10
Asset Pricing and Macroeconomics

10.1 Introduction

Assets—physical, human, and financial capital—play a crucial role in macroeconomics. They are required for production and for generating income, and they are central to the intertemporal allocation of resources through the processes of saving, lending, and borrowing. In this chapter we focus on how financial assets are priced in general equilibrium. In chapter 11 we apply these theories to three financial assets: bonds, equity, and foreign exchange. Each asset has specific features that require the theories to be applied separately.

We began our macroeconomic analysis by discussing the decision about whether to consume today or in the future. This gave us our theories of physical capital accumulation and savings. We argued that people plan their consumption both for today and for the rest of their lives with the aim of maintaining their standard of living even though income may vary through time. During periods when income is low—in retirement or in periods of unemployment, for example—their standard of living would fall unless they had saved some of their income and could draw on this. In order to consume more in the future, people must consume less today, i.e., they substitute intertemporally between consumption today and consumption in the future. The decision of whether to consume or to save depends on the rate of return to savings relative to the rate of time preference: in other words, on the price of financial assets.

Future consumption requires output and hence physical capital. The decision on whether to invest and accumulate capital or to disburse profits depends on the rate of return to capital and the cost of borrowing from households. In general equilibrium, the rate of return to capital and the rate of interest on savings are related because firms will not be willing to borrow at rates higher than their rate of return to capital, and households will not be willing to lend to firms, or anyone else, such as government, unless the rate of return to savings is greater than or equal to their rate of time preference. Moreover, no matter the type of asset, we want to price them in a consistent way. We therefore seek a theory of asset pricing that reflects these intertemporal general equilibrium considerations.

So far we have treated asset returns as though they are all risk free. In practice, however, nearly all assets are risky, having uncertain payoffs in the future and hence risky returns. We therefore need our theory of asset pricing to take account of risk. One way of classifying the various theories of asset pricing is through the way they account for risk, and hence in their measure of the risk premium—the additional expected return in excess of the certain return required to compensate for bearing risk or uncertainty. We consider four theories of asset pricing: contingent-claims analysis, general equilibrium asset pricing, the consumption-based capital-asset-pricing model, and the traditional capital-asset-pricing model. We also show how they are related. As risky returns are random variables, we use stochastic dynamic programming instead of Lagrange multiplier analysis though, as explained, Lagrange multiplier analysis could still be used. We begin by considering some preliminaries: expected utility and risk aversion, the risk premium, arbitrage and no arbitrage, and their implications for efficient market theory. We then consider contingent-claims theory before turning to intertemporal asset pricing. In chapter 11 we apply this theory to the stock, bond, and foreign exchange markets.

A helpful general reference for the basic concepts of the theory of finance covered here is Ingersoll (1987). An excellent recent reference covering asset pricing theory based on the discount-factor approach is Cochrane (2005). For discussion of the links between finance and macroeconomics see Lucas (1978) and Altug and Labadie (1994). In keeping with the rest of this book, our discussion of finance will use discrete time. For an account of the intertemporal capital-asset-pricing model in continuous time see Merton (1973).

10.2 Expected Utility and Risk

10.2.1 Risk Aversion

We begin by establishing a definition of risk aversion. Consider a gamble with a random payoff (value of wealth after the gamble) W in which there are two possible outcomes (payoffs or prospects) x_1 and x_2. Let the probabilities of the two outcomes be π and $(1 - \pi)$, respectively. The issue is whether to avoid the gamble and receive with certainty the actuarial value of the gamble (i.e., its expected or average value), or to take the gamble even though this involves an uncertain outcome.

A person who prefers the gamble is a *risk lover*, one who is indifferent is *risk neutral*, and one who prefers the actuarial value with certainty is *risk averse*. The expected (or actuarial) value of the gamble is

$$E(W) = \pi x_1 + (1 - \pi)x_2.$$

A *fair gamble* is one where $E(W) = 0$.

For the utility function $U(W)$ with $U'(W) \geqslant 0$ and $U''(W) \leqslant 0$ we can define attitudes to risk more precisely as follows:

$$\begin{aligned} &\text{risk aversion:} && U[E(W)] > E[U(W)]; \\ &\text{risk neutrality:} && U[E(W)] = E[U(W)]; \\ &\text{risk loving:} && U[E(W)] < E[U(W)]. \end{aligned}$$

It can be shown that

$$E[U(W)] \gtreqless U[E(W)] \quad \text{as } U'' \gtreqless 0.$$

This is known as Jensen's inequality. To prove this, consider a Taylor series expansion about $E(W)$:

$$\begin{aligned} E[U(W)] &= E(U[E(W)]) + E(W - E[W])U' + \tfrac{1}{2}E[W - E(W)]^2 U'' \\ &= U[E(W)] + \tfrac{1}{2}E[W - E(W)]^2 U'' \\ &\gtreqless U[E(W)] \quad \text{as } U'' \gtreqless 0. \end{aligned}$$

Consider now the case where there is a single risky asset with return r and variance $V(r)$ and a risk-free asset with return r^{f}. If the initial stock of wealth is W_0, then wealth after investing in the risky asset is $W^{\text{r}} = W_0(1 + r)$ and after investing in the risk-free asset it is $W^{\text{f}} = W_0(1+r^{\text{f}})$. Expanding $E[U(W^{\text{r}})]$ about $r = r^{\text{f}}$ we obtain

$$E[U(W^{\text{r}})] \simeq U[W_0(1 + r^{\text{f}})] + W_0^2 \tfrac{1}{2} V(r) U''[W_0(1 + r^{\text{f}})] \tag{10.1}$$

$$\gtreqless U[W_0(1 + r^{\text{f}})] = U(E[W^{\text{f}}]) \quad \text{as } U'' \gtreqless 0. \tag{10.2}$$

Hence, for a risk-averse investor (i.e., $U'' < 0$) the expected utility of investing in the risky asset (taking a gamble) $E[U(W^{\text{r}})]$ is less than the certain utility of investing in the risk-free asset $U(E[W^{\text{f}}])$. We conclude that when the expected returns are the same, a risk-averse investor would prefer not to take a gamble. On the other hand, the investor who is risk neutral ($U'' = 0$) would be indifferent between the two assets.

10.2.2 Risk Premium

We now ask how much compensation a risk-averse investor would need in order to be willing to take the gamble or hold the risky asset. We assume that this compensation can take the form of a known additional payment, or of a higher expected return than the risk-free rate. The additional (certain) payoff (return) required to compensate for the risk arising from taking the fair gamble is called the risk premium. We consider only the case of a single risky asset and a single risk-free asset.

In equation (10.2) we showed that for a risk-averse investor, $E[U(W^{\text{r}})] < U(E[W^{\text{f}}])$. We define the risk premium as the certain value of ρ that satisfies $E(r) = r^{\text{f}} + \rho$ and

$$E[U(W^{\text{r}})] = U(W^{\text{f}}). \tag{10.3}$$

We now take a Taylor series expansion of $E[U(W^{\text{r}})]$ about $r = r^{\text{f}} + \rho$ to obtain

$$E[U(W^{\text{r}})] \simeq U[W_0(1 + r^{\text{f}} + \rho)] + \tfrac{1}{2}W_0^2 E(r - r^{\text{f}} - \rho)^2 U''. \tag{10.4}$$

Expanding $U[W_0(1 + r^{\text{f}} + \rho)]$ about $\rho = 0$ we obtain

$$U[W_0(1 + r^{\text{f}} + \rho)] \simeq U[W_0(1 + r^{\text{f}})] + W_0 \rho U'. \tag{10.5}$$

Combining equations (10.4) and (10.5) gives

$$E[U(W^{\mathrm{r}})] \simeq U(E[W^{\mathrm{f}}]) + W_0 \rho U' + \tfrac{1}{2} W_0^2 V(r) U''. \tag{10.6}$$

It follows that if equation (10.3) is satisfied, then

$$\rho = -\frac{V(r)}{2} \frac{W_0 U''}{U'}.$$

Thus, the risk premium ρ will be larger, the larger the coefficient of relative risk aversion $-(W_0 U''/U')$ (i.e., the curvature of the utility function) and the larger the variance (or volatility) of the risky return, $V(r)$.

10.3 No-Arbitrage and Market Efficiency

10.3.1 Arbitrage and No-Arbitrage

Whether or not assets are correctly priced by a market relates to the concepts of arbitrage and no-arbitrage.

1. An *arbitrage* portfolio is a self-financing portfolio with a zero or negative cost that has a positive payoff.
2. An *arbitrage-free*, or *no-arbitrage*, portfolio is a self-financing portfolio with a zero payoff.

Crudely put, an arbitrage portfolio gives the investor something for nothing. Such opportunities are therefore rare. The financial market, seeing the existence of an arbitrage opportunity, would compete for the assets, thereby raising their price and eliminating the arbitrage opportunity. It is therefore common in the theory of asset pricing to assume that arbitrage opportunities do not exist and to impose this as a restriction. The implication is that if a market is efficient, then it is pricing assets correctly and quickly eliminates arbitrage opportunities.

10.3.2 Market Efficiency

A market is said to be efficient if there are no unexploited arbitrage opportunities. This requires that all new information is instantly impounded in market prices. This is an exacting standard. In practice, fully and correctly reflecting all relevant information so that new information, or new ways of processing this information, have no effect on an asset or any other price is almost impossible to achieve. In principle, the concept should be extended even further to become a criterion of general equilibrium.

