

Quartz crystal wrist watch (1971),
0.2 sec/2 mos.,
1 sec/yr.

Cesium beam (atomic resonance, Cs^{133}), (1952–1955),
0.1 μsec/day,
0.5 μsec/mo.

Rubidium gas cell (Rb^{87} resonance), (1957),
0.1 μsec/day,
1–5 μsec/mo.

Hydrogen maser (1960),
0.01 μsec/2 hr,
0.1 μsec/day.

Methane stabilized laser (1969),
0.01 μsec/100 sec.

Recent measurements [Evenson *et al.* (1972)] change the details of the foregoing 1967 argument, but not the principles.

§1.6. CURVATURE

Gravitation seems to have disappeared. Everywhere the geometry of spacetime is locally Lorentzian. And in Lorentz geometry, particles move in a straight line with constant velocity. Where is any gravitational deflection to be seen in that? For answer, turn back to the apple (Figure 1.1). Inspect again the geodesic tracks of the ants on the surface of the apple. Note the reconvergence of two nearby geodesics that originally diverged from a common point. What is the analog in the real world of physics? What analogous concept fits Einstein's injunction that physics is only simple when analyzed locally? Don't look at the distance from the spaceship to the Earth. Look at the distance from the spaceship to a nearby spaceship! Or, to avoid any possible concern about attraction between the two ships, look at two nearby test particles in orbit about the Earth. To avoid distraction by the nonlocal element (the Earth) in the situation, conduct the study in the interior of a spaceship, also in orbit about the Earth. But this region has already been counted as a local inertial frame! What gravitational physics is to be seen there? None. Relative to the spaceship and therefore relative to each other, the two test particles move in a straight line with uniform velocity, to the precision of measurement that is contemplated (see Box 1.5, "Test for Flatness"). Now the key point begins to appear: precision of measurement. Increase it until one begins to discern the gradual acceleration of the test particles away from each other, if they lie along a common radius through the center of the Earth; or toward each other, if their separation lies perpendicular to that line. In Newtonian language, the source of these accelerations is the tide-producing action of the Earth. To the observer in the spaceship, however, no Earth is to be seen. And following Einstein, he knows it is important to analyze motion locally. He represents the separation of the new test particle from the fiducial test particle by the vector ξ^k($k = 1, 2, 3$; components measured in a local Lorentz frame). For the acceleration of this separation, one knows from Newtonian physics what he will find: if the Cartesian z-axis is in the radial direction, then

Gravitation is manifest in relative acceleration of neighboring test particles

$$
\begin{aligned}
\frac{d^2\xi^x}{dt^2} &= -\frac{Gm_{\rm conv}}{c^2r^3}\,\xi^x;\\
\frac{d^2\xi^y}{dt^2} &= -\frac{Gm_{\rm conv}}{c^2r^3}\,\xi^y;\\
\frac{d^2\xi^z}{dt^2} &= \frac{2Gm_{\rm conv}}{c^2r^3}\,\xi^z.
\end{aligned}
\tag{1.5}
$$

Proof: In Newtonian physics the acceleration of a single particle toward the center of the Earth in conventional units of time is $Gm_{\rm conv}/r^2$, where G is the Newtonian constant of gravitation, $6.670 \times 10^{-8}\ {\rm cm}^3/{\rm g\ sec}^2$ and $m_{\rm conv}$ is the mass of the Earth in conventional units of grams. In geometric units of time (cm of light-travel time), the acceleration is Gm_{conv}/c^2r^2. When the two particles are separated by a distance ξ perpendicular to r, the one downward acceleration vector is out of line with the other by the angle ξ/r. Consequently one particle accelerates toward the other by the stated amount. When the separation is parallel to r, the relative acceleration is given by evaluating the Newtonian acceleration at r and at $r + \xi$, and taking the difference (ξ times d/dr) Q.E.D. In conclusion, the "local tide-producing acceleration" of Newtonian gravitation theory provides the local description of gravitation that Einstein bids one to seek.

Relative acceleration is caused by curvature

What has this tide-producing acceleration to do with curvature? (See Box 1.6.) Look again at the apple or, better, at a sphere of radius a (Figure 1.10). The separation of nearby geodesics satisfies the "equation of geodesic deviation,"

$$d^2\xi/ds^2 + R\xi = 0. \tag{1.6}$$

Here $R = 1/a^2$ is the so-called Gaussian curvature of the surface. For the surface of the apple, the same equation applies, with the one difference that the curvature R varies from place to place.

Box 1.5 TEST FOR FLATNESS

1. Specify the extension in space L (cm or m) and extension in time T (cm or m of light travel time) of the region under study.

2. Specify the precision $\delta\xi$ with which one can measure the separation of test particles in this region.

3. Follow the motion of test particles moving along initially parallel world lines through this region of spacetime.

4. When the world lines remain parallel to the precision $\delta\xi$ for all directions of travel, then one says that "in a region so limited and to a precision so specified, spacetime is flat."

EXAMPLE: Region just above the surface of the earth, 100 m × 100 m × 100 m (space extension), followed for 10^9 m of light-travel time ($T_{\text{conv}} \sim 3$ sec). Mass of Earth, $m_{\text{conv}} = 5.98 \times 10^{27}$ g, $m = (0.742 \times 10^{-28}\ \text{cm/g}) \times (5.98 \times 10^{27}\ \text{g}) = 0.444$ cm [see eq. (1.12)]. Tide-producing acceleration $R^z{}_{0z0}$ (relative acceleration in z-direction of two test particles initially at rest and separated from each other by 1 cm of vertical elevation) is

$$\begin{aligned}(d/dr)(m/r^2) &= -2m/r^3\\ &= -0.888\ \text{cm}/(6.37 \times 10^8\ \text{cm})^3\\ &= -3.44 \times 10^{-27}\ \text{cm}^{-2}\end{aligned}$$

("cm of relative displacement per cm of light-travel time per cm of light-travel time per cm of vertical separation"). Two test particles with a vertical separation $\xi^z = 10^4$ cm acquire in the time $t = 10^{11}$ cm (difference between time and proper time negligible for such slowly moving test particles) a relative displacement

$$\begin{aligned}\delta\xi^z &= -\tfrac{1}{2}R^z{}_{0z0}t^2\xi^z\\ &= 1.72 \times 10^{-27}\ \text{cm}^{-2}(10^{11}\ \text{cm})^2\ 10^4\ \text{cm}\\ &= 1.72\ \text{mm}.\end{aligned}$$

(Change in relative separation less for other directions of motion). When the minimum uncertainty $\delta\xi$ attainable in a measurement over a 100 m spacing is "worse" than this figure (exceeds 1.72 mm), then to this level of precision the region of spacetime under consideration can be treated as flat. When the uncertainty in measurement is "better" (less) than 1.72 mm, then one must limit attention to a smaller region of space or a shorter interval of time or both, to find a region of spacetime that can be regarded as flat to that precision.

