Chapter One

A Model of Rayleigh-Bénard Convection

There are two basic types of fluid flows within planets and stars that are driven by thermally produced buoyancy forces: thermal convection and internal gravity waves. The type depends on the thermal stratification within the fluid region. The Earth's atmosphere and ocean, for example, are in most places *convectively stable*, which means that they support internal gravity waves, not (usually) convection (but see Chapter 7). On warm afternoons, however, the sun can heat the ground surface, which changes the vertical temperature gradient in the troposphere and makes the atmosphere *convectively unstable*; the appearance of cumulus clouds is an indication of the resulting convective heat (and moisture) flux. Thermal convection likely also occurs in the Earth's liquid outer core, which generates the geomagnetic field, and, on a much longer time scale, in the Earth's mantle, which drives plate tectonics and, on a much shorter time scale, initiates earthquakes and volcanic eruptions. Thermal convection is seen on the surface of the sun and likely occurs in the outer 30% of the solar radius, where solar magnetic field is generated. Below this depth buoyancy likely drives internal gravity waves. Rotation strongly influences the style of the convection and waves in all of these examples except the mantle, which is dominated by viscous forces.

Computer simulation studies, over the past few decades, have significantly improved our understanding of these phenomena. Some studies, like those for the atmospheres of the Earth and sun, have provided physical explanations and predictions of the observations. Others, like those for the deep interiors of the Earth and sun, have provided detailed theories and predictions of the dynamics that cannot be directly observed. As computers continue to improve in speed and memory, computer programs are able to run at greater spatial and temporal resolutions, which improves the quality of and confidence in the simulations. Numerical and programming methods have also improved and need to continue to improve to take full advantage of the improvements in computer hardware.

1.1 BASIC THEORY

We begin with a simple description of the fundamental dynamics expected in a fluid that is convectively stable and in one that is convectively unstable. Then we review the equations that govern fluid dynamics based on conservation of mass, momentum, and energy.

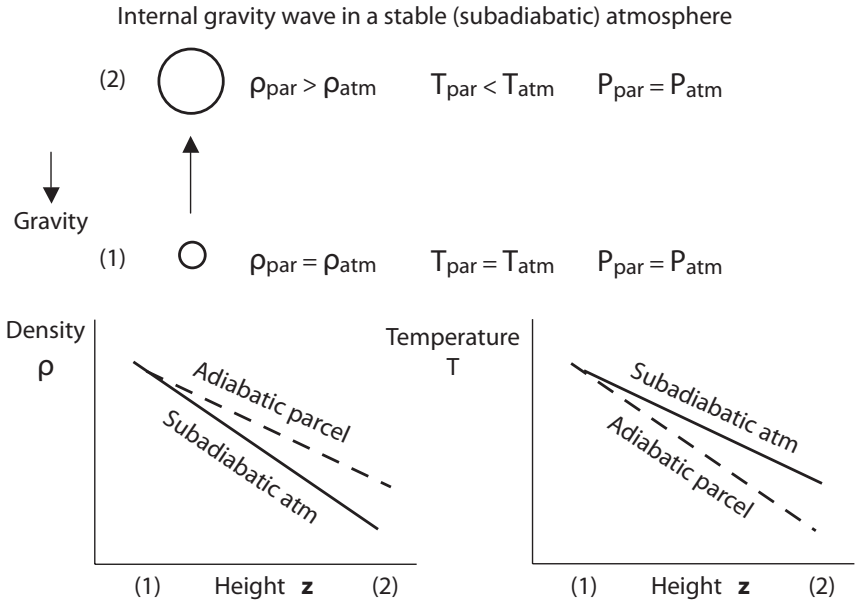

Figure 1.1 A schematic of a test parcel that is raised adiabatically from position (1) to position (2) in an atmosphere that has *subadiabatic* temperature and density stratifications.

1.1.1 Thermal Convection and Internal Gravity Waves

The thermal stability of a fluid within a gravitational field is determined by its horizontal-mean (i.e., ambient) vertical temperature gradient. The classic way of describing this is to consider a fluid in *hydrostatic equilibrium*, i.e., the weight of the fluid above a given height (per cross-sectional area) is supported by the pressure at that height. Therefore, the vertical pressure gradient is negative. (As usual, "vertical" here and throughout this book refers to the direction of increasing height or radius, opposite to that of the gravitational acceleration.) In the interiors of planets and stars the horizontal-mean density and temperature also decrease with height. The question is how does the vertical temperature gradient of this fluid (atmosphere) compare with what an adiabatic temperature gradient would be.