The return on an asset may be written as

$$r_{t+1} = \frac{X_{t+1}}{P_t},$$

where P_t is the price of the asset at the start of period t and X_{t+1} is its value or payoff at the start of period $t + 1$. For any risky asset i with return $r_{i,t+1}$ the absence of arbitrage opportunities implies that

$$E_t r_{i,t+1} = r_t^{\mathrm{f}} + \rho_{it}, \tag{10.7}$$

where r_t^{f}, the return on the risk-free asset, is known with certainty at the start of period t, and ρ_{it} is the risk premium for the ith asset, also known at the start of period t.

Equation (10.7) shows that asset pricing consists of pricing one asset relative to another, namely, the risk-free rate, then adding the risk premium. Traditional finance commonly does this by relating the risk premium to a set of factors determined from the past behavior of asset prices. An example is the use of affine factor models to determine the prices of bonds with different times to maturity, i.e., the term structure of interest rates. In contrast, the general equilibrium theory of asset pricing is based on identifying the fundamental sources of risk generated by macroeconomic fluctuations and their uncertainty. These are due largely to unanticipated fluctuations in output and inflation in both the domestic and the international economies. We begin our discussion of asset pricing by considering contingent-claims analysis.

10.4 Asset Pricing and Contingent Claims

Contingent-claims analysis provides a very general theory of asset pricing to which all other theories may be related. A typical asset can be thought of as comprising a combination of primitive assets called contingent claims. Once we know the prices of the primitive assets we can calculate the price of any other asset. We state the problem of pricing an asset using contingent claims as follows:

1. The price of an asset depends on its payoff.
2. Payoffs are typically unknown today. They depend on the state of the world tomorrow.
3. All assets can be considered as a bundle of primitive assets called *contingent claims*.
4. The difference between assets arises from the way the contingent claims are combined.
5. If we can price each contingent claim, then we can price any combination of them, i.e., any asset.

10.4.1 A Contingent Claim

Suppose there are $s = 1, \dots, S$ states of the world. A contingent claim is an asset that has a payoff of \$1 if state s occurs and a payoff of 0 otherwise. Let $q(s)$ denote today's price of a contingent claim with a payoff of \$1 in state s. Also let $x(s)$ be the quantity of this contingent claim that is purchased at date t. Finally, let p denote today's price of an asset whose payoff depends on which state of the world $s = 1, \dots, S$ occurs.

10.4.2 The Price of an Asset

Provided the state prices exist, the price p of any asset can now be expressed as

$$p = \sum_{s=1}^{S} q(s)x(s). \tag{10.8}$$

The vector $q = [q(1)q(2)\cdots q(S)]'$ is then known as a *state-price* vector. This relation says that the price of the asset is simply equal to the sum of the price in a given state s multiplied by the quantity of contingent claims held in that state.

10.4.3 The Stochastic Discount-Factor Approach to Asset Pricing

Suppose that $\pi(s)$ is the probability of state s occurring. The $\pi(s)$ therefore define the state-density function. Next we define

$$m(s) = \frac{q(s)}{\pi(s)}, \quad s = 1, \dots, S. \tag{10.9}$$

Thus $m(s)$ is the price in state s divided by the probability of state s occurring; $m(s)$ is nonnegative because state prices and probabilities are both nonnegative. We can now write the price of the asset as

$$\begin{aligned} p &= \sum_{s=1}^{S} \pi(s)m(s)x(s) \\ &= E(mx). \end{aligned} \tag{10.10}$$

$m(s)$ can therefore be interpreted as the value of the stochastic discount factor of \$1 in state s, $x(s)$ can be interpreted as the payoff in state s, and the price of the asset as the average, or expected, discounted value of these payoffs. If $m(s)$ is small, then state s is "cheap" in the sense that investors are unwilling to pay a high price to receive the payoff in state s. An asset that delivers in cheap states tends to have a payoff that covaries negatively with $m(s)$, i.e., $\text{Cov}(m, x) < 0$.

Equation (10.10) is a completely general pricing formula applicable to all assets, including derivatives such as options. It is called the stochastic discount-factor approach. All other asset-pricing theories can be expressed in this form. The differences between them are in the way that the stochastic discount factor m is specified. The reader is referred to Cochrane (2005) for a more detailed treatment of the stochastic discount-factor approach to asset pricing.

10.4.4 Asset Returns

Equation (10.10) can be expressed in terms of returns instead of the asset price. Dividing equation (10.8) through by p and defining $1 + r(s) = x(s)/p$ for $s = 1, \dots, S$, we can rewrite equation (10.8) as

$$1 = \sum_{s=1}^{S} q(s)[1 + r(s)]. \tag{10.11}$$

It follows that

$$\begin{aligned}1 &= \sum_{s=1}^{S} \pi(s)m(s)[1+r(s)] \\ &= E[m(1+r)],\end{aligned} \tag{10.12}$$

where r is the return on the asset. This is the stochastic discount-factor representation of returns, whether risky or risk free.

10.4.5 Risk-Free Return

Since equation (10.12) applies to all assets, it applies to risk-free assets. If the asset is risk free, then it has the same payoff in all states of the world. Thus, $x(s)$ is independent of s, and we can write $x(s) = x$ for all s. The price of the risk-free asset is then

$$\begin{aligned}p^{\text{f}} &= \sum q(s)x = x\sum q(s) \\ &= x\sum \pi(s)m(s) = xE(m),\end{aligned} \tag{10.13}$$

otherwise an arbitrage opportunity would exist.

If, for example, $x = 1$, then the price of an asset today that pays one unit in all states of nature next period is given by

$$p^{\text{f}} = E(m),$$

or

$$\begin{aligned}1 &= \frac{1}{p^{\text{f}}}E(m) \\ &= (1+r^{\text{f}})E(m),\end{aligned}$$

where r^{f} is the risk-free rate. Further,

$$E(m) = \frac{1}{1+r^{\text{f}}}. \tag{10.14}$$

10.4.6 The No-Arbitrage Relation

We can derive the no-arbitrage relation, equation (10.7), from equations (10.12) and (10.14). From the definition of a covariance between two random variables x and y,

$$\text{Cov}(x, y) = E(xy) - E(x)E(y),$$

and noting that in general m and r are stochastic, we may rewrite equation (10.12) as

$$1 = E(m)E(1+r) + \text{Cov}(m, 1+r);$$

hence

$$E(1+r) = \frac{1}{E(m)} - \frac{\text{Cov}(m, 1+r)}{E(m)}. \tag{10.15}$$

From equations (10.14) and (10.15) we obtain the no-arbitrage relation:

$$E(r) = r^{\mathrm{f}} - \frac{\operatorname{Cov}(m, 1+r)}{1+r^{\mathrm{f}}}. \tag{10.16}$$

Thus the risk premium ρ—the expected return in excess of the risk-free rate—is

$$\rho = -\frac{\operatorname{Cov}(m, 1+r)}{1+r^{\mathrm{f}}}. \tag{10.17}$$

For $\rho > 0$ we require that $\operatorname{Cov}(m, 1+r) = \operatorname{Cov}(m, r) < 0$. In other words, risk arises when low returns coincide with a high discount factor. We will consider how to determine the stochastic discount factor below.

10.4.7 Risk-Neutral Valuation

Having introduced the concept of a risk premium, before pursuing the issue of how to determine it, we consider how to avoid considerations of risk by using risk-neutral valuation. This requires us to use the concept of a risk-neutral probability $\pi^{\mathrm{N}}(s)$ instead of the state probability $\pi(s)$, which is the actual probability of state s occurring. The price of any asset can be represented as the expected value of its future random payoffs using these risk-neutral probabilities. Risk-neutral (or risk-adjusted) probability is crucial to many results in the theory of finance, particularly in pricing options.

10.4.7.1 Risk-Neutral Probability

Given a positive state-price vector $[q(1)q(2)\cdots q(S)]'$, we may define the risk-neutral probability $\pi^*(s)$ as

$$\begin{aligned}\pi^{\mathrm{N}}(s) &= (1+r^{\mathrm{f}})\pi(s)m(s)\\ &= \frac{1}{\sum q(s)}\pi(s)\left(\frac{q(s)}{\pi(s)}\right) = \frac{q(s)}{\sum q(s)},\end{aligned}$$

where

$$\sum_{s=1}^{S} \pi^{\mathrm{N}}(s) = 1 \quad \text{and} \quad 0 < \pi^*(s) < 1.$$

10.4.7.2 Asset Pricing Using Risk-Neutral Probabilities

We can convert the price of an asset written in terms of state probabilities into one written in terms of risk-neutral probabilities as follows:

$$\begin{aligned}p &= \sum \pi(s)m(s)x(s)\\ &= \frac{1}{1+r^{\mathrm{f}}}\sum \pi^{\mathrm{N}}(s)x(s)\\ &= \frac{1}{1+r^{\mathrm{f}}}E^{\mathrm{N}}[x(s)]\\ &= E(m)E^{\mathrm{N}}(x),\end{aligned}$$

where we have substituted $1/E(m)$ for $1 + r^{\mathrm{f}}$ and $E^{\mathrm{N}}(\cdot)$ denotes an expectation taken with respect to the risk-neutral probabilities. Hence, by using risk-neutral probabilities, we can express the price of an asset as

$$p = E[mx] = E(m)E^{\mathrm{N}}(x) \tag{10.18}$$

$$= \frac{1}{1 + r^{\mathrm{f}}}E^{\mathrm{N}}(x). \tag{10.19}$$

Equation (10.18) implies that using risk-neutral probabilities, m and x are uncorrelated. Equation (10.19) shows that the price of any asset can be written as the expected discounted value of its future payoffs, where the discounting is done by means of the stochastic discount factor $m(s)$. It then follows that the price of the asset is proportional to the risk-neutral expectation of its random payoff.