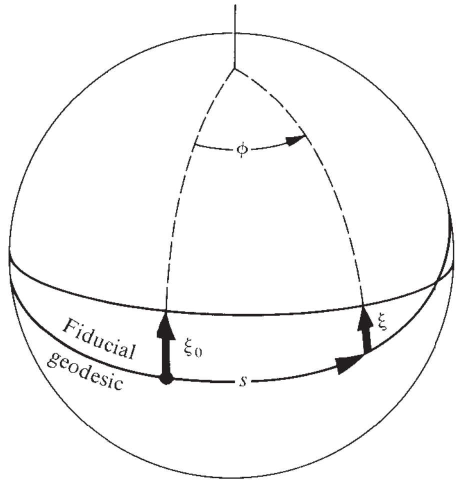


Figure 1.10.
Curvature as manifested in the "acceleration of the separation" of two nearby geodesics. Two geodesics, originally parallel, and separated by the distance ("geodesic deviation") ξ_0, are no longer parallel when followed a distance s. The separation is $\xi = \xi_0 \cos \phi = \xi_0 \cos (s/a)$, where a is the radius of the sphere. The separation follows the equation of simple harmonic motion, $d^2\xi/ds^2 + (1/a^2)\,\xi = 0$ ("equation of geodesic deviation").

The direction of the separation vector, $\boldsymbol{\xi}$, is fixed fully by its orthogonality to the fiducial geodesic. Hence, no reference to the direction of $\boldsymbol{\xi}$ is needed or used in the equation of geodesic deviation; only the magnitude ξ of $\boldsymbol{\xi}$ appears there, and only the magnitude, not direction, of the relative acceleration appears.

In a space of more than two dimensions, an equation of the same general form applies, with several differences. In two dimensions the *direction* of acceleration of one geodesic relative to a nearby, fiducial geodesic is fixed uniquely by the demand that their separation vector, $\boldsymbol{\xi}$, be perpendicular to the fiducial geodesic (see Figure 1.10). Not so in three dimensions or higher. There $\boldsymbol{\xi}$ can remain perpendicular to the fiducial geodesic but rotate about it (Figure 1.11). Thus, to specify the relative acceleration uniquely, one must give not only its magnitude, but also its direction.

The relative acceleration in three dimensions and higher, then, is a vector. Call it "$D^2\boldsymbol{\xi}/ds^2$," and call its four components "$D^2\xi^\alpha/ds^2$." Why the capital D? Why not "$d^2\xi^\alpha/ds^2$"? Because our coordinate system is completely arbitrary (cf. §1.2). The twisting and turning of the coordinate lines can induce changes from point to point in the components ξ^α of $\boldsymbol{\xi}$, even if the vector $\boldsymbol{\xi}$ is not changing at all. Consequently, the accelerations of the components $d^2\xi^\alpha/ds^2$ are generally not equal to the components $D^2\xi^\alpha/ds^2$ of the acceleration!

How, then, in curved spacetime can one determine the components $D^2\xi^\alpha/ds^2$ of the relative acceleration? By a more complicated version of the equation of geodesic deviation (1.6). Differential geometry (Part III of this book) provides us with a geometrical object called the *Riemann curvature tensor,* "***Riemann***." ***Riemann*** is

Curvature is characterized by Riemann tensor

(*continued on page 34*)

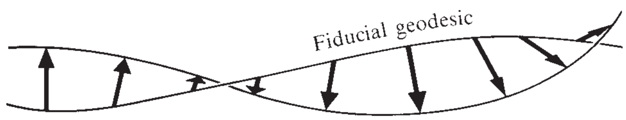
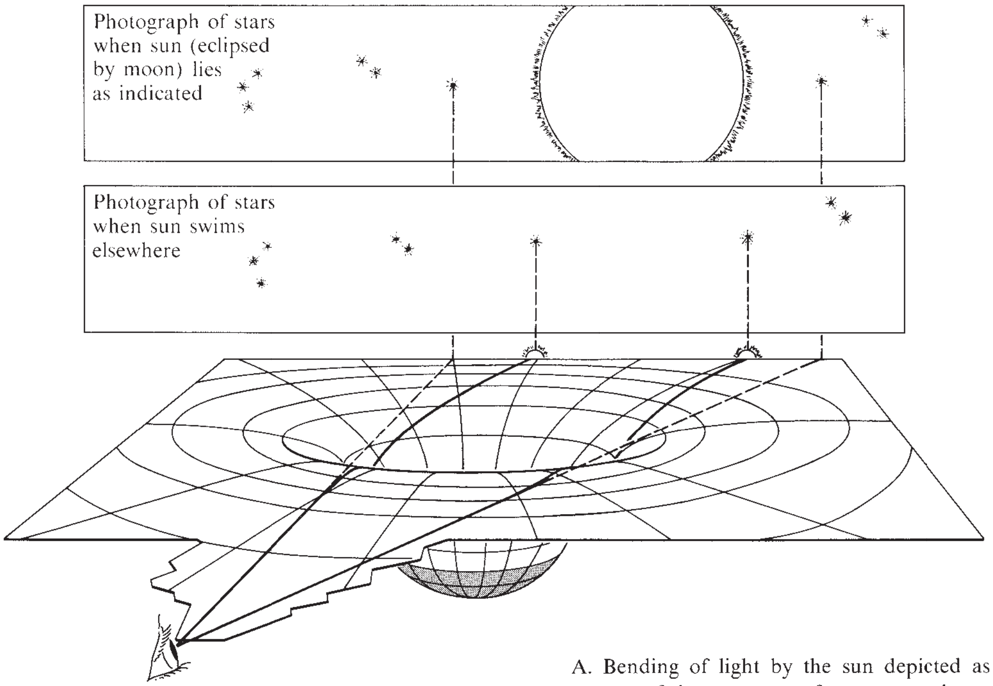