Consider a small (test) parcel of fluid (Fig. 1.1) that, at its initial position (1), has the same pressure, density, and temperature as the surrounding atmosphere at that position. Imagine raising the parcel to a new height (2), fast enough so there is no heat transfer between it and the surrounding atmosphere but slowly enough that it remains in pressure equilibrium with its surroundings; that is, its upward velocity is much less than the local sound speed. Assuming this process is reversible and also adiabatic since there is no heat transfer, the parcel's entropy remains constant while rising; that is, this is an isentropic process. However, since it remains in pressure equilibrium with the surroundings, its density and temperature both decrease as it

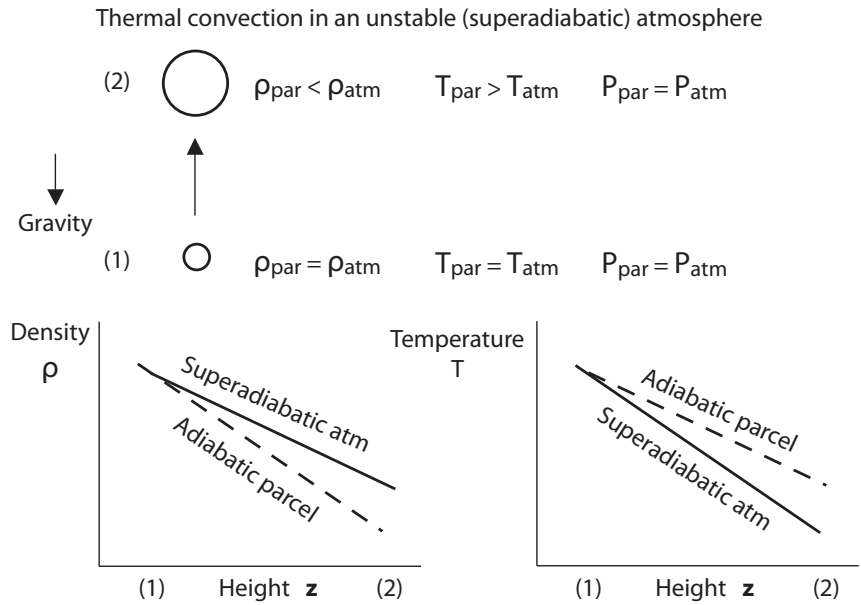

Figure 1.2 A schematic of a test parcel that is raised adiabatically from position (1) to position (2) in an atmosphere that has *superadiabatic* temperature and density stratifications.

rises because the decrease in pressure causes it to expand. If, when reaching its new higher position (2), its temperature has decreased more than the temperature of the surrounding atmosphere has decreased over that change in height, its density there will be greater than the density of the surrounding atmosphere there (assuming a typical coefficient of thermal expansion). Therefore, the parcel will be antibuoyant and, when no longer externally supported, will fall. As the parcel falls its temperature increases faster than the surrounding temperature and when it passes the initial position its temperature exceeds the temperature of surrounding atmosphere, causing the now buoyant parcel to eventually stop falling and then to start rising. This process of accelerating downward when it is above the initial position and accelerating upward when below the initial position is called an *internal gravity wave* and the surrounding atmosphere is said to be *convectively stable*. Recall that this occurs when the surrounding temperature decreases less rapidly with height than an adiabatic temperature profile would, since the test parcel moves adiabatically. That is, the surrounding temperature gradient is *subadiabatic*. The temperature stratification would be extremely stable if the surrounding temperature increased with height. In reality, thermal and viscous diffusion cause internal gravity waves to decay with time unless they are continually being excited.

Now consider the case for which the changes in the parcel's temperature as it moves up and down are less than that of the surrounding atmosphere (Fig. 1.2). That is, consider a surrounding atmosphere with a *superadiabatic* temperature gradient.

In this case, when reaching its new higher position (2), the parcel's temperature will be higher than the temperature of the surroundings there and so its density will be less than that of the surroundings. This makes the parcel buoyant and so, when no longer externally supported, it continues to rise. Likewise, if the parcel were initially lowered, it would continue to sink. Typically, a rising parcel will eventually encounter a cold impermeable top boundary where it gives up heat by conduction, contracts, and becomes antibuoyant. This causes it to fall until it encounters a hot impermeable bottom boundary where it absorbs heat by conduction, expands, becomes buoyant, and rises. This process is called *thermal convection* and in this case the surrounding atmosphere is said to be *convectively unstable*. In reality, thermal and viscous diffusion would cause this process to decay with time unless the excess temperature drop maintained across the region (compared to the adiabatic temperature drop) is larger than a critical value (Section 3.3). That is, if thermal diffusion (i.e., conduction) is not efficient enough at transferring heat upward through the atmosphere, thermal convection will occur, which will transfer heat as *convective heat flux*.