From equation (10.19), the no-arbitrage equation for returns can now be written without a risk premium as

$$E^{\mathrm{N}}(r) = r^{\mathrm{f}}. \tag{10.20}$$

Comparing equation (10.7) or equation (10.16) with (10.20) we deduce that

$$E^{\mathrm{N}}(r) = E(r) - \rho,$$

where ρ is the risk premium. Thus risk-neutral valuation risk-adjusts the risky return. It does not, of course, eliminate the risk itself, which remains. The advantage is that it can simplify asset pricing.

10.5 General Equilibrium Asset Pricing

In general equilibrium, asset prices are determined jointly with all other variables in the economy. Previously in our discussion of the real macroeconomy, we determined the real rate of return to capital jointly with consumption, investment, and capital. But in our discussion of households and life-cycle theory we treated the rate of return to financial assets as given. We now reconsider the analysis of the household, who we take to be the representative investor in financial assets. We begin by examining the problem using contingent-claims analysis. We then extend the discussion to the type of formulation of the macroeconomy that we have used until this chapter.

10.5.1 Using Contingent-Claims Analysis

Consider a representative household that is deciding between consumption today and consumption tomorrow, where current income is known with certainty but income next period is random and hence uncertain. We do not specify the source of this income, which could be from working or from asset income. We assume that the household maximizes the value of current plus discounted expected future utility—both derived from consumption—subject to a budget constraint that depends on the state of the world in the second period. Thus the problem is to maximize

$$\begin{aligned} V &= U(c_t) + \beta E_t U(c_{t+1}) \\ &\equiv U(c) + \beta \sum_s \pi(s) U[c(s)] \end{aligned}$$

subject to

$$c + \sum_s q(s)c(s) = y + \sum_s q(s)y(s),$$

where c is current consumption and y is current income and both are known with certainty in the current period, $c(s)$ is next period's consumption and $y(s)$ is next period's income and both are unknown in the current period as s, the state of the economy next period, is unknown. The $q(s)$ are the state prices for contingent claims that are used to value future random consumption and income streams.

Although the problem is stochastic, it can be analyzed using Lagrange multiplier analysis. The problem is to maximize the Lagrangian

$$\mathcal{L} = U(c) + \beta \sum_s \pi(s) U[c(s)] + \lambda \Big[y + \sum_s q(s)y(s) - c - \sum_s q(s)c(s) \Big].$$

The first-order conditions are given by

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c} &= U'(c) - \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial c(s)} &= \beta \pi(s) U'[c(s)] - \lambda q(s) = 0, \quad s = 1, \dots, S. \end{aligned}$$

Combining these conditions yields the set of conditions

$$q(s) = \beta \pi(s) \frac{U'[c(s)]}{U'(c)}, \quad s = 1, \dots, S.$$

From equation (10.9),

$$q(s) = \pi(s)m(s);$$

hence

$$m(s) = \frac{\beta U'[c(s)]}{U'(c)}. \tag{10.21}$$

Thus, the stochastic discount factor is the intertemporal marginal rate of substitution in consumption between two consecutive periods. As consumption in the second period is a random variable, so is the stochastic discount factor.

It also follows that the state prices $q(s)$ are defined as the product of the state probabilities and the intertemporal marginal rate of substitution in consumption between two consecutive periods. If we are willing to formulate a well-specified underlying economic model, we can then obtain explicit expressions for the state prices $q(s)$. The expected value of any random consumption stream is given by

$$\sum_s q(s)c(s) = \sum_s \pi(s)m(s)c(s) = E(mc).$$

Having determined the stochastic discount factors $m(s)$, we can price any asset in this economy using equation (10.8). The resulting price is

$$p = \sum_s \pi(s)m(s)x(s) = \frac{\beta \sum_s \pi(s)U'[c(s)]x(s)}{U'(c)}. \tag{10.22}$$

In particular, we can price the income stream $y(s)$ by setting $x(s) = y(s)$.

We can also rewrite equation (10.22) in terms of the stochastic rate, or return, $r(s)$ as

$$\sum_s \pi(s)\frac{\beta U'[c(s)]}{U'(c)}[1 + r(s)] = 1, \tag{10.23}$$

where $1 + r(s) = x(s)/p$. Equation (10.23) is just an Euler equation defined for stochastic returns.

If we denote the current period as time t and the second period as $t + 1$, then we can rewrite the no-arbitrage equation (10.16) for the return on any risky income stream as

$$E(r_{t+1}) = r_t^{\mathrm{f}} - \frac{1}{1 + r_t^{\mathrm{f}}}\operatorname{Cov}\left[\frac{\beta U'(c_{t+1})}{U'(c_t)}, r_{t+1}\right]. \tag{10.24}$$

This will also be the no-arbitrage equation for the return on any asset in the economy. The last term is the risk premium.

10.5.2 Asset Pricing Using the Consumption-Based Capital Asset-Pricing Model (C-CAPM)

Equation (10.24) is commonly known as the asset-pricing equation for the consumption-based capital-asset-pricing model (see Breedon 1979). We now derive this equation using the formulation of the macroeconomy that we have adopted in previous chapters. We demonstrate that the Euler equation that determines the optimal path of consumption is also used to price assets. This result shows that the DGE model provides a single theoretical framework for use in both macroeconomics and finance, and hence unifies the two subjects.

In chapter 4 we defined the household's problem as maximizing

$$V_t = \sum_{s=0}^{\infty} \beta^s U(c_{t+s}) \tag{10.25}$$

subject to the budget constraint

$$\Delta a_{t+1} + c_t = x_t + r_t a_t, \tag{10.26}$$

where x_t is income and a_t is the real stock of assets. It was assumed that the future was known with certainty. We now assume that the future is uncertain so that $\{x_{t+s}, r_{t+s};\ s > 0\}$ are random variables. We therefore replace equation (10.25) by its conditional expectation based on information available at time t:

$$V_t = \sum_{s=0}^{\infty} \beta^s E_t[U(c_{t+s})]. \tag{10.27}$$

Previously, we solved the optimization using the method of Lagrange multipliers. As explained in the mathematical appendix, the problem with applying this method to the stochastic case is that the Lagrange multipliers are also random variables. As a result, the Euler equation is expressed in terms of the conditional expectation of the product of the rate of change in the Lagrange multipliers and r_{t+1}, and we are unable to substitute the marginal utility of consumption for the Lagrange multipliers and so solve for consumption. Instead, therefore, we use the method of stochastic dynamic programming, the details of which are given in the mathematical appendix.

First, we rewrite equation (10.27) as the recursion

$$V_t = U(c_t) + \beta E_t[V_{t+1}]. \tag{10.28}$$

More generally, we could have a time-nonseparable utility function

$$V_t = G\{U(c_t), E_t[V_{t+1}]\}.$$

The advantage of such a formulation is that it enables attitudes to risk to be distinguished from attitudes to time (see Kreps and Porteus 1978). For simplicity, we shall confine ourselves to time-separable utility as in equation (10.28). We note that the assumed time horizon in equation (10.27) is infinity. We can justify this by noting that, although people live finite lives, provided their effective time horizon is long enough, the assumption of an infinite horizon will provide a very good approximation. We may also note that people do not know when they will die and act for most of their lives as though they have many more years to live. Further, if reoptimization takes place each period, only the first period (period t) would be carried out. A common reformulation of equation (10.27) with a finite horizon of T is

$$V_t = \sum_{i=0}^{T} \beta^i E_t[U(c_{t+i})] + \beta^T E_t[B(a_T)], \tag{10.29}$$

where $B(a_T)$ can be interpreted as a bequest motive. The familiar capital-asset-pricing model (CAPM) of Sharpe (1964), Lintner (1965), and Mossin (1966) is another special case of equation (10.29) involving only the last term. In other words, in CAPM the aim is to maximize the value of the stock of financial assets at some point in the future, thereby ignoring intermediate consumption.

10.5.2.1 The Stochastic Dynamic Programming Solution

The problem is to maximize the value-function equation (10.28) subject to the dynamic-constraint equation (10.26). As shown in the mathematical appendix, provided a solution exists, the first-order condition can be obtained by differentiating with respect to c_t to give

$$\frac{\partial V_t}{\partial c_t} = \frac{\partial U_t}{\partial c_t} - \beta E_t\left(\frac{\partial V_{t+1}}{\partial c_t}\right) = 0 \tag{10.30}$$

with

$$\frac{\partial V_{t+1}}{\partial c_t} = \frac{\partial V_{t+1}}{\partial c_{t+1}} \frac{\partial c_{t+1}}{\partial c_t}.$$

In order to evaluate $E_t(\partial V_{t+1}/\partial c_{t+1})$ we note that

$$V_{t+1} = U(c_{t+1}) + \beta E_{t+1}(V_{t+2});$$

hence,

$$\frac{\partial V_{t+1}}{\partial c_{t+1}} = \frac{\partial U_{t+1}}{\partial c_{t+1}}.$$

To evaluate $\partial c_{t+1}/\partial c_t$ we express c_{t+1} as a function of c_t. This requires using the two-period budget constraint obtained by combining the budget constraints for periods t and $t+1$ to eliminate a_{t+1} so that

$$c_t + \frac{c_{t+1}}{1+r_{t+1}} + \frac{a_{t+2}}{1+r_{t+1}} = x_t + \frac{x_{t+1}}{1+r_{t+1}} + a_t(1+r_t).$$

Hence,

$$\frac{\partial c_{t+1}}{\partial c_t} = -(1+r_{t+1}).$$

The first-order condition (10.30) can now be written as

$$\frac{\partial V_t}{\partial c_t} = \frac{\partial U_t}{\partial c_t} - \beta E_t\left[\frac{\partial U_{t+1}}{\partial c_{t+1}}(1+r_{t+1})\right] = 0.$$

We therefore obtain the Euler equation for the stochastic case as

$$E_t\left[\frac{\beta U'_{t+1}}{U'_t}(1+r_{t+1})\right] = 1. \tag{10.31}$$

This is equivalent to equations (10.23) and (10.12) with

$$E_t[M_{t+1}(1+r_{t+1})] = 1 \tag{10.32}$$

and $M_{t+1} \equiv (\beta U'_{t+1}/U'_t)$.