Figure 1.11.
The separation vector $\boldsymbol{\xi}$ between two geodesics in a curved three-dimensional manifold. Here $\boldsymbol{\xi}$ can not only change its length from point to point, but also rotate at a varying rate about the fiducial geodesic. Consequently, the relative acceleration of the geodesics must be characterized by a direction as well as a magnitude; it must be a vector, $D^2\boldsymbol{\xi}/ds^2$.

Box 1.6 CURVATURE OF WHAT?

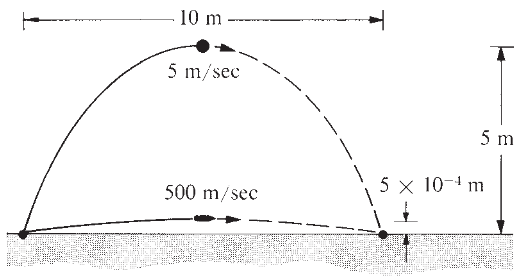
Nothing seems more attractive at first glance than the idea that gravitation is a manifestation of the curvature of space (A), and nothing more ridiculous at a second glance (B). How can the tracks of a ball and of a bullet be curved so differently if that curvature arises from the geometry of space? No wonder that great Riemann did not give the world a geometric theory of gravity. Yes, at the age of 28 (June 10, 1854) he gave the world the mathematical machinery to define and calculate curvature (metric and Riemannian geometry). Yes, he spent his dying days at 40 working to find a unified account of electricity and gravitation. But if there was one reason more than any other why he failed to make the decisive connection between gravitation and curvature, it was this, that he thought of space and the curvature of space, not of spacetime and the curvature of spacetime. To make that forward step took the forty years to special relativity (1905: time on the same footing as space) and then another ten years (1915: general relativity). Depicted in spacetime (C), the tracks of ball and bullet appear to have comparable curvature. In fact, however, neither track has any curvature at all. They both look curved in (C) only because one has forgotten that the spacetime they reside in is itself curved—curved precisely enough to make these tracks the straightest lines in existence ("geodesics").

If it is at first satisfying to see curvature, and curvature of spacetime at that, coming to the fore in so direct a way, then a little more reflection produces a renewed sense of concern. Curvature with respect to what? Not with respect to the labo-

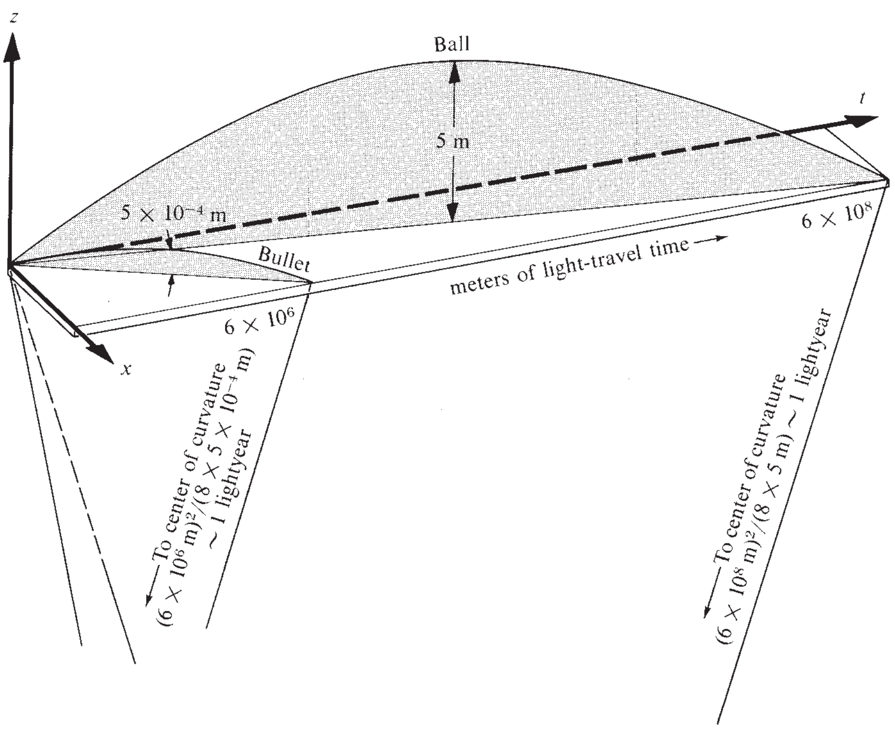


A. Bending of light by the sun depicted as a consequence of the curvature of space near the sun. Ray of light pursues geodesic, but geometry in which it travels is curved (actual travel takes place in spacetime rather than space; correct deflection is twice that given by above elementary picture). Deflection inversely proportional to angular separation between star and center of sun. See Box 40.1 for actual deflections observed at time of an eclipse.

ratory. The earth-bound laboratory has no simple status whatsoever in a proper discussion. First, it is no Lorentz frame. Second, even to mention the earth makes one think of an action-at-a-distance version of gravity (distance from center of earth to ball or bullet). In contrast, it was the whole point of Einstein that physics looks simple only when analyzed locally. To look at local physics, however, means to compare one geodesic of one test particle with geodesics of other test particles traveling (1) nearby with (2) nearly the same directions and (3) nearly the same speeds. Then one can "look at the separations between these nearby test particles and from the second time-rate of change of these separations and the 'equation of geodesic deviation' (equation 1.8) read out the curvature of spacetime."