Thermal convection exists in planets and stars and takes on a variety of forms. Thermal convection likely occurs throughout the mantles of terrestrial planets, like the Earth, and also throughout most if not all of their liquid cores. However, because of their very different viscosities, the time scale for mantle convection is typically a hundred million times longer than that for core convection and, whereas the style of mantle convection is unaffected by planetary rotation (Coriolis forces), core convection is dominated by rotation. The interactions between the Earth's mantle and its fluid core are discussed in Buffett (2007). Where, within the interior of stars and giant planets, thermal convection occurs depends on the interior structure. Most one-dimensional (1D) evolutionary models of giant gas planets predict convection throughout their liquid/gas interiors. Thermal conduction in stars is by radiative transfer, which is less efficient at heat transfer in regions where atomic excitation and ionization occur, i.e., where the opacity is large and the adiabatic gradient is less steep, respectively. Convection typically occurs in these regions because the temperature gradient would need to be steeper than the adiabatic gradient to conduct all of the heat upward (i.e., outward). For example, the sun has a convection zone in roughly the outer 30% of its radius; radiative transfer is sufficient to carry the heat flux within the inner 70% of its radius, where the gas is fully ionized. Lower mass stars, which are cooler than the sun and therefore have a larger fraction of un-ionized gas, have much deeper convection zones; stars with less than 30% of a solar mass are fully convective. Stars more massive (and so hotter) than the sun have very shallow outer convection zones, if any, and much larger stars have convection within their central cores where the strong temperature dependence of the nuclear energy generation rate maintains a sufficiently steep temperature gradient.

Computer modeling has made and will continue to make significant contributions to our understanding of the interior dynamics of planets and stars. Our study of computer modeling begins with simple models in this Part 1. In Chapters 1–5 we focus on modeling thermal convection. In Chapter 6 we discuss the relatively simple changes to the model needed to simulate internal gravity waves. A combination

of convection and gravity waves occurs in double-diffusive convection, which we describe in Chapter 7.

1.1.2 Equations of Motion

The fluid dynamics and thermal dynamics of these processes are governed by the classical conservation laws for mass, momentum, and energy. However, since we are considering a continuous fluid, these laws are written in terms of the densities of mass, momentum, energy, and force.

The mass conservation equation,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{v}\,, \tag{1.1a}$$

says that the local (Eulerian) time rate of change of mass density (ρ) is determined by the convergence (i.e., negative divergence) of mass flux ($\rho\mathbf{v}$) at that location and time (t); $\mathbf{v}$ is the fluid velocity. Note that the Lagrangian time derivative of density ($d\rho/dt$), which is the rate at which the density of a fluid parcel changes as it moves with the flow, is the sum of the Eulerian time derivative ($\partial\rho/\partial t$) and the advection of density $(\mathbf{v}\cdot\nabla)\rho$. (The Lagrangian derivative is also called the material derivative.) Therefore, Eq. 1.1a can also be written as

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \mathbf{v}\,. \tag{1.1b}$$

Newton's Second Law of motion applied to a fluid describes momentum conservation: mass density times acceleration equals the net force density on a fluid parcel as it moves. This equation,

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \mathbf{g}\rho\,, \tag{1.2}$$

is called the *Navier-Stokes equation* after Claude-Louis Navier and George Gabriel Stokes.

The first two terms on the right side are the macroscopic representation of the effects due to molecules (or atoms). The first is the negative pressure gradient, a force density from high to low pressure, p, which is due to static normal stress. The second is the divergence of the viscous stress tensor, $\boldsymbol{\sigma}$ or "σ_{ij}", to indicate that it is a tensor. Unless noted, we assume a *Newtonian fluid*; that is, viscous stress is proportional to the rate of strain of the fluid:

$$\sigma_{ij} = 2\rho\nu \left(e_{ij} - 1/3\, e_{kk}\, \delta_{i,j}\right) \tag{1.3}$$

where

$$e_{ij} \equiv 1/2 \left(\partial v_i/\partial x_j + \partial v_j/\partial x_i\right) \tag{1.4}$$

is the rate of strain tensor (for $i = 1, 2, 3$ or x, y, z in cartesian coordinates), ν is the viscous diffusivity (also called the kinematic shear viscosity), and the kronecker delta function $\delta_{i,j}$ is one for $i = j$ and zero if not. Note, $e_{kk} = \nabla \cdot \mathbf{v}$. As is usually done for subsonic convection problems, we have neglected the small contribution to the viscous stress due to bulk viscosity: $\lambda e_{kk}\delta_{ij}$, where λ is called the of kinematic bulk viscosity. If the dynamic shear viscosity $\rho\nu$ were constant in space, the

divergence of the viscous stress tensor (when neglecting the bulk viscosity) would reduce to

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = \rho\nu\left(\nabla^2\mathbf{v} + 1/3\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v})\right) . \tag{1.5}$$

The last term on the right of Eq. 1.2 is the gravitational force density, $\mathbf{g}$ being the gravitational acceleration.

By doing a Taylor expansion about both position and time, the left side of Eq. 1.2 can be written in the Eulerian form as $\rho(\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v})$, which by using Eq. 1.1 also equals $\partial\rho\mathbf{v}/\partial t + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}\mathbf{v})$. Above we called $\rho\mathbf{v}$ mass flux; here we call it momentum density. The Reynolds stress tensor, $\rho\mathbf{v}\mathbf{v}$, is the momentum flux due to the flow. That is, it states how each of the three components of momentum is being transported in each of the three directions. For example, $\rho v_x v_z$ is the rate that the x-component of momentum is being transported in the z-direction, which is also the rate that the z-component of momentum is being transported in the x-direction. The divergence of this tensor is a vector equal to the net rate that each of the three components of momentum is diverging at the given position and time.