10.5.2.2 Pricing Risky Assets

Previously we solved the Euler equation for consumption, taking the rate of return as given. To obtain the asset price we reverse this by solving instead for r_{t+1} in terms of consumption. In this way we use the same stochastic dynamic general equilibrium model to determine the macroeconomic variables and the asset prices. We therefore unify economics and finance within a single theoretical framework.

In general, the Euler equation will be highly nonlinear. We therefore take a Taylor series expansion of the Euler equation about $c_{t+1} = c_t$. Expanding $U'(c_{t+1})$ about $c_{t+1} = c_t$ to give

$$U'(c_{t+1}) \simeq U'(c_t) + (c_{t+1} - c_t)U''_t$$

leads to the approximation

$$\frac{\beta U'_{t+1}}{U'_t} \simeq \beta\left[\frac{U'_t + \Delta c_{t+1} U''_t}{U'_t}\right] = \beta\left[1 + \frac{\Delta c_{t+1}}{c_t}\frac{c_t U''_t}{U'_t}\right]$$
$$= \beta\left[1 - \sigma_t \frac{\Delta c_{t+1}}{c_t}\right],$$

where

$$\sigma_t = -\frac{c_t U''_t}{U'_t} \geqslant 0, \quad \text{as } U''_t \leqslant 0,$$

is the coefficient of relative risk aversion (CRRA); the greater the CRRA is, the more risk averse the investor. We recall that in the special case of power utility,

$$U(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma \geqslant 0;$$

the CRRA is a constant, i.e., $\sigma_t = \sigma$.

We can now write the Euler equation as

$$E_t\left[\beta\left(1 - \sigma_t \frac{\Delta c_{t+1}}{c_t}\right)(1 + r_{t+1})\right] = 1.$$

Recalling that $\beta = 1/(1+\theta)$ this can be rewritten as

$$1 - \sigma_t E_t\left(\frac{\Delta c_{t+1}}{c_t}\right) + E_t(r_{t+1}) - \sigma_t E_t\left(\frac{\Delta c_{t+1}}{c_t} r_{t+1}\right) = 1 + \theta.$$

Using the fact that[1]

$$E_t\left(\frac{\Delta c_{t+1}}{c_t} r_{t+1}\right) = \operatorname{Cov}_t\left(\frac{\Delta c_{t+1}}{c_t}, r_{t+1}\right) + E_t\left(\frac{\Delta c_{t+1}}{c_t}\right) E_t(r_{t+1}),$$

we obtain an expression for the expected rate of return:

$$E_t(r_{t+1}) = \frac{\theta + \sigma_t E_t(\Delta c_{t+1}/c_t) + \sigma_t \operatorname{Cov}_t((\Delta c_{t+1}/c_t), r_{t+1})}{1 - \sigma_t E_t(\Delta c_{t+1}/c_t)}. \tag{10.33}$$

10.5.2.3 Pricing the Risk-Free Asset

Equation (10.33) holds for any asset, whether risky or risk free. But if the asset is risk free, we can replace r_t in the budget constraint by the risk-free rate r^{f}_{t-1}. Furthermore, r_{t+1} can be replaced in equation (10.33) by r^{f}_t, which is known at time t. Hence, $E_t(r^{\mathrm{f}}_t) = r^{\mathrm{f}}_t$ and $\operatorname{Cov}_t((\Delta c_{t+1}/c_t), r^{\mathrm{f}}_t) = 0$. Equation (10.33) therefore becomes

$$r^{\mathrm{f}}_t = \frac{\theta + \sigma_t E_t(\Delta c_{t+1}/c_t)}{1 - \sigma_t E_t(\Delta c_{t+1}/c_t)}. \tag{10.34}$$

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[1] If, for example, X_{t+1} is a vector of random variables with conditional mean AX_t so that $X_{t+1} = AX_t + e_t$, where the e_t are independently distributed with zero mean and variance Σ_t, then for any two elements $x_{i,t+1}$ and $x_{j,t+1}$ we have $\operatorname{Cov}_t(x_{i,t+1}, x_{j,t+1}) = \sigma_{ij,t}$, where $\Sigma_t = \{\sigma_{ij,t}\}$. Thus the conditional covariance is the covariance of the error terms in the forecasting model of X_t based on information available at time t and is itself a forecast of the covariance between $e_{i,t+1}$ and $e_{j,t+1}$. Clearly, a vector autoregressive (VAR) model is a convenient vehicle for constructing the conditional covariance.

For future reference we also note that if z_t is another random variable, then $\operatorname{Cov}_t(a + bx_{t+1}, c + dy_{t+1} + z_t) = bd \operatorname{Cov}_t(x_{t+1}, y_{t+1})$.

10.5.2.4 The No-Arbitrage Relation

Combining equations (10.33) and (10.34) yields the no-arbitrage equation

$$\begin{aligned} E_t r_{t+1} &= r_t^{\mathrm{f}} + \frac{\sigma_t \operatorname{Cov}_t((\Delta c_{t+1}/c_t), r_{t+1})}{1 - \sigma_t E_t(\Delta c_{t+1}/c_t)} \\ &= r_t^{\mathrm{f}} + \beta\sigma_t(1 + r_t^{\mathrm{f}}) \operatorname{Cov}_t\left(\frac{\Delta c_{t+1}}{c_t}, r_{t+1}\right). \end{aligned} \tag{10.35}$$

The last term in equation (10.35) is the risk premium. Thus, an asset is risky if for states of nature in which returns are low, the intertemporal marginal rate of substitution in consumption, $M_{t+1} \equiv (\beta U'_{t+1}/U'_t)$, is high. Since M_{t+1} will be high if future consumption is low, a risky asset is one which yields low returns in states for which consumers also have low consumption. This situation is typical of what happens in a business cycle. For example, in the recession phase both returns and consumption growth are low, whereas in the boom phase both are high. This generates a positive correlation between the two and hence a positive risk premium.

We note that we can also write equation (10.35) in terms of the excess return

$$E_t r_{t+1} - r_t^{\mathrm{f}} = \beta\sigma_t(1 + r_t^{\mathrm{f}}) \operatorname{Cov}_t\left(\frac{\Delta c_{t+1}}{c_t}, r_{t+1} - r_t^{\mathrm{f}}\right) \tag{10.36}$$

as

$$\operatorname{Cov}_t\left(\frac{\Delta c_{t+1}}{c_t}, r_{t+1}\right) = \operatorname{Cov}_t\left(\frac{\Delta c_{t+1}}{c_t}, r_{t+1} - r_t^{\mathrm{f}}\right)$$

due to r_t^{f} being part of the information set at time t.

Taken together, the above results show that in an efficient market risky assets are priced off the risk-free asset plus an asset-specific risk premium that reflects macroeconomic sources of risk.

10.5.2.5 Pricing Nominal Returns

The analysis above assumes that all variables are measured in real terms, including asset returns. This implies that there is a real risk-free asset. In practice, the closest we get to a real risk-free asset is an index-linked bond. As even this is not fully indexed it is not a true risk-free asset. In contrast, if we ignore default risk, there are nominal risk-free bonds. A short-term Treasury bill is an example.

In order to price nominal returns we need to modify our previous analysis a little by restating the budget constraint in nominal terms as

$$P_t c_t + \Delta A_{t+1} = P_t x_t + R_t A_t,$$

where P_t is the price level, A_t is nominal wealth, and R_t is a risky nominal return. It can be shown that the Euler equation is now

$$E_t\left[\frac{\beta U'_{t+1}}{U'_t}\frac{P_t}{P_{t+1}}(1 + R_{t+1})\right] = 1$$

or

$$E_t\left[\frac{\beta U'_{t+1}}{U'_t}\frac{1+R_{t+1}}{1+\pi_{t+1}}\right] = 1, \tag{10.37}$$

where $\pi_{t+1} = \Delta P_{t+1}/P_t$ is the rate of inflation. Noting that $1 + r_{t+1} = (1 + R_{t+1})/(1+\pi_{t+1})$, where r_{t+1} is the real risky return, equation (10.37) is identical to (10.31).