B. Tracks of ball and bullet through space as seen in laboratory have very different curvatures.



C. Tracks of ball and bullet through spacetime, as recorded in laboratory, have comparable curvatures. Track compared to arc of circle: (radius) = (horizontal distance)^2/8 (rise).

the higher-dimensional analog of the Gaussian curvature R of our apple's surface. ***Riemann*** is the mathematical embodiment of the bends and warps in spacetime. And ***Riemann*** is the agent by which those bends and warps (curvature of spacetime) produce the relative acceleration of geodesics.

Riemann, like the metric tensor $\boldsymbol{g}$ of Box 1.3, can be thought of as a family of machines, one machine residing at each event in spacetime. Each machine has three slots for the insertion of three vectors:

$$\begin{array}{cccc} & \text{slot 1} & \text{slot 2} & \text{slot 3} \\ & \downarrow & \downarrow & \downarrow \\ \textbf{\textit{Riemann}}\ (& \quad , & \quad , & \quad). \end{array}$$

Choose a fiducial geodesic (free-particle world line) passing through an event $\mathscr{Q}$, and denote its unit tangent vector (particle 4-velocity) there by

$$\boldsymbol{u} = \boldsymbol{dx}/d\tau;\ \text{components},\ u^{\alpha} = dx^{\alpha}/d\tau. \tag{1.7}$$

Choose another, neighboring geodesic, and denote by $\boldsymbol{\xi}$ its perpendicular separation from the fiducial geodesic. Then insert $\boldsymbol{u}$ into the first slot of ***Riemann*** at $\mathscr{Q}$, $\boldsymbol{\xi}$ into the second slot, and $\boldsymbol{u}$ into the third. ***Riemann*** will grind for awhile; then out will pop a new vector,

$$\textbf{\textit{Riemann}}\ (\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{u}).$$

Riemann tensor, through equation of geodesic deviation, produces relative accelerations

The equation of geodesic deviation states that this new vector is the negative of the relative acceleration of the two geodesics:

$$D^2\boldsymbol{\xi}/d\tau^2 + \textbf{\textit{Riemann}}\ (\boldsymbol{u}, \boldsymbol{\xi}, \boldsymbol{u}) = 0. \tag{1.8}$$

The Riemann tensor, like the metric tensor (Box 1.3), and like all other tensors, is a linear machine. The vector it puts out is a linear function of each vector inserted into a slot:

$$\begin{aligned} &\textbf{\textit{Riemann}}\ (2\boldsymbol{u}, a\boldsymbol{w} + b\boldsymbol{v}, 3\boldsymbol{r}) \\ &= 2 \times a \times 3\ \textbf{\textit{Riemann}}\ (\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{r}) + 2 \times b \times 3\ \textbf{\textit{Riemann}}\ (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{r}). \end{aligned} \tag{1.9}$$

Consequently, in any coordinate system the components of the vector put out can be written as a "trilinear function" of the components of the vectors put in:

$$\boldsymbol{r} = \textbf{\textit{Riemann}}\ (\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \Longleftrightarrow r^{\alpha} = R^{\alpha}{}_{\beta\gamma\delta}\, u^{\beta}\, v^{\gamma}\, w^{\delta}. \tag{1.10}$$

(Here there is an implied summation on the indices β, γ, δ; cf. Box 1.1.) The $4 \times 4 \times 4 \times 4 = 256$ numbers $R^{\alpha}{}_{\beta\gamma\delta}$ are called the "components of the Riemann tensor in the given coordinate system." In terms of components, the equation of geodesic deviation states

$$\frac{D^2\xi^{\alpha}}{d\tau^2} + R^{\alpha}{}_{\beta\gamma\delta}\frac{dx^{\beta}}{d\tau}\xi^{\gamma}\frac{dx^{\delta}}{d\tau} = 0. \tag{1.8$'$}$$

In Einstein's geometric theory of gravity, this equation of geodesic deviation summarizes the entire effect of geometry on matter. It does for gravitation physics what the Lorentz force equation,

Equation of geodesic deviation is analog of Lorentz force law

$$\frac{D^2 x^\alpha}{d\tau^2} - \frac{e}{m} F^\alpha{}_\beta \frac{dx^\alpha}{d\tau} = 0, \tag{1.11}$$

does for electromagnetism. See Box 1.7.

The units of measurement of the curvature are cm^{-2} just as well in spacetime as on the surface of the apple. Nothing does so much to make these units stand out clearly as to express mass in "geometrized units":

Geometrized units

$$\begin{aligned} m(\mathrm{cm}) &= (G/c^2) m_{\mathrm{conv}}(\mathrm{g}) \\ &= (0.742 \times 10^{-28}\ \mathrm{cm/g}) m_{\mathrm{conv}}(\mathrm{g}). \end{aligned} \tag{1.12}$$

Box 1.7 EQUATION OF MOTION UNDER THE INFLUENCE OF A GRAVITATIONAL FIELD AND AN ELECTROMAGNETIC FIELD, COMPARED AND CONTRASTED

	Electromagnetism [*Lorentz force, equation (1.11)*]	*Gravitation* [*Equation of geodesic deviation (1.8′)*]
Acceleration is defined for one particle?	Yes	No
Acceleration defined how?	Actual world line compared to world line of uncharged "fiducial" test particle passing through same point with same 4-velocity.	Already an uncharged test particle, which can't accelerate relative to itself! Acceleration measured relative to a nearby test particle as fiduciary standard.
Acceleration depends on all four components of the 4-velocity of the particle?	Yes	Yes
Universal acceleration for all test particles in same locations with same 4-velocity?	No; is proportional to e/m	Yes
Driving field	Electromagnetic field	Riemann curvature tensor
Ostensible number of distinct components of driving field	$4 \times 4 = 16$	$4^4 = 256$
Actual number when allowance is made for symmetries of tensor	6	20
Names for more familiar of these components	3 electric 3 magnetic	6 components of local Newtonian tide-producing acceleration

This conversion from grams to centimeters by means of the ratio

$$G/c^2 = 0.742 \times 10^{-28}\ \text{cm/g}$$

is completely analogous to converting from seconds to centimeters by means of the ratio

$$c = \frac{9.4605460000 \times 10^{17}\ \text{cm}}{31{,}556{,}925.974\ \text{sec}}$$

(see end of §1.5). The sun, which in conventional units has $m_{\text{conv}} = 1.989 \times 10^{33}$ g, has in geometrized units a mass $m = 1.477$ km. Box 1.8 gives further discussion.