A few more words may be appropriate about the Eulerian and Lagrangian time derivatives. In an Eulerian representation we ask how the properties of the fluid are changing in time on, for example, a set of grid points in space, without keeping track of where the current fluid parcels at these locations originated. In a Lagrangian representation, on the other hand, there are no set grid points in space. Instead, we ask how the properties *and* the coordinate locations of a given set of fluid parcels change with time. The Eulerian approach is preferred for a continuous fluid that fills a defined volume. The Lagrangian approach is preferred for a discontinuous set of particles interacting within an otherwise empty volume of space. We are adopting the Eulerian approach.

The first law of thermodynamics describes internal energy conservation: the rate of change of the internal energy of a fluid parcel plus the rate the fluid parcel does work equals the rate it absorbs heat. Note that "work" and "heat transfer" are *process* functions, not properties of the fluid. However, internal energy conservation can also be described in terms of state functions. The rate the fluid does work per mass is pressure times the rate of change of the volume per mass (i.e., specific volume, which is $1/\rho$); therefore, using Eq. 1.1b, the rate fluid does work per volume is $p\boldsymbol{\nabla}\cdot\mathbf{v}$. The rate the fluid absorbs heat per volume can also be written in terms of state variables as $\rho T dS/dt$, where S is specific entropy (i.e., entropy per mass). Therefore, conservation of internal energy density is

$$\rho\frac{de}{dt} + p\boldsymbol{\nabla}\cdot\mathbf{v} = \rho T\frac{dS}{dt} = \boldsymbol{\nabla}\cdot(k\boldsymbol{\nabla}T) + Q\,, \tag{1.6}$$

where e is internal energy per mass (i.e., specific internal energy). The first heating term on the far right side of Eq. 1.6 is the convergence of diffusive heat flux, $-k\boldsymbol{\nabla}T$, where T is temperature, $k = c_p\rho\kappa$ is thermal conductivity, c_p is specific heat capacity at constant pressure, and κ is thermal diffusivity. The remaining term, Q, represents viscous and ohmic heating and any other heating or cooling, e.g., nuclear.

This relationship between process and state functions is valid within the very good approximation of local thermodynamic equilibrium (LTE). That is, since

"fluid dynamics" is a macroscopic description of the state and evolution of a fluid averaged over length and time scales large compared to the molecular structure and processes, state variables like temperature, pressure, density, specific internal energy, and specific entropy are defined as continuous functions of space and time that usually vary slowly enough on macroscopic length and time scales that thermodynamic equilibrium can be assumed in small neighborhoods around every location and time within the domain of study. For example, although temperature can vary in space (and therefore drive a diffusive heat flux), around any point within the fluid there is a small neighborhood in which the velocities of the particles have a well-defined Maxwellian distribution defined by the local temperature.

In addition to the internal energy density equation (Eq. 1.6), a useful equation is one that describes the rate of change of kinetic and gravitational potential energy densities. This is obtained by taking the dot product of fluid velocity and the momentum equation (Eq. 1.2), which gives

$$\rho\left(\frac{\partial}{\partial t}\left(\tfrac{1}{2}v^2\right)+(\mathbf{v}\cdot\nabla)\left(\tfrac{1}{2}v^2\right)\right)=-\mathbf{v}\cdot\nabla p+\mathbf{v}\cdot(\nabla\cdot\boldsymbol{\sigma})-\rho\mathbf{v}\cdot\nabla\Phi\,, \tag{1.7}$$

where we have written the gravitational acceleration as $\mathbf{g}=-\nabla\Phi$, Φ being the gravitational potential energy per mass. Assuming Φ is time independent and using Eq. 1.1, the gravitational work term in Eq. 1.7 can be written as

$$-\rho\mathbf{v}\cdot\nabla\Phi=-\frac{\partial}{\partial t}(\rho\Phi)-\nabla\cdot(\rho\Phi\mathbf{v})\,.$$

Equation 1.1 can also be used to write the Lagrangian time derivative on the left side of Eq. 1.7 as

$$\rho\left(\frac{\partial}{\partial t}\left(\tfrac{1}{2}v^2\right)+(\mathbf{v}\cdot\nabla)\left(\tfrac{1}{2}v^2\right)\right)=\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right)+\nabla\cdot(\tfrac{1}{2}\rho v^2\mathbf{v})\,.$$