However, the asset-pricing equation is different because we now have a nominal instead of a real stochastic discount factor. This is $(\beta U'_{t+1}/U'_t)(1/(1+\pi_{t+1}))$. A Taylor series expansion of this gives

$$\frac{\beta U'_{t+1}}{U'_t}\frac{1}{1+\pi_{t+1}} \simeq \beta\left[1 - \sigma_t\frac{\Delta c_{t+1}}{c_t} - \pi_{t+1}\right].$$

The asset-pricing equation for a nominal risky asset is now

$$E_t(R_{t+1}) = \frac{\theta + \sigma_t E_t(\Delta c_{t+1}/c_t) + E_t\pi_{t+1} + \sigma_t \operatorname{Cov}_t((\Delta c_{t+1}/c_t), R_{t+1}) + \operatorname{Cov}_t(\pi_{t+1}, R_{t+1})}{1 - \sigma_t E_t(\Delta c_{t+1}/c_t) - E_t\pi_{t+1}}. \tag{10.38}$$

For a nominal risk-free asset it is

$$E_t(R^{\mathrm{f}}_{t+1}) = \frac{\theta + \sigma_t E_t(\Delta c_{t+1}/c_t) + E_t\pi_{t+1}}{1 - \sigma_t E_t(\Delta c_{t+1}/c_t) - E_t\pi_{t+1}}. \tag{10.39}$$

And the no-arbitrage equation is

$$E_t R_{t+1} = R^{\mathrm{f}}_t + \beta\sigma_t(1 + R^{\mathrm{f}}_t)\left[\operatorname{Cov}_t\left(\frac{\Delta c_{t+1}}{c_t}, R_{t+1}\right) + \operatorname{Cov}_t(\pi_{t+1}, R_{t+1})\right]. \tag{10.40}$$

Thus the risk premium for a nominal risky asset involves two terms: the conditional covariances between the nominal risky rate and consumption growth, and between the nominal risky rate and inflation. If inflation is low when nominal returns are low, as happens in a business cycle caused by negative demand shocks, then the inflation component of the risk premium will be positive. In contrast, a negative supply shock, such as an oil-price shock, would be likely to cause inflation to rise and returns to fall, which would make the inflation component of the risk premium, and possibly even the whole risk premium, negative.

10.5.2.6 The Assumption of Log-Normality

An assumption that is widely used, due partly to its convenience and partly to the fact that it is a reasonably good approximation, is that the stochastic discount factor and the gross return have a joint log-normal distribution. We note that a random variable x is said to be log-normally distributed if $\ln(x)$ follows a normal distribution with mean μ and variance σ^2. If $\ln x$ is $N(\mu, \sigma^2)$, then the expected value of x is given by

$$E(x) = \exp(\mu + \tfrac{1}{2}\sigma^2),$$

and hence

$$\ln E(x) = \mu + \tfrac{1}{2}\sigma^2.$$

As a result, equation (10.32) can be written as

$$\begin{aligned}1 &= E_t[M_{t+1}(1+r_{t+1})]\\ &= \exp\{E_t[\ln(M_{t+1}(1+r_{t+1}))] + V_t[\ln(M_{t+1}(1+r_{t+1}))]/2\}.\end{aligned}$$

Taking logarithms yields

$$\begin{aligned}0 &= \ln E_t[M_{t+1}(1+r_{t+1})]\\ &= E_t[\ln(M_{t+1}(1+r_{t+1}))] + V_t[\ln(M_{t+1}(1+r_{t+1}))]/2\\ &= E_t[\ln M_{t+1} + \ln(1+r_{t+1})] + V_t[\ln M_{t+1} + \ln(1+r_{t+1})]/2\\ &\simeq E_t(\ln M_{t+1}) + E_t(r_{t+1}) + V_t(\ln M_{t+1})/2 + V_t(r_{t+1})/2 + \mathrm{cov}_t(\ln M_{t+1}, r_{t+1})\\ &= 0. \end{aligned} \tag{10.41}$$

When the asset is risk free, equation (10.41) becomes

$$E_t(\ln M_{t+1}) + r_t^{\mathrm{f}} + \tfrac{1}{2}V_t(\ln M_{t+1}) = 0. \tag{10.42}$$

Subtracting (10.42) from (10.41) produces the no-arbitrage condition under log-normality:

$$E_t(r_{t+1} - r_t^{\mathrm{f}}) + \tfrac{1}{2}V_t(r_{t+1}) = -\mathrm{Cov}_t(\ln M_{t+1}, r_{t+1}), \tag{10.43}$$

where $\frac{1}{2}V_t(r_{t+1})$ is the Jensen effect, which arises because expectations are being taken of a nonlinear function and $E[f(x)] \neq f[E(x)]$ unless $f(x)$ is linear.

If $M_{t+1} = \beta(U'(c_{t+1})/U'(c_t))$, then

$$\ln M_{t+1} \simeq -(\theta + \sigma_t \Delta \ln c_{t+1}). \tag{10.44}$$

The no-arbitrage condition can now be written as

$$E_t r_{t+1} - r_t^{\mathrm{f}} + \tfrac{1}{2}V_t(r_{t+1}) = \sigma_t\,\mathrm{Cov}_t(\Delta \ln C_{t+1}, r_{t+1}). \tag{10.45}$$

This may be compared with equation (10.36), which does not assume log-normality.

10.5.2.7 Multi-Factor Models

There is a more general way of expressing asset-pricing models: namely, as multi-factor models. If the stochastic discount factor is written as

$$M_{t+1} = a + \sum_i b_i z_{i,t+1},$$

then the no-arbitrage condition becomes

$$\begin{aligned}E_t r_{t+1} - r_t^{\mathrm{f}} &= -(1 + r_t^{\mathrm{f}})\,\mathrm{Cov}_t(M_{t+1}, r_{t+1})\\ &= -(1 + r_t^{\mathrm{f}})\sum_i b_i\,\mathrm{Cov}_t(z_{i,t+1}, r_{t+1}).\end{aligned} \tag{10.46}$$

For example, CAPM assumes that there is a single stochastic discount factor $M_{t+1} = \sigma_t(1 + r^{\mathrm{m}}_{t+1})$, where r^{m}_{t+1}, the return on the market, is the single factor; C-CAPM assumes that there is a single stochastic discount factor given by $M_{t+1} = \beta(U'_{t+1}/U'_t) \simeq \beta[1 - \sigma_t(\Delta c_{t+1}/c_t)]$ and so consumption growth is the single factor. Thus, for $\sigma_t = \sigma$, a constant,

$$z_{t+1} = \begin{cases} r^{\mathrm{m}}_{t+1}, & \text{CAPM,} \\ \dfrac{\Delta C_{t+1}}{C_t}, & \text{C-CAPM.} \end{cases}$$

Assuming log-normality and that

$$\ln M_{t+1} = a + \sum_i b_i z_{i,t+1},$$

the asset pricing relation is

$$\begin{aligned} E_t(r_{t+1} - r^{\mathrm{f}}_t) + \tfrac{1}{2}V_t(r_{t+1}) &= \sigma_t \operatorname{Cov}_t(\ln M_{t+1}, r_{t+1}) \\ &= \sigma_t \sum_i b_i \operatorname{Cov}_t(z_{i,t+1}, r_{t+1}). \end{aligned} \tag{10.47}$$

Equations (10.46) and (10.47) are known as multi-factor affine models (affine means linear). In practice in finance, often the factors are not chosen to satisfy general equilibrium pricing kernels like the intertemporal marginal rate of substitution but are determined from the data.

10.6 Asset Allocation

We have said that the theory above applies to any asset. If there are several assets in which to hold financial wealth, we must consider in what proportion each asset is to be held in the portfolio. This is the problem of asset allocation or portfolio selection. We begin by examining the case of two assets: a risky and a risk-free asset.

Again the only change that we need to make to the model is to the budget constraint. Let the stocks of risky and risk-free assets be a_t and b_t, respectively, then the budget constraint can be written

$$c_t + a_{t+1} + b_{t+1} = x_t + a_t(1 + r_t) + b_t(1 + r^{\mathrm{f}}_{t-1}).$$

If we define $W_t = a_t + b_t$ and the portfolio shares as $w_t = a_t/W_t$ and $1 - w_t = b_t/W_t$, then the budget constraint can also be written as

$$\begin{aligned} c_t + W_{t+1} &= x_t + W_t[1 + r^{\mathrm{f}}_{t-1} + w_t(r_t - r^{\mathrm{f}}_{t-1})] \\ &= x_t + W_t(1 + r^{\mathrm{p}}_t), \end{aligned}$$

where $r^{\mathrm{p}}_t = r^{\mathrm{f}}_{t-1} + w_t(r_t - r^{\mathrm{f}}_{t-1})$ is the return on the portfolio. The problem now is to maximize V_t with respect to $\{c_{t+s}, a_{t+s+1}, b_{t+s+1};\ s \geqslant 0\}$ or equivalently $\{c_{t+s}, W_{t+s+1}, w_{t+s+1};\ s \geqslant 0\}$.