Components of Riemann tensor evaluated from relative accelerations of slowly moving particles

Using geometrized units, and using the Newtonian theory of gravity, one can readily evaluate nine of the most interesting components of the Riemann curvature tensor near the Earth or the sun. The method is the gravitational analog of determining the electric field strength by measuring the acceleration of a slowly moving test particle. Consider the separation between the geodesics of two nearby and slowly moving ($v \ll c$) particles at a distance r from the Earth or sun. In the standard, nearly inertial coordinates of celestial mechanics, all components of the 4-velocity of the

Box 1.8 GEOMETRIZED UNITS

Throughout this book, we use "geometrized units," in which the speed of light c, Newton's gravitational constant G, and Boltzman's constant k are all equal to unity. The following alternative ways to express the number 1.0 are of great value:

$$1.0 = c = 2.997930 \cdots \times 10^{10}\ \text{cm/sec}$$

$$1.0 = G/c^2 = 0.7425 \times 10^{-28}\ \text{cm/g};$$

$$1.0 = G/c^4 = 0.826 \times 10^{-49}\ \text{cm/erg};$$

$$1.0 = Gk/c^4 = 1.140 \times 10^{-65}\ \text{cm/K};$$

$$1.0 = c^2/G^{1/2} = 3.48 \times 10^{24}\ \text{cm/gauss}^{-1}.$$

One can multiply a factor of unity, expressed in any one of these ways, into any term in any equation without affecting the validity of the equation. Thereby one can convert one's units of measure from grams to centimeters to seconds to ergs to . . . . For example:

$$\begin{aligned}\text{Mass of sun} = M_\odot &= 1.989 \times 10^{33}\ \text{g}\\ &= (1.989 \times 10^{33}\ g) \times (G/c^2)\\ &= 1.477 \times 10^{5}\ \text{cm}\\ &= (1.989 \times 10^{33}\ g) \times (c^2)\\ &= 1.788 \times 10^{54}\ \text{ergs.}\end{aligned}$$

The standard unit, in terms of which everything is measured in this book, is centimeters. However, occasionally conventional units are used; in such cases a subscript "conv" is sometimes, but not always, appended to the quantity measured:

$$M_{\odot\text{conv}} = 1.989 \times 10^{33}\ \text{g}.$$

fiducial test particle can be neglected except $dx^0/d\tau = 1$. The space components of the equation of geodesic deviation read

$$d^2\xi^k/d\tau^2 + R^k{}_{0j0}\xi^j = 0. \tag{1.13}$$

Comparing with the conclusions of Newtonian theory, equations (1.5), we arrive at the following information about the curvature of spacetime near a center of mass:

$$\begin{Vmatrix} R^{\hat{x}}{}_{\hat{0}\hat{x}\hat{0}} & R^{\hat{y}}{}_{\hat{0}\hat{x}\hat{0}} & R^{\hat{z}}{}_{\hat{0}\hat{x}\hat{0}} \\ R^{\hat{x}}{}_{\hat{0}\hat{y}\hat{0}} & R^{\hat{y}}{}_{\hat{0}\hat{y}\hat{0}} & R^{\hat{z}}{}_{\hat{0}\hat{y}\hat{0}} \\ R^{\hat{x}}{}_{\hat{0}\hat{z}\hat{0}} & R^{\hat{y}}{}_{\hat{0}\hat{z}\hat{0}} & R^{\hat{z}}{}_{\hat{0}\hat{z}\hat{0}} \end{Vmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{Vmatrix} m/r^3 & 0 & 0 \\ 0 & m/r^3 & 0 \\ 0 & 0 & -2m/r^3 \end{Vmatrix} \tag{1.14}$$

(units cm^{-2}). Here and henceforth the caret or "hat" is used to indicate the components of a vector or tensor in a local Lorentz frame of reference ("physical components," as distinguished from components in a general coordinate system). Einstein's theory will determine the values of the other components of curvature (e.g., $R^{\hat{x}}{}_{\hat{z}\hat{x}\hat{z}} = -m/r^3$); but these nine terms are the ones of principal relevance for many applications of gravitation theory. They are analogous to the components of the electric field in the Lorentz equation of motion. Many of the terms not evaluated are analogous to magnetic field components—ordinarily weak unless the source is in rapid motion.

This ends the survey of the effect of geometry on matter ("effect of curvature of apple in causing geodesics to cross"—especially great near the dimple at the top, just as the curvature of spacetime is especially large near a center of gravitational attraction). Now for the effect of matter on geometry ("effect of stem of apple in causing dimple")!

§1.7. EFFECT OF MATTER ON GEOMETRY

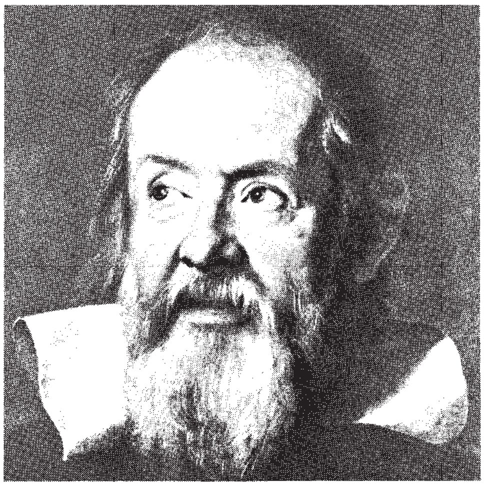
The weight of any heavy body of known weight at a particular distance from the center of the world varies according to the variation of its distance therefrom; so that as often as it is removed from the center, it becomes heavier, and when brought near to it, is lighter. On this account, the relation of gravity to gravity is as the relation of distance to distance from the center.