It is also convenient to write the pressure work term in Eq. 1.7 as

$$-\mathbf{v}\cdot\nabla p=-\nabla\cdot(p\mathbf{v})+p\nabla\cdot\mathbf{v}\,.$$

Also it can be shown (e.g., Batchelor, 1967) that part of the work done by viscous forces goes into viscous heating and the remaining part into the convergence of viscous energy flux. That is,

$$\mathbf{v}\cdot(\nabla\cdot\boldsymbol{\sigma})=-2\rho\nu\left(e_{ij}e_{ij}-1/3(\nabla\cdot\mathbf{v})^2\right)+\nabla\cdot(\mathbf{v}\cdot\boldsymbol{\sigma})\,. \tag{1.8}$$

Therefore, substituting these expressions into Eq. 1.7 gives the mechanical energy density equation, i.e., the rate of change of the sum of the kinetic and gravitational potential energy densities:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2+\rho\Phi\right)=&-\nabla\cdot\left[(\tfrac{1}{2}\rho v^2+\rho\Phi+p)\mathbf{v}-\mathbf{v}\cdot\boldsymbol{\sigma}\right]\\&+p\nabla\cdot\mathbf{v}-2\rho\nu\left(e_{ij}e_{ij}-1/3(\nabla\cdot\mathbf{v})^2\right)\,.\end{aligned} \tag{1.9}$$

Combining this equation with the internal energy density equation (Eq. 1.6) and using Eq. 1.1 to write

$$\rho\frac{de}{dt}=\frac{\partial\rho e}{\partial t}+\nabla\cdot(\rho e\mathbf{v})\,,$$

and setting Q to just the viscous heating rate (i.e., the negative of the first term on the right of Eq. 1.8) gives the rate of change of the internal, kinetic, and potential energy densities:

$$\frac{\partial}{\partial t}\left(\rho e + \tfrac{1}{2}\rho v^2 + \rho\Phi\right) = -\boldsymbol{\nabla}\cdot\left[(\rho e + \tfrac{1}{2}\rho v^2 + \rho\Phi + p)\mathbf{v} - k\boldsymbol{\nabla}T - \mathbf{v}\cdot\boldsymbol{\sigma}\right]. \tag{1.10}$$

Note that the "PdV" work and the viscous heating, which occur in Eq. 1.6, cancel those in Eq. 1.9. Equation 1.10 says that the local rate of change of total energy density equals the convergence of the fluxes of enthalpy density ($\rho h = \rho e + p$), kinetic energy density ($\rho v^2/2$), and gravitational potential energy density ($\rho\Phi$) plus the thermal diffusive heat flux ($-k\boldsymbol{\nabla}T$) and the viscous energy flux ($-\mathbf{v}\cdot\boldsymbol{\sigma}$). For stress-free impermeable boundaries that do not move or deform, which we will usually assume, the volume integral of Eq. 1.10 over the domain of the fluid (i.e., the rate of change of the total energy in the fluid) reduces to minus the surface integral over the boundary of the fluid of the thermal heat flux out of the domain. Note, additional energy and energy flux terms would be included if magnetic fields exist within the fluid (Eq. 11.16); however, if the fluid were rotating, Coriolis forces (measured within a rotating frame of reference) would do no work because they are always perpendicular to the local flow.

In this Part 1 we make a very common and traditional approximation to these conservation equations, the *Boussinesq approximation*, which simplifies these equations to a form very similar to that of an incompressible fluid. In Part 3 we discuss a more realistic approximation for planets and certainly stars, the *anelastic approximation*, which accounts for the effects of density stratification. For a review of the fully compressible equations, briefly described here, and of their Boussinesq and anelastic approximations see, for example, Braginsky & Roberts (2007).

1.2 BOUSSINESQ EQUATIONS

Since density, temperature, and pressure increase with depth in the atmospheres and interiors of planets and stars, understanding the conditions under which the Boussinesq equations represent a valid approximation for these compressible fluids is critical. The Boussinesq approximation to the fluid flow equations (Boussinesq, 1903; Spiegel & Veronis, 1960) assumes that the vertical extent of the modeled domain is small relative to the hydrostatic scale heights of density, temperature, and pressure. A *density scale height*, for example, at a given location within the domain is $-(d\ln\rho/dr)^{-1}$ and is roughly the distance over which density decreases by a factor of e. This assumption is fairly valid for convection in an ocean, mantle, or liquid core of a terrestrial planet. It may also be valid near the center of a giant planet or star, where the local density scale height is much greater than the radius of the body. However, it is not valid in the outer part of the interior or in the atmosphere of a giant planet or star where the density scale height is much smaller.

The Boussinesq and anelastic (Chapter 12) approximations filter out sound waves, which typically represent even smaller perturbations in density and pressure. Sound waves usually travel much faster than the fluid flows in planetary and

stellar interiors and would therefore require much smaller computational time steps to resolve. This huge reduction in the number of computational time steps needed to simulate convection (or gravity waves) compared to what would be required for a compressible model is the main reason for employing the Boussinesq (or anelastic) approximation in numerical models. Effectively, the speed of sound is assumed to be infinite. That is, a computational time step in a Boussinesq (or anelastic) simulation is assumed to be long compared to the time it would take for changes in pressure to be communicated throughout the modeled domain. Therefore, another condition for the Boussinesq (and anelastic) approximation to be valid is that the fluid velocity be small relative to the local sound speed (Eq. 12.8); that is, the Mach number (the ratio of the fluid velocity to the local sound speed) needs to be less than, say, 0.1. If, on the other hand, the fluid velocity for a particular problem were comparable to the local sound speed, a fully compressible model would be needed.