Using previous results, the first-order conditions are

$$\frac{\partial V_t}{\partial c_t} = U'_t - \beta E_t[U'_{t+1}(1 + r^{\mathrm{p}}_{t+1})] = 0$$

and

$$\begin{aligned}\frac{\partial V_t}{\partial w_{t+1}} &= -\beta E_t\left[\frac{\partial V_{t+1}}{\partial c_{t+1}}\frac{\partial c_{t+1}}{\partial w_{t+1}}\right] = 0\\ &= -\beta E_t[U'_{t+1}W_{t+1}(r_{t+1} - r^{\mathrm{f}}_t)] = 0.\end{aligned}$$

The first condition is the same as before except that r^{p}_{t+1} replaces r_{t+1}. Thus the consumption/savings decision is unchanged, except that it is now based on the portfolio return. From the budget constraint, W_{t+1} is determined by time t variables, hence the second condition can be written as

$$E_t[U'_{t+1}(r_{t+1} - r^{\mathrm{f}}_t)] = 0. \tag{10.48}$$

We note that

$$\begin{aligned}U'_{t+1} &= U'\{x_{t+1} + W_{t+1}[1 + r^{\mathrm{f}}_t + w_{t+1}(r_{t+1} - r^{\mathrm{f}}_t)] - W_{t+2}\}\\ &\simeq U'^{*}_t + W_{t+1}w_{t+1}(r_{t+1} - r^{\mathrm{f}}_t)U''^{*}_{t+1},\end{aligned}$$

where we have used a Taylor series approximation about $w_{t+1} = 0$ and defined $U'^{*}_t = U'(x_{t+1} + W_{t+1}[1 + r^{\mathrm{f}}_t] - W_{t+2})$. Equation (10.48) can now be written

$$\begin{aligned}0 &= E_t[U'_{t+1}(r_{t+1} - r^{\mathrm{f}}_t)]\\ &\simeq U'^{*}_t E_t(r_{t+1} - r^{\mathrm{f}}_t) + W_{t+1}U''^{*}_{t+1}w_{t+1}E_t(r_{t+1} - r^{\mathrm{f}}_t)^2,\end{aligned}$$

and so the share of the risky asset in the portfolio is

$$w_{t+1} = \frac{E_t c_{t+1}}{W_{t+1}}\frac{E_t(r_{t+1} - r^{\mathrm{f}}_t)}{\sigma_t E_t(r_{t+1} - r^{\mathrm{f}}_t)^2}, \tag{10.49}$$

where $\sigma_t = -(E_t c_{t+1}U''^{*}_{t+1}/U'^{*}_{t+1})$ is the CRRA. The higher the proportion of total wealth that is consumed, $E_t c_{t+1}/W_{t+1}$, and the expected excess return, $E_t(r_{t+1} - r^{\mathrm{f}}_t)$, and the lower the conditional volatility of the excess return, $E_t(r_{t+1} - r^{\mathrm{f}}_t)^2$, and the degree of aversion to risk, σ_t, the larger the share invested in the risky asset is. If there were no risky asset, then in effect $E_t(r_{t+1} - r^{\mathrm{f}}_t) = 0$ and so $w_{t+1} = 0$, i.e., the portfolio would be completely risk free.

The analysis can be generalized to many risky assets. In this case r_t and w_t become vectors $\boldsymbol{r}_t$ and $\boldsymbol{w}_t$ with the share in the risk-free asset given by $1 - \boldsymbol{\ell}'\boldsymbol{w}_t$, where $\boldsymbol{\ell}' = (1, 1, \dots, 1)$. The solution has the same form as equation (10.49) and is the vector of shares

$$\boldsymbol{w}_{t+1} = \sigma_t^{-1}\boldsymbol{\Sigma}_t^{-1}E_t(\boldsymbol{r}_{t+1} - \boldsymbol{\ell}r^{\mathrm{f}}_t),$$

where $\boldsymbol{\Sigma}_t$ is the conditional covariance matrix of risky returns.

The excess return on the optimal portfolio is given by premultiplying by $\boldsymbol{w}'_t$,

$$E_t(r^{\mathrm{p}}_{t+1} - r^{\mathrm{f}}_t) = \boldsymbol{w}'_t E_t(\boldsymbol{r}_{t+1} - \boldsymbol{\ell}r^{\mathrm{f}}_t) = \sigma_t\boldsymbol{w}'_t\boldsymbol{\Sigma}_t\boldsymbol{w}_t = \sigma_t V_t(r^{\mathrm{p}}_{t+1}),$$

where $V_t(r^{\mathrm{p}}_{t+1})$ is the conditional variance of the portfolio return. It follows that

$$\frac{E_t(r^{\mathrm{p}}_{t+1} - r^{\mathrm{f}}_t)}{V_t(r^{\mathrm{p}}_{t+1})} = \sigma_t. \tag{10.50}$$

Eliminating σ_t using equation (10.50) yields the excess return for each individual asset as

$$\begin{aligned} E_t(\boldsymbol{r}_{t+1}) - \boldsymbol{\ell} r^{\mathrm{f}}_t &= \sigma_t \boldsymbol{\Sigma}_t \boldsymbol{w}_t \\ &= E_t(r^{\mathrm{p}}_{t+1} - r^{\mathrm{f}}_t)\frac{V_t(\boldsymbol{r}_{t+1})\boldsymbol{w}_t}{V_t(r^{\mathrm{p}}_{t+1})} \\ &= E_t(r^{\mathrm{p}}_{t+1} - r^{\mathrm{f}}_t)\frac{\mathrm{Cov}_t(\boldsymbol{r}_{t+1}, r^{\mathrm{p}}_{t+1})}{V_t(r^{\mathrm{p}}_{t+1})}. \end{aligned} \tag{10.51}$$

Using equation (10.50) we can also write this as

$$E_t(\boldsymbol{r}_{t+1} - \boldsymbol{\ell} r^{\mathrm{f}}_t) = \sigma_t \,\mathrm{Cov}_t(\boldsymbol{r}_{t+1}, r^{\mathrm{p}}_{t+1}). \tag{10.52}$$

For the ith asset this becomes

$$E_t(r_{i,t+1} - r^{\mathrm{f}}_t) = \sigma_t \,\mathrm{Cov}_t(r_{i,t+1}, r^{\mathrm{p}}_{t+1}). \tag{10.53}$$

Equation (10.53) can be shown to be identical to equation (10.36) if $r^{\mathrm{f}}_t = \theta$ and if

$$c_{t+1} = r^{\mathrm{p}}_{t+1} W_{t+1}, \tag{10.54}$$

i.e., consumption is equal to the permanent income arising from wealth, as in life-cycle theory.

It is instructive to consider the implications of this solution for expected utility. The conditional expectation of the instantaneous utility function for period $t + 1$ may be approximated by a second-order Taylor series expansion about $E_t c_{t+1}$ to give

$$\begin{aligned} E_t U(c_{t+1}) &\simeq U(E_t c_{t+1}) + U'_t E_t(c_{t+1} - E_t c_{t+1}) + U''_t E_t(c_{t+1} - E_t c_{t+1})^2 \\ &\simeq U'_t\left[E_t c_{t+1} - \frac{\sigma_t}{2}\frac{V_t(c_{t+1})}{E_t c_{t+1}}\right]. \end{aligned} \tag{10.55}$$

Thus, expected utility is approximately a trade-off between expected consumption and the expected volatility of consumption evaluated at marginal utility.

This can be rewritten in terms of returns. Since

$$\begin{aligned} E_t c_{t+1} &= W_{t+1} E_t r^{\mathrm{p}}_{t+1}, \\ V_t(c_{t+1}) &= W^2_{t+1} V_t(r^{\mathrm{p}}_{t+1}), \end{aligned}$$

from equations (10.50) and (10.55), the maximized value of $E_t U(c_{t+1})$ is approximately

$$\begin{aligned}\max E_t U(c_{t+1}) &\simeq U_t' W_{t+1}\left[E_t r^{\mathrm{p}}_{t+1} - \frac{\sigma_t}{2}\frac{V_t(r^{\mathrm{p}}_{t+1})}{E_t r^{\mathrm{p}}_{t+1}}\right]\\ &= U_t' W_{t+1}\left[E_t r^{\mathrm{p}}_{t+1} - \frac{1}{2}\frac{E_t(r^{\mathrm{p}}_{t+1} - r^{\mathrm{f}}_t)}{E_t r^{\mathrm{p}}_{t+1}}\right]\\ &= U_t' W_{t+1}\left[r^{\mathrm{f}}_t + \rho_t\left(1 - \frac{1}{2(r^{\mathrm{f}}_t + \rho_t)}\right)\right],\end{aligned}$$

where ρ_t is the risk premium:

$$\begin{aligned}\rho_t &= \beta\sigma_t(1 + f_t)\,\mathrm{Cov}_t(\Delta\ln c_{t+1}, r^{\mathrm{p}}_{t+1})\\ &= \beta\sigma_t(1 + f_t)W_{t+1}V_t(r^{\mathrm{p}}_{t+1}).\end{aligned}$$

An increase in risk causes the following change in expected utility:

$$\frac{\partial\{\max E_t U(c_{t+1})\}}{\partial\rho_t} = U_t' W_{t+1}\left(1 - \frac{r^{\mathrm{f}}_t}{2(r^{\mathrm{f}}_t + \rho_t)^2}\right).$$

Although the sign is ambiguous, it is likely to be negative if r^{f}_t and ρ_t are not large. Usually, therefore, an increase in risk may be expected to reduce utility.

10.6.1 The Capital Asset-Pricing Model (CAPM)

The CAPM, due to Sharpe (1964), Lintner (1965), and Mossin (1966), is a special case of equation (10.51) that assumes that every market investor is identical and will therefore hold identical portfolios. As a result, r^{p}_{t+1} will also be the *market* return r^{m}_{t+1}. Thus

$$E_t(\boldsymbol{r}_{t+1} - \boldsymbol{\ell} r^{\mathrm{f}}_t) = E_t(r^{\mathrm{m}}_{t+1} - r^{\mathrm{f}}_t)\frac{\mathrm{Cov}_t(\boldsymbol{r}_{t+1}, r^{\mathrm{m}}_{t+1})}{V_t(r^{\mathrm{m}}_{t+1})}.$$

For the ith risky asset we obtain

$$\begin{aligned}E_t(r_{i,t+1} - r^{\mathrm{f}}_t) &= E_t(r^{\mathrm{m}}_{t+1} - r^{\mathrm{f}}_t)\frac{\mathrm{Cov}_t(r_{i,t+1}, r^{\mathrm{m}}_{t+1})}{V_t(r^{\mathrm{m}}_{t+1})}\\ &= E_t(r^{\mathrm{m}}_{t+1} - r^{\mathrm{f}}_t)\beta_{it},\end{aligned}\tag{10.56}$$

where β_{it} is the *market beta* for asset i and is defined by

$$\beta_{it} = \frac{\mathrm{Cov}_t(r_{i,t+1}, r^{\mathrm{m}}_{t+1})}{V_t(r^{\mathrm{m}}_{t+1})}.\tag{10.57}$$

Equation (10.56) therefore gives another expression for the risk premium. It says that the expected excess return on a risky asset is proportional to the expected excess return on the market portfolio. The proportionality coefficient beta varies over time and across assets. The beta for the risk-free asset is zero and the beta for the market portfolio is unity. An implication of CAPM is that an investor can hold one unit of asset i, or β_{it} units of the market portfolio. The expected excess return is the same in each case.