AL KHĀZINĪ (*Merv, A.D. 1115*), *Book of the Balance of Wisdom*

Figure 1.12 shows a sphere of the same density, $\rho = 5.52\ \mathrm{g/cm^3}$, as the average density of the Earth. A hole is bored through this sphere. Two test particles, A and B, execute simple harmonic motion in this hole, with an 84-minute period. Therefore their geodesic separation $\boldsymbol{\xi}$, however it may be oriented, undergoes a simple periodic motion with the same 84-minute period:

$$d^2\xi^j/d\tau^2 = -\left(\frac{4\pi}{3}\rho\right)\xi^j, \qquad j = x \text{ or } y \text{ or } z. \tag{1.15}$$

Box 1.9 GALILEO GALILEI
Pisa, February 15, 1564—Arcetri, Florence, January 8, 1642


Uffizi Gallery, Florence

"In questions of science the authority of a thousand is not worth the humble reasoning of a single individual."

GALILEO GALILEI (1632)

"The spaces described by a body falling from rest with a uniformly accelerated motion are to each other as the squares of the time intervals employed in traversing these distances."

GALILEO GALILEI (1638)

"Everything that has been said before and imagined by other people [about the tides] *is in my opinion completely invalid. But among the great men who have philosophised about this marvellous effect of nature the one who surprised me the most is Kepler. More than other people he was a person of independent genius, sharp, and had in his hands the motion of the earth. He later pricked up his ears and became interested in the action of the moon on the water, and in other occult phenomena, and similar childishness."*

GALILEO GALILEI (1632)

"It is a most beautiful and delightful sight to behold [with the new telescope] *the body of the Moon . . . the Moon certainly does not possess a smooth and polished surface, but one rough and uneven . . . full of vast protuberances, deep chasms and sinuosities . . . stars in myriads, which have never been seen before and which surpass the old, previously known, stars in number more than ten times. I have discovered four planets, neither known nor observed by any one of the astronomers before my time . . . got rid of disputes about the Galaxy or Milky Way, and made its nature clear to the very senses, not to say to the understanding . . . the galaxy is nothing else than a mass of luminous stars planted together in clusters . . . the number of small ones is quite beyond determination—the stars which have been called by every one of the astronomers up to this day* nebulous *are groups of small stars set thick together in a wonderful way."*

GALILEO GALILEI IN *SIDEREUS NUNCIUS* (1610)

"So the principles which are set forth in this treatise will, when taken up by thoughtful minds, lead to many another more remarkable result; and it is to be believed that it will be so on account of the nobility of the subject, which is superior to any other in nature."

GALILEO GALILEI (1638)

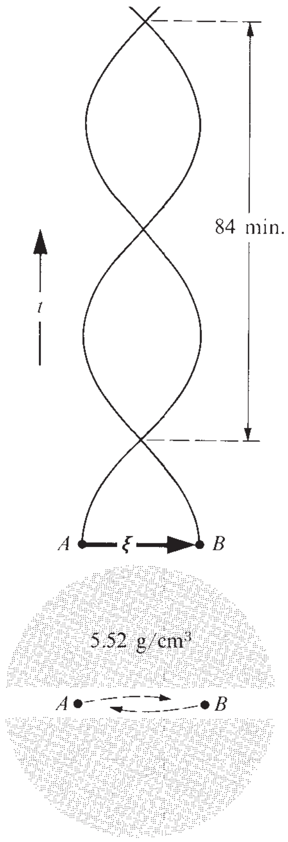


Figure 1.12.
Test particles A and B move up and down a hole bored through the Earth, idealized as of uniform density. At radius r, a particle feels Newtonian acceleration

$$\begin{aligned}\frac{d^2r}{d\tau^2} &= \frac{1}{c^2}\,\frac{d^2r}{dt_{\text{conv}}{}^2} \\ &= -\frac{G}{c^2}\,\frac{(\text{mass inside radius } r)}{r^2} \\ &= -\left(\frac{G}{r^2c^2}\right)\left(\frac{4\pi}{3}\,\rho_{\text{conv}}r^3\right) \\ &= -\omega^2 r.\end{aligned}$$

Consequently, each particle oscillates in simple harmonic motion with precisely the same angular frequency as a satellite, grazing the model Earth, traverses its circular orbit:

$$\omega^2(\text{cm}^{-2}) = \frac{4\pi}{3}\,\rho(\text{cm}^{-2}),$$

$$\omega^2{}_{\text{conv}}(\text{sec}^{-2}) = \frac{4\pi G}{3}\,\rho_{\text{conv}}(\text{g/cm}^3).$$

Comparing this actual motion with the equation of geodesic deviation (1.13) for slowly moving particles in a nearly inertial frame, we can read off some of the curvature components for the interior of this model Earth.

The Riemann tensor inside the Earth

$$\left\|\begin{matrix} R^{\hat x}{}_{\hat 0\hat x\hat 0} & R^{\hat y}{}_{\hat 0\hat x\hat 0} & R^{\hat z}{}_{\hat 0\hat x\hat 0} \\ R^{\hat x}{}_{\hat 0\hat y\hat 0} & R^{\hat y}{}_{\hat 0\hat y\hat 0} & R^{\hat z}{}_{\hat 0\hat y\hat 0} \\ R^{\hat x}{}_{\hat 0\hat z\hat 0} & R^{\hat y}{}_{\hat 0\hat z\hat 0} & R^{\hat z}{}_{\hat 0\hat z\hat 0}\end{matrix}\right\| = (4\pi\rho/3)\left\|\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\right\| \tag{1.16}$$

This example illustrates how the curvature of spacetime is connected to the distribution of matter.