The objective here is to describe how to develop a model and the corresponding code that can be run on a computer to simulate thermal convection in a nearly incompressible liquid within a uniform gravitational field, heated on the bottom boundary and cooled on the top boundary, i.e., *Rayleigh-Bénard convection*. Laboratory experiments of Rayleigh-Bénard convection were first done by Henri Bénard in 1900 and later the linear stability analysis (Section 3.4) was described by Rayleigh (1916).

In this book, "model" refers to the equations, numerical methods, and the assumptions and approximations upon which these are based; "code" refers to the computer program that translates the model into computer language; and "simulation" refers to the numerical results obtained when the code is run on a computer.

As usual, the independent variables, for this Eulerian representation, are the time, t, and the cartesian spatial coordinates, x, y, and z. The gravitational acceleration, $\mathbf{g} = -g_o\hat{z}$, is directed downward (i.e., g_o is positive and $\hat{\boldsymbol{z}}$ is the unit vector in the positive vertical direction).

For simplicity, we make the Boussinesq approximation; that is, when the change in hydrostatic density across the domain is small relative to the volume-averaged density, the background density (ρ_o) is taken to be constant in space and time. Therefore, to first order, the mass conservation equation, 1.1b, is simply

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,. \tag{1.11}$$

This does not imply that density is exactly constant in space and time, but only that the amplitude of its rate of change is small, i.e., of the order of the relative change in hydrostatic density across the domain, compared to the amplitudes of the three individual contributions to the divergence of velocity.

The local effects of pressure on density are also assumed to be small. That is, density perturbations (ρ) are assumed to be produced only by temperature perturbations (T) according to the equation of state:

$$\rho = -\rho_o\alpha T\ , \tag{1.12}$$

where α is the constant coefficient of thermal expansion.

These perturbations produce buoyancy forces that drive the convection according to the momentum equation 1.2, which to first order within the Boussinesq

approximation reduces to

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \rho_o^{-1}\nabla p + \alpha g_o T\hat{\mathbf{z}} + \nu\nabla^2\mathbf{v}\,. \tag{1.13}$$

This equation was obtained from Eq. 1.2 by first subtracting from it the momentum equation of the hydrostatic background state:

$$0 = -\frac{dp_o}{dz} - g_o\rho_o\,.$$

Density, in the remaining terms, is, to first order, the constant background density ρ_o.

The viscous force in Eq. 1.13 simplifies to a constant viscous diffusivity, ν, times the Laplacian of velocity. This can be thought of as the convergence of a molecular momentum flux that is modeled as being proportional to the negative gradient of fluid velocity. That is, viscous forces tend to smooth out gradients in velocity by transporting momentum from regions of high momentum to those of low momentum.

Updating density perturbations via the equation of state 1.12 instead of the mass conservation equation 1.1 is why sound waves are filtered out in a Boussinesq model. That is, the divergence of Eq. 1.13 removes the time derivative because of Eq. 1.11; therefore the resulting pressure Poisson equation determines the pressure everywhere within the domain at each time step given the updated velocity and temperature perturbations.

The rate of change of internal energy density, $\rho de/dt$, for a perfect gas (Section 12.1.1) is $\rho c_v dT/dt$, where c_v is specific heat capacity at constant volume. The internal energy density equation 1.6 within the Boussinesq approximation reduces to

$$\frac{\partial T}{\partial t} = -(\mathbf{v}\cdot\nabla)T + \kappa\nabla^2 T \tag{1.14}$$

when only retaining the heating due to the convergence of diffusive heat flux. (Viscous heating is usually neglected when using the Boussinesq approximation because it is argued that it should be small (Spiegel & Veronis, 1960); mantle convection may be an exception (Jarvis & McKenzie, 1980).) Note that the work term, $p\nabla\cdot\mathbf{v}$, in Eq. 1.6 has not been neglected here because $\nabla\cdot\mathbf{v}$ is the same order as the other terms in this equation. Spiegel & Veronis (1960) show that (for a perfect gas) the two terms on the left of Eq. 1.6 add, to first order, to simply $\rho c_p dT/dt$ (i.e., c_v has been changed to c_p) when T is taken to be the temperature perturbation (here including the depth-dependent horizontal mean) relative to an adiabatic temperature profile (Section 1.1.1). The adiabatic temperature gradient (Eq. 12.10) and the parameters c_v, c_p, and κ are all assumed to be constants. Equation 1.14 also works for a perfectly incompressible liquid, in which case $c_p = c_v$ and there is no "PdV" work since $\nabla\cdot\mathbf{v}$ vanishes.