Our results also imply that CAPM can be given a general equilibrium interpretation if we define β_{it} as in equation (10.57) and equate the portfolio return with the market return ($r^{\mathrm{p}}_{t+1} = r^{\mathrm{m}}_{t+1}$) such that the market return satisfies equation (10.54).

10.7 Consumption under Uncertainty

We now examine the implications of the presence of risky assets for the consumption/savings decision. Previously we assumed perfect foresight and no uncertainty so that the return on the financial asset in which savings were held was given. We now assume uncertainty about the future and that savings are held in a risky asset.

In analyzing consumption when the asset is risky we recall our earlier remark that the same model may be used for determining consumption as is used for determination of the price of the risky asset. Hence we use equation (10.35). Solving this equation for consumption gives

$$E_t\frac{\Delta c_{t+1}}{c_t} = \frac{E_t r_{t+1} - \theta}{\sigma_t(1 + E_t r_{t+1})} - \frac{\mathrm{Cov}_t((\Delta c_{t+1}/c_t), r_{t+1})}{1 + E_t r_{t+1}} \tag{10.58}$$

$$\simeq \frac{[E_t r_{t+1} - \sigma_t \,\mathrm{Cov}_t((\Delta c_{t+1}/c_t), r_{t+1})] - \theta}{\sigma_t}. \tag{10.59}$$

Compared with the case of perfect foresight, the optimal rate of growth of consumption under uncertainty involves an extra term in $\mathrm{cov}_t((\Delta c_{t+1}/c_t), r_{t+1})$. As previously noted, this term is expected to be positive.

If households hold their savings in a risk-free asset with a certain return r^{f}_t, then, as $\mathrm{cov}_t((\Delta c_{t+1}/c_t), r^{\mathrm{f}}_t) = 0$, from equation (10.58) we obtain

$$E_t\frac{\Delta c_{t+1}}{c_t} \simeq \frac{r^{\mathrm{f}}_t - \theta}{\sigma_t}. \tag{10.60}$$

Thus perfect foresight and investing in a risk-free asset produce exactly the same result.

If we assume that $r^{\mathrm{f}}_t \simeq \theta$, then equation (10.35) may be written as

$$E_t r_{t+1} - r^{\mathrm{f}}_t = \sigma_t \,\mathrm{cov}_t\left(\frac{\Delta c_{t+1}}{c_t}, r_{t+1}\right). \tag{10.61}$$

The last term is the risk premium for the risky asset. It then follows that equations (10.59) and (10.60) are the same. This implies that in determining consumption it does not matter whether we assume that investors hold the risk-free asset or the risky asset as we have to risk-adjust the risky return in the equation for consumption. This is an important result as it suggests that we may continue to work with the simpler assumption of perfect foresight in our macroeconomic analysis. An alternative would be to evaluate all expectations in the DGE model using risk-neutral probabilities. This would also eliminate the need to take risk into account.

We note, however, that if we assume log-normality, then equation (10.58) is replaced by

$$E_t \Delta \ln c_{t+1} = \frac{r_t^{\mathrm{f}} - \theta}{\sigma_t} + \tfrac{1}{2}\sigma_t V_t(\Delta \ln c_{t+1}).$$

Thus the expected rate of growth of consumption along the optimal path is positively related to the difference between the risk-free rate and the consumer's subjective rate of time preference, and it varies positively with the variance of consumption growth (since σ_t is equal to the CRRA). If consumers are risk averse, higher variability of consumption growth is accompanied by higher expected consumption growth along the optimal path.

10.8 Complete Markets

The concept of a complete market is very important in finance as it determines whether arbitrage opportunities exist, and in macroeconomics it determines whether risk sharing is possible, i.e., whether it is possible to fully insure against risk.

If there is a contingent claim for each possible state of nature, and if there are at least as many assets as states, then the price of each asset is uniquely defined. If not, then there would be arbitrage possibilities. Unfortunately, in practice, there are almost certainly more states of nature than contingent claims.

Let $p(i)$ denote the price of the ith asset and let $x_i(s)$ denote the payoff in state s. It follows that

$$p(i) = \sum_s q(s)x_i(s), \quad i = 1, \dots, n.$$

Combining these equations for all n assets gives the matrix equation

$$\begin{bmatrix} p(1) \\ \vdots \\ p(n) \end{bmatrix} = \begin{bmatrix} x_1(1) & \cdots & x_1(S) \\ \vdots & \ddots & \vdots \\ x_n(1) & \cdots & x_n(S) \end{bmatrix} \begin{bmatrix} q(1) \\ \vdots \\ q(S) \end{bmatrix}.$$

In vector notation with $\tilde{p} = (p(1), p(2), \dots, p(n))'$,

$$\tilde{p} = X\tilde{q}.$$

There are three cases to consider.

1. The number of assets equals the number of states, i.e., $n = S$. Thus X is a square matrix and can be inverted to give the contingent-claims prices $q(s)$ for each possible state of nature $s = 1, \dots, S$:

$$\tilde{q} = X^{-1}\tilde{p}.$$

Given information on the market prices $p(s)$ and payoffs $x(s)$, we can then infer the state prices $q(s)$. In this case markets are said to be complete as market prices contain the complete information needed to obtain the state prices uniquely. Another important implication is that the stochastic discount factors $m(s)$ are uniquely defined and are the same for all investors.

2. The number of assets is greater than the number of states, i.e., $n > S$, and so X is an $n \times S$ matrix. Therefore a unique inverse of X no longer exists, only a generalized inverse. (Note: if X^* is an inverse of X, then $X^*X = I$. If there is a unique inverse, then $X^* = X^{-1}$; this requires X to be a square matrix. But if X is $n \times S$ and $n > S$, then there is more than one X^* matrix that satisfies $X^*X = I$.) It follows that $\tilde{q}$ cannot be obtained uniquely from $\tilde{p}$. In fact there are an infinite number of ways of deriving $\tilde{q}$, and hence an infinite number of pricing functions or stochastic discount factors, $m(s)$.
3. The number of states is greater than the number of contingent claims, i.e., $S > n$ and so X is an $n \times S$ matrix. It follows that now no inverse of X exists. It is not therefore possible to derive $\tilde{q}$ from $\tilde{p}$. This is the case that is considered when pricing derivative securities in terms of the prices of some underlying security.

10.8.1 Risk Sharing and Complete Markets

In practice, investors are heterogeneous. Each investor is subject to different sources of risk, called idiosyncratic risk. Investors may wish to diversify away this risk. In actual markets, however, insurance opportunities are typically imperfect. While there are numerous financial instruments that allow consumers to insure against various types of idiosyncratic shocks, such instruments typically do not allow for the complete diversification of idiosyncratic risk. In contrast, in a complete-markets equilibrium, consumers would be able to purchase contingent claims for each realization of such idiosyncratic shocks and would therefore be able to diversify away all idiosyncratic risk.

We have seen that an implication of the existence of a complete set of contingent claims is that consumers will value future random payoffs using the same pricing function, i.e., they will have the same stochastic discount factors. Suppose that consumer i invests in an asset that has a random payoff $1 + r_{t+1}$ at date $t + 1$. The Euler equation for the ith investor is

$$E_t\left[\frac{\beta_i U'(c^i_{t+1})}{U'(c^i_t)}(1 + r_{t+1})\right] = 1,$$

where β_i is the rate of time preference and c^i_t the consumption of the ith investor. This can be written as

$$E_t[m_{i,t+1}(1 + r_{t+1})] = 1,$$

where $m_{i,t+1} = (\beta_i U'(c^i_{t+1})/U'(c^i_t))$.

In a complete-markets equilibrium, the intertemporal marginal rate of substitution that is used to value future random payoffs will be the same for all consumers, i.e., $m_i = m_j$ for all i, j. Hence,

$$\frac{\beta_i U'_{i,t+1}}{U'_{i,t}} = \frac{\beta_j U'_{j,t+1}}{U'_{j,t}}.$$

If all investors have the same rate of time preference β and the same utility function $U(\cdot)$, then the growth rate of consumption for all consumers will be the same:

$$\frac{c_{i,t+1}}{c_{i,t}} = \frac{c_{j,t+1}}{c_{j,t}} \quad \text{for all } i, j.$$

This implies that in a complete-markets equilibrium only aggregate consumption shocks affect asset prices, and an individual income shock can be insured away through asset markets.