Let a gravitational wave from a supernova pass through the Earth. Idealize the Earth's matter as so nearly incompressible that its density remains practically unchanged. The wave is characterized by ripples in the curvature of spacetime, propagating with the speed of light. The ripples will show up in the components $R^{\hat j}{}_{\hat 0\hat k\hat 0}$ of the Riemann tensor, and in the relative acceleration of our two test particles. The left side of equation (1.16) will ripple; but the right side will not. Equation (1.16) will break down. No longer will the Riemann curvature be generated directly and solely by the Earth's matter.

Effect of gravitational wave on Riemann tensor

Nevertheless, Einstein tells us, a part of equation (1.16) is undisturbed by the waves: its trace

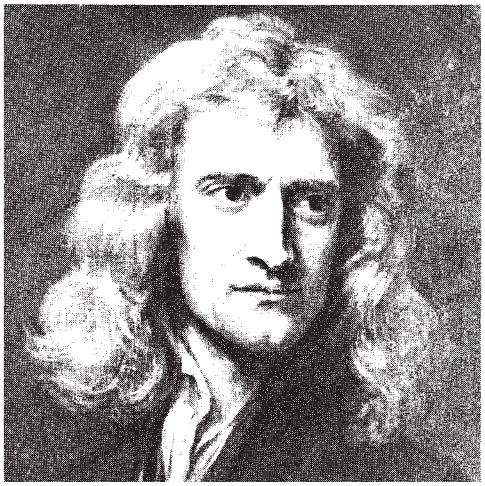
$$R_{\hat{0}\hat{0}} \equiv R^{\hat{x}}{}_{\hat{0}\hat{x}\hat{0}} + R^{\hat{y}}{}_{\hat{0}\hat{y}\hat{0}} + R^{\hat{z}}{}_{\hat{0}\hat{z}\hat{0}} = 4\pi\rho. \tag{1.17}$$

Even in the vacuum outside the Earth this is valid; there both sides vanish [cf. (1.14)].

Einstein tensor introduced

More generally, a certain piece of the Riemann tensor, called the *Einstein tensor* and denoted ***Einstein*** or ***G***, is always generated directly by the local distribution of matter. ***Einstein*** is the geometric object that generalizes $R_{\hat{0}\hat{0}}$, the lefthand side

Box 1.10 ISAAC NEWTON
Woolsthorpe, Lincolnshire, England, December 25, 1642—
Kensington, London, March 20, 1726



"The description of right lines and circles, upon which geometry is founded, belongs to mechanics. Geometry does not teach us to draw these lines, but requires them to be drawn."

[FROM P. 1 OF NEWTON'S PREFACE TO THE FIRST (1687) EDITION OF THE *PRINCIPIA*]

"Absolute space, in its own nature, without relation to anything external, remains always similar and immovable
"Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external."

[FROM THE *SCHOLIUM* IN THE *PRINCIPIA*]

"I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses; for whatever is not reduced from the phenomena is to be called an hypothesis; and hypotheses . . . have no place in experimental philosophy. . . . And to us it is enough that gravity does really exist, and act according to the laws which we have explained, and abundantly serves to account for all the motions of the celestial bodies, and of our sea."

[FROM THE *GENERAL SCHOLIUM* ADDED AT THE END OF THE THIRD BOOK OF THE *PRINCIPIA* IN THE SECOND EDITION OF 1713; ESPECIALLY FAMOUS FOR THE PHRASE OFTEN QUOTED FROM NEWTON'S ORIGINAL LATIN, *"HYPOTHESES NON FINGO."*]

"And the same year [*1665 or 1666*] *I began to think of gravity extending to the orb of the Moon, and having found out. . . . All this was in the two plague years of 1665 and 1666, for in those days I was in the prime of my age for invention, and minded Mathematicks and Philosophy more than at any time since."*

[FROM MEMORANDUM IN NEWTON'S HANDWRITING ABOUT HIS DISCOVERIES ON FLUXIONS, THE BINOMIAL THEOREM, OPTICS, DYNAMICS, AND GRAVITY, BELIEVED TO HAVE BEEN WRITTEN ABOUT 1714, AND FOUND BY ADAMS ABOUT 1887 IN THE "PORTSMOUTH COLLECTION" OF NEWTON PAPERS] of equation (1.17). Like $R_{\hat{0}\hat{0}}$, ***Einstein*** is a sort of average of ***Riemann*** over all directions. Generating ***Einstein*** and generalizing the righthand side of (1.16) is a geometric object called the *stress-energy tensor* of the matter. It is denoted $\boldsymbol{T}$. No coordinates are need to define ***Einstein***, and none to define $\boldsymbol{T}$; like the Riemann tensor, ***Riemann***, and the metric tensor, $\boldsymbol{g}$, they exist in the complete absence of coordinates. Moreover, in nature they are always equal, aside from a factor of 8π:

Stress-energy tensor introduced

$$\textbf{\textit{Einstein}} \equiv \boldsymbol{G} = 8\pi \boldsymbol{T}. \tag{1.18}$$

"For hypotheses ought . . . to explain the properties of things and not attempt to predetermine them except in so far as they can be an aid to experiments."

[FROM LETTER OF NEWTON TO I. M. PARDIES, 1672, AS QUOTED IN THE CAJORI NOTES AT THE END OF NEWTON (1687), P. 673]

"That one body may act upon another at a distance through a vacuum, without the mediation of any thing else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man, who has in philosophical matters a competent faculty of thinking, can ever fall into it."

[PASSAGE OFTEN QUOTED BY MICHAEL FARADAY FROM LETTERS OF NEWTON TO RICHARD BENTLY, 1692–1693, AS QUOTED IN THE NOTES OF THE CAJORI EDITION OF NEWTON (1687), P. 643]

"The attractions of gravity, magnetism, and electricity, reach to very sensible distances, and so have been observed . . . ; and there may be others which reach to so small distances as hitherto escape observation; . . . some force, which in immediate contract is exceeding strong, at small distances performs the chemical operations above-mentioned, and reaches not far from the particles with any sensible effect."

