Chapter One

Preliminaries

Math is a formal language useful in clarifying and exploring connections between concepts. Like any language, it has a syntax that must be understood before its meaning can be parsed. We discuss the building blocks of this syntax in this chapter. The first is the variables that translate concepts into mathematics, and we begin here. Next we cover groupings of these variables into sets, and then operators on both variables and sets. Most data in political science are ordered, and relations, the topic of our fourth section, provide this ordering. In the fifth section we discuss the level of measurement of variables, which will aid us in conceptual precision. In the sixth section we offer an array of notation that will prove useful throughout the book; the reader may want to bookmark this section for easy return. Finally, the seventh section discusses methods of proof, through which we learn new things about our language of mathematics. This section is the most difficult, is useful primarily to those doing formal theory or devising new methods in statistics, and can be put aside for later reading or skipped entirely.

1.1 VARIABLES AND CONSTANTS

Political scientists are interested in concepts such as participation, voting, democracy, party discipline, alliance commitment, war, etc. If scholars are to communicate meaningfully, they must be able to understand what one another is arguing. In other words, they must be specific about their theories and their empirical evaluation of the hypotheses implied by their theories.

A theory is a set of statements that involve concepts. The statements comprise assumptions, propositions, corollaries, and hypotheses. Typically, assumptions are asserted, propositions and corollaries are deduced from these assumptions, and hypotheses are derived from these deductions and then empirically challenged.[1] Concepts are inventions that human beings create to help them understand the world. They can generally take different values: high or low, present or absent, none or few or many, etc.

Throughout the book we use the term "concept," not "variable," when discussing theory. Theories (and the hypotheses they imply) concern relationships among abstract concepts. Variables are the indicators we develop to measure

[1] Of course, assumptions and the solution concepts from which deductions are made may be empirically challenged as well, but this practice is rarer in the discipline.

our concepts. Current practice in political science does not always honor this distinction, but it can be helpful, particularly when first developing theory, to speak of concepts when referring to theories and hypotheses, and reserve the term variables for discussion of indicators or measures.

We assign *variables* and *constants* to concepts so that we may use them in formal mathematical expressions. Both variables and constants are frequently represented by an upper- or lowercase letter. Y or y is often used to represent a concept that one wishes to explain, and X or x is often used to represent a concept that causes Y to take different values (i.e., vary). The letter one chooses to represent a concept is arbitrary—one could choose m or z or h, etc. There are some conventions, such as the one about x and y, but there are no hard-and-fast rules here.

Variables and constants can be anything one believes to be important to one's theory. For example, y could represent voter turnout and x the level of education. They differ only in the degree to which they vary across some set of cases. For example, students of electoral politics are interested in the gender gap in participation and/or party identification. Gender is a variable in the US electorate because its value varies across individuals who are typically identified as male or female.[2] In a study of voting patterns among US Supreme Court justices between 1850 and 1950, however, gender is a constant (all the justices were male).

More formally, a **constant** is a concept or a measure[3] that has a single value for a given set. We define sets shortly, but the sets that interest political scientists tend to be *the characteristics of* individuals (e.g., eligible voters), collectives (e.g., legislatures), and countries. So if the values for a given concept (or its measure) do not vary across the individuals, collectives, or countries, etc., to which it applies, then the value is a constant. A **variable** is a concept or a measure that takes different values in a given set. Coefficients on variables (i.e., the parameters that multiply the variables) are usually constants.

1.1.1 Why Should I Care?

Concepts and their relationships