A simple way to summarize the Boussinesq approximation is that it ignores variations in density except in the buoyancy force and in the equation of state.

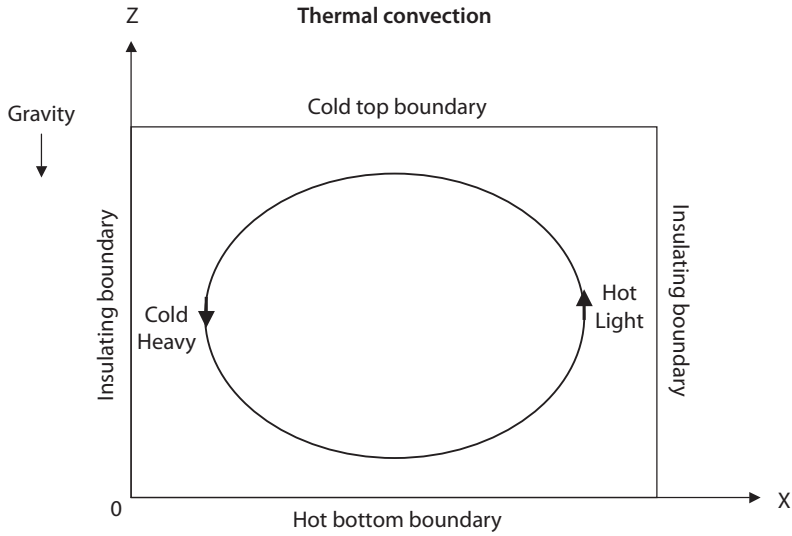

Figure 1.3 A schematic of a simple one-cell thermal convection pattern in the fluid box domain of unstably stratified fluid.

1.3 MODEL DESCRIPTION

Here in Part 1 we define the fluid domain as a rectangular region bounded by walls that are impermeable and stress-free (Fig. 1.3). One could think of this rectangular box as a small region within a global planet or star. Modifications made to the geometry and physics of this problem later in Parts 2 and 3 build upon this first scenario, improving the physical realism of the simulations.

The bottom and top boundaries of our fluid box are maintained at constant temperatures by external heaters and coolers; the bottom boundary temperature is higher than the top boundary temperature by an amount ΔT. Heat diffuses in through the bottom boundary and out through the top boundary. The side boundaries are set to be thermally insulating; that is, there is no heat flow through them.

To keep the problem more manageable, we allow fluid flow and gradients only in two directions, the horizontal (x-direction) and the vertical (z-direction). Therefore, $\partial/\partial y = 0$ and the y-component of the fluid velocity, v_y, vanishes. In this two-dimensional (2D) problem the linear terms have eight spatial derivatives on the fluid velocity and pressure, four in z and four in x. Therefore, eight boundary conditions are formally required to maintain a unique solution of the velocity and pressure. We apply all eight on the velocity, none on the pressure perturbation, because the amplitude and gradient of the pressure perturbation are both expected to vary with time and location on the impermeable boundaries.

Let's consider the velocity boundary conditions more carefully. Impermeable means fluid cannot pass through the boundaries. Therefore, the vertical component of the flow, v_z, has to vanish on the top ($z = D$) and bottom ($z = 0$) boundaries.

Likewise, the horizontal component of the flow, v_x, vanishes on the side boundaries ($x = 0$ and L). Stress-free means the rates of tangential strain vanish at the boundaries. That is, the fluid slips without resistance along the boundaries instead of the more physical condition that requires the fluid at the boundary not to move relative to the boundary. Our stress-free condition allows fluid at the boundary to flow parallel to the boundary but requires the gradient, normal to the boundary, of this parallel flow to vanish at the boundary. Therefore, at the top and bottom boundaries, $\partial v_x/\partial z$ vanishes and at the side boundaries $\partial v_z/\partial x$ vanishes.

The heat equation has four spatial derivatives on the temperature perturbation, two in z and two in x, and therefore four boundary conditions are required. The isothermal top and bottom boundaries are forced by setting the temperature perturbation to be zero at $z = D$ and at a prescribed value, ΔT, at the bottom boundary, $z = 0$. No heat flows through the insulating side boundaries. There is no advective heat flux through them since they are impermeable. The diffusive (conductive) heat flux through a side boundary is proportional to the horizontal gradient of the temperature there, which is set to zero at the side boundaries to prevent any diffusive heat flux through them. That is, the other two boundary conditions are that $\partial T/\partial x$ vanishes at $x = 0$ and $x = L$.