To illustrate this, suppose that there are two consumers in an economy that lasts for one period, and that each consumer $i = \text{A}, \text{B}$ has the same utility function $U(C) = \ln(C)$ but different income streams. In particular, consumer A is employed when consumer B is unemployed, and vice versa. This gives two (idiosyncratic) states. In state 1 the incomes are $y^{\text{A}} = y$ and $y^{\text{B}} = 0$ with probability π and in state 2 they are $y^{\text{A}} = 0$ and $y^{\text{B}} = y$ with probability $1 - \pi$, where $y > 0$ is a constant. Each consumer maximizes $E[\ln(C^i)]$, the expected value of utility from consumption. The problem is to find the consumption allocations for each consumer in a complete contingent-claims equilibrium.

As there are two possible states of the world, we have two state prices $q(1)$ and $q(2)$. Consumption and income for each consumer are indexed by the state of the world. Thus, the budget constraints for consumers A and B are given by

$$q(1)C^{\text{A}}(1) + q(2)C^{\text{A}}(2) = q(1)y,$$
$$q(1)C^{\text{B}}(1) + q(2)C^{\text{B}}(2) = q(2)y.$$

Each consumer maximizes utility subject to these two budget constraints. The Lagrangian for consumer A is given by

$$\mathcal{L}^{\text{A}} = \pi \ln[C^{\text{A}}(1)] + (1 - \pi) \ln[C^{\text{A}}(2)] + \lambda^{\text{A}}[q(1)y - q(1)C^{\text{A}}(1) - q(2)C^{\text{A}}(2)].$$

The first-order conditions are

$$\frac{\partial \mathcal{L}^{\text{A}}}{\partial C^{\text{A}}(1)} = \frac{\pi}{C^{\text{A}}(1)} - \lambda^{\text{A}} q(1) = 0,$$
$$\frac{\partial \mathcal{L}^{\text{A}}}{\partial C^{\text{A}}(2)} = \frac{1 - \pi}{C^{\text{A}}(2)} - \lambda^{\text{A}} q(2) = 0.$$

Eliminating the Lagrange multipliers from these expressions gives the condition

$$\frac{C^{\text{A}}(2)}{C^{\text{A}}(1)} = \frac{(1 - \pi)q(1)}{\pi q(2)}.$$

Consumer B's problem is similar to that of consumer A: the Lagrangian has the same form, but the budget constraint is different. It follows that

$$\frac{C^{\mathrm{B}}(2)}{C^{\mathrm{B}}(1)} = \frac{(1-\pi)q(1)}{\pi q(2)}.$$

Hence

$$\frac{C^{\mathrm{A}}(2)}{C^{\mathrm{A}}(1)} = \frac{C^{\mathrm{B}}(2)}{C^{\mathrm{B}}(1)} = \bar{c}.$$

Suppose now that y, the income received by each consumer, varies with the aggregate state of the economy. When the economy is in a boom, income is high, and is $\bar{y}$ with probability ϕ. When the economy is in a recession, income is low—perhaps due to unemployment—and is $\underline{y}$ with probability $1-\phi$, where $\underline{y} < \bar{y}$. Thus there are now four states of the world:

state 1: $y^{\mathrm{A}} = \bar{y}$ and $y^{\mathrm{B}} = 0$ with probability $\pi\phi$;
state 2: $y^{\mathrm{A}} = 0$ and $y^{\mathrm{B}} = \bar{y}$ with probability $(1-\pi)\phi$;
state 3: $y^{\mathrm{A}} = \underline{y}$ and $y^{\mathrm{B}} = 0$ with probability $\pi(1-\phi)$;
state 4: $y^{\mathrm{A}} = 0$ and $y^{\mathrm{B}} = \underline{y}$ with probability $(1-\pi)(1-\phi)$.

The solution could be obtained from the first-order conditions as before. A simpler way is to note that within a given aggregate state, consumers will equate their marginal rates of substitution for consumption across the idiosyncratic states. However, their marginal rates of substitution for consumption across the idiosyncratic states will vary with the aggregate state. Thus, the earlier conditions now become

$$\text{boom state:} \quad \frac{C^{\mathrm{A}}(2)}{C^{\mathrm{A}}(1)} = \frac{C^{\mathrm{B}}(2)}{C^{\mathrm{B}}(1)} = \bar{c}_1,$$

$$\text{recession state:} \quad \frac{C^{\mathrm{A}}(4)}{C^{\mathrm{A}}(3)} = \frac{C^{\mathrm{B}}(4)}{C^{\mathrm{B}}(3)} = \bar{c}_2,$$

where $\bar{c}_1$ and $\bar{c}_2$ differ because there are now different amounts of aggregate resources in the economy depending on whether the economy is in a boom or a recession.

Consider the possibility of insuring against aggregate versus idiosyncratic income shocks. In the absence of aggregate shocks, the ratios of consumption across the employment/unemployment states are equated for both consumers. This is equivalent to insurance against idiosyncratic shocks. However, insurance against variations in the aggregate economy is not possible. Hence, the ratios of consumption across the employment/unemployment states vary with the aggregate state of the economy.

Finally, consider the implications of a complete-markets equilibrium for asset pricing. Suppose that consumer $i = \mathrm{A}, \mathrm{B}$ invests in an asset that has a random payoff $1 + r_{t+1}$ at date $t+1$. The consumption/saving problem implies that at the optimum, the ith investor sets

$$E_t\left[\frac{\beta_i U'(c_{t+1}^i)}{U'(c_t^i)}(1 + r_{t+1})\right] = 1.$$

But we showed that in a complete-markets equilibrium, the intertemporal marginal rate of substitution that is used to value future random payoffs will be the same for all consumers, i.e., $m^{\mathrm{A}} = m^{\mathrm{B}}$. Hence,

$$\frac{\beta_{\mathrm{A}} U'_{\mathrm{A},t+1}}{U'_{\mathrm{A},t}} = \frac{\beta_{\mathrm{B}} U'_{\mathrm{B},t+1}}{U'_{\mathrm{B},t}}.$$

If all consumers have the same discount factor β and the same utility function $U(\cdot)$, then their rates of consumption growth will be identical:

$$\frac{c_{\mathrm{A},t+1}}{c_{\mathrm{A},t}} = \frac{c_{\mathrm{A},t+1}}{c_{\mathrm{B},t}}.$$

We have shown, therefore, that in a complete-markets equilibrium only aggregate consumption shocks affect asset prices and an individual income shock can be insured away through asset markets. Although in practice we do not have complete markets, this is an important concept in finance and in macroeconomics.

10.8.2 Market Incompleteness

Even if markets are not complete and individuals have different marginal rates of substitution, if there is a risk-free asset to which all consumers have access, then, from equation (10.14), the expected marginal rates of substitution for each investor will be the same and, from equation (10.60), the rates of growth of consumption for all consumers will be the same (see Heaton and Lucas 1995, 1996).

The concept of market completeness is particularly relevant in an open economy as it implies that an economy can insure away its idiosyncratic risk by holding a portfolio of internationally traded assets. In this case the marginal rates of substitution are the same for all countries and all countries have the same rates of growth of consumption. If, however, international markets are incomplete, as countries have different marginal rates of substitution, then, provided each country has access to an internationally traded real risk-free asset at the same rate, expected marginal rates of substitution in each country will be the same, as will consumption growth rates. The problem here is that for the real risk-free rate to be the same in each country, PPP is required, and we have already seen that this does not hold. We return to this point in our discussion of foreign exchange markets.

10.9 Conclusions

Increasingly, asset pricing has become associated exclusively with finance. We have shown that it is, in fact, an important branch of economics, and plays a central role in general equilibrium macroeconomics. In particular, we have demonstrated that the same DGE model used to determine macroeconomic variables also provides a general equilibrium theory of asset pricing. We have therefore unified macroeconomics and finance.

Assets are priced as the discounted value of future payoffs. The key difference between finance and economics is in the choice of discount factor. In traditional finance the risk free rate is often preferred; in general equilibrium economics the marginal rate of substitution M_{t+1} is used. We have shown that the connection between the two is that $E_t M_{t+1} = 1/(1 + r_t^f)$. In finance the risk-free rate is sometimes supplemented with other variables, which are referred to as factors. The problem is how to choose these factors. In economics M_{t+1} is stochastic and is based on the variables that determine marginal utility: typically consumption growth and, for nominal returns, inflation. It will be shown in chapter 11 that, depending on the choice of utility function, other variables may also be used as factors.

We have shown that where each household has the same discount factor, markets are complete, implying that it is then possible to insure against risks and that optimal consumption growth is the same for all households. This can be extended to the open economy when we require that each country has the same discount factor.

We have also shown that returns must satisfy a no-arbitrage condition—otherwise markets would not be efficient and would allow unlimited profit-making opportunities. Given the different characteristics of risky assets, no-arbitrage is brought about by adjusting returns for risk, with the result that, after being risk-adjusted, the expected values of all returns are the same. The problem then is how to determine the risk premium for each asset. We have shown that the answer to this problem is linked to the choice of the discount factor. This presents a problem for traditional finance, as discounting using the risk-free rate does not produce a risk premium; this is why additional factors are sought.

The no-arbitrage condition provides a restriction often ignored in financial econometrics, where univariate time-series methods are commonly used. Testing this restriction enables asset-pricing theories to be evaluated. As it is necessary to jointly model risky returns, the risk-free rate, and the stochastic discount in order to take account of the no-arbitrage condition and model the risk premium, multivariate methods are required. We discuss such tests for particular financial markets in chapter 11.

Although it is necessary to take account of risk when determining asset prices, we have argued that it may not be necessary to include risk premia in stochastic macroeconomic relations. If a real risk-free rate exists, this may be used instead of risky returns because, when they are used in macroeconomic relations, risky returns should be risk-adjusted. An alternative is to evaluate expectations using risk-neutral valuation. This also results in the use of the risk-free rate.