[FROM QUERY 31 AT THE END OF NEWTON'S *OPTICKS* (1730)]

"What is there in places almost empty of matter, and whence is it that the sun and planets gravitate towards one another, without dense matter between them? Whence is it that nature doth nothing in vain; and whence arises all that order and beauty which we see in the world? To what end are comets, and whence is it that planets move all one and the same way in orbs concentrick, while comets move all manner of ways in orbs very excentrick; and what hinders the fixed stars from falling upon one another?"

[FROM QUERY 28]

"He is not eternity or infinity, but eternal and infinite; He is not duration or space, but He endures and is present. He endures forever, and is everywhere present; and by existing always and everywhere, He constitutes duration and space. . . . And thus much concerning God; to discourse of whom from the appearances of things, does certainly belong to natural philosophy."

[FROM THE *GENERAL SCHOLIUM* AT THE END OF THE *PRINCIPIA* (1687)]

Einstein field equation: how matter generates curvature

This *Einstein field equation,* rewritten in terms of components in an arbitrary coordinate system, reads

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}. \tag{1.19}$$

The Einstein field equation is elegant and rich. No equation of physics can be written more simply. And none contains such a treasure of applications and consequences.

Consequences of Einstein field equation

The field equation shows how the stress-energy of matter generates an average curvature (***Einstein*** $\equiv$ ***G***) in its neighborhood. Simultaneously, the field equation is a propagation equation for the remaining, anisotropic part of the curvature: it governs the external spacetime curvature of a static source (Earth); it governs the generation of gravitational waves (ripples in curvature of spacetime) by stress-energy in motion; and it governs the propagation of those waves through the universe. The field equation even contains within itself the equations of motion ("Force =

Box 1.11
ALBERT EINSTEIN
Ulm, Germany,
March 14, 1879—
Princeton, New Jersey,
April 18, 1955

Library of E. T. Hochschule, Zürich

Académie des Sciences, Paris

Archives of California Institute of Technology

SEAL: Courtesy of the Lewis and Rosa Strauss Foundation and Princeton University Press mass × acceleration") for the matter whose stress-energy generates the curvature.

Those were some consequences of $\boldsymbol{G} = 8\pi \boldsymbol{T}$. Now for some applications.

Applications of Einstein field equation

The field equation governs the motion of the planets in the solar system; it governs the deflection of light by the sun; it governs the collapse of a star to form a black hole; it determines uniquely the external spacetime geometry of a black hole ("a black hole has no hair"); it governs the evolution of spacetime singularities at the end point of collapse; it governs the expansion and recontraction of the universe. And more; much more.

In order to understand how the simple equation $\boldsymbol{G} = 8\pi \boldsymbol{T}$ can be so all powerful, it is desirable to backtrack, and spend a few chapters rebuilding the entire picture of spacetime, of its curvature, and of its laws, this time with greater care, detail, and mathematics.

Thus ends this survey of the effect of geometry on matter, and the reaction of matter back on geometry, rounding out the parable of the apple.

"What really interests me is whether God had any choice in the creation of the world"

EINSTEIN TO AN ASSISTANT, AS QUOTED BY G. HOLTON (1971), P. 20

"But the years of anxious searching in the dark, with their intense longing, their alternations of confidence and exhaustion, and the final emergence into the light—only those who have experienced it can understand that"

EINSTEIN, AS QUOTED BY M. KLEIN (1971), P. 1315

"Of all the communities available to us there is not one I would want to devote myself to, except for the society of the true searchers, which has very few living members at any time. . ."

EINSTEIN LETTER TO BORN, QUOTED BY BORN (1971), P. 82

"I am studying your great works and—when I get stuck anywhere—now have the pleasure of seeing your friendly young face before me smiling and explaining"

EINSTEIN, LETTER OF MAY 2, 1920, AFTER MEETING NIELS BOHR

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

EINSTEIN (1921), P. 28

"The most incomprehensible thing about the world is that it is comprehensible."

EINSTEIN, IN SCHILPP (1949), P. 112

EXERCISES

Exercise 1.1. CURVATURE OF A CYLINDER

Show that the Gaussian curvature R of the surface of a cylinder is zero by showing that geodesics on that surface (unroll!) suffer no geodesic deviation. Give an independent argument for the same conclusion by employing the formula $R = 1/\rho_1\rho_2$, where ρ_1 and ρ_2 are the principal radii of curvature at the point in question with respect to the enveloping Euclidean three-dimensional space.

Exercise 1.2. SPRING TIDE VS. NEAP TIDE

Evaluate (1) in conventional units and (2) in geometrized units the magnitude of the Newtonian tide-producing acceleration $R^m{}_{0n0}(m, n = 1, 2, 3)$ generated at the Earth by (1) the moon ($m_{\text{conv}} = 7.35 \times 10^{25}$ g, $r = 3.84 \times 10^{10}$ cm) and (2) the sun ($m_{\text{conv}} = 1.989 \times 10^{33}$ g, $r = 1.496 \times 10^{13}$ cm). By what factor do you expect spring tides to exceed neap tides?

Exercise 1.3. KEPLER ENCAPSULATED

A small satellite has a circular frequency $\omega(\text{cm}^{-1})$ in an orbit of radius r about a central object of mass m(cm). From the known value of ω, show that it is possible to determine neither r nor m individually, but only the effective "Kepler density" of the object as averaged over a sphere of the same radius as the orbit. Give the formula for ω^2 in terms of this Kepler density.

It is a reminder of the continuity of history that Kepler and Galileo (Box 1.9) wrote back and forth, and that the year that witnessed the death of Galileo saw the birth of Newton (Box 1.10). After Newton the first dramatically new synthesis of the laws of gravitation came from Einstein (Box 1.11).

And what the dead had no speech for, when living,
They can tell you, being dead; the communication
Of the dead is tongued with fire beyond
the language of the living.

T. S. ELIOT, in *LITTLE GIDDING* (1942)

I measured the skies
Now the shadows I measure
Skybound was the mind
Earthbound the body rests

JOHANNES KEPLER, d. November 15, 1630.
He wrote his epitaph in Latin;
it is translated by Coleman (1967), p. 109.

Ubi materia, ibi geometria.

JOHANNES KEPLER