At this point readers need to decide if they wish to use dimensional or nondimensional variables. Nondimensional variables, which are dimensional variables scaled by values representative of the chosen problem, have traditionally been used so the equations and solutions can easily be characterized and readily be applied to many different physical problems. On the other hand, variables represented in centimeters, grams, seconds (CGS units) or meters, kilograms, seconds (MKS units) are easily monitored without first having to divide each variable by its chosen scaling parameter. In addition, using dimensional variables facilitates testing and debugging. For example, if a term in an equation is supposed to contain the radius of a planet squared and one mistakenly had the radius cubed, it would be relatively easy to spot if the radius were written in centimeters instead of scaled to something near unity. Also, the characteristic nondimensional numbers that would appear in the nondimensional equations can easily be constructed and presented along with the dimensional values of the simulation. What often happens, however, when people present just their nondimensional results for a study of the internal dynamics of a particular planet or star is that the degree of agreement (or disagreement) with observations is not obvious to the readers (or audience).

In this Part 1 of the book we use nondimensional variables and equations (for the most part) because the problems described are relatively simple and not meant to be realistic simulations of any particular type of planet or star. Later in the book we use dimensional variables and equations, for writing codes and presenting results, when simulating a particular problem with more realistic geometry and physics.

As a side note, it is interesting that researchers in some communities, like those in the Earth sciences, have traditionally used MKS units; whereas those in other communities, like astronomy and astrophysics (which deal with much greater masses and lengths), have traditionally used CGS units (which are smaller mass and length units). Planetary science seems to be a mixed bag. Those studying planetary atmospheres and surfaces tend to use MKS units, probably because they started

in the Earth sciences; whereas many studying the dynamics of the deep interiors of giant gas planets tend to use CGS units because they also study stellar interiors.

In this chapter we choose a traditional set of scales for the problem. Length, time, and temperature are scaled by the depth of the box (D), the thermal diffusion time (D^2/κ), and the temperature drop across the depth (ΔT), respectively. For convenience, we choose $\rho_o \kappa^2/D^2$ as the pressure scale. Then Eq. 1.11 is multiplied by D^2/κ, Eq. 1.13 by D^3/κ^2 and Eq. 1.14 by $D^2/\kappa\Delta T$. This results in the following nondimensional versions of these equations:

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,, \tag{1.15}$$

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} - \boldsymbol{\nabla} p + \mathrm{RaPr}\, T\hat{\boldsymbol{z}} + \mathrm{Pr}\,\nabla^2\mathbf{v}\,, \tag{1.16}$$

$$\frac{\partial T}{\partial t} = -(\mathbf{v}\cdot\boldsymbol{\nabla})T + \nabla^2 T\ . \tag{1.17}$$

All variables are now nondimensional and $0 \le z \le 1$ and $0 \le x \le a$, where $a = L/D$, the aspect ratio of the box.

This scaling results in two nondimensional numbers, which characterize the type of flow based on prescribed fluid properties and boundary conditions:

$$\mathrm{Ra} \equiv \frac{g_o \alpha \Delta T D^3}{\nu\kappa}\,, \tag{1.18}$$

$$\mathrm{Pr} \equiv \frac{\nu}{\kappa}\,. \tag{1.19}$$

The *Rayleigh number*, Ra, is a measure of the convective driving; the terms in the numerator promote convection, whereas those in the denominator inhibit convection. The *Prandtl number*, Pr, is the ratio of viscous to thermal diffusion; a small Pr usually means flow structures are smaller scale than thermal structures and vice versa for large Pr.

Note that an alternative choice would have been to scale the time by the viscous diffusion time, D^2/ν, which would have resulted in a slightly different arrangement of nondimensional numbers in the equations. However, neither of these scalings are particularly appropriate for simulations that are strongly driven by buoyancy because the resulting convective velocities are typically much greater than the thermal and viscous diffusion velocities, κ/D and ν/D, respectively.

SUPPLEMENTAL READING

Batchelor (1967)

EXERCISES

1. *Viscous force density*
   Derive Eq. 1.5 starting from Eqs. 1.3 and 1.4 assuming a constant dynamic viscosity.

2. *Nondimensional equations*
   Demonstrate how scaling length by the depth of the box (D), time by the thermal diffusion time (D^2/κ), temperature by the temperature drop across the depth (ΔT), and pressure by $\rho_o \kappa^2/D^2$ transforms Eqs. 1.11, 1.13, and 1.14 into Eqs. 1.15, 1.16, and 1.17, respectively.
3. *An alternative set of nondimensional equations*
   Find another set of nondimensional equations by again scaling length by the depth of the box (D) and temperature by the temperature drop across the depth (ΔT) but time by the *viscous* diffusion time (D^2/ν) and pressure by $\rho_o \nu^2/D^2$.
4. *The Boussinesq equation of state for a perfect gas*
   Explain how the Boussinesq approximation to the equation of state, 1.12, can be appropriate for a perfect gas, assuming the domain spans much less than a density scale height. See Spiegel & Veronis (1960).
5. *The Boussinesq internal energy equation for a perfect gas*
   Explain how the Boussinesq approximation to the internal energy equation, 1.14, can be appropriate for a perfect gas, assuming the domain spans much less than a density scale height. See Spiegel & Veronis (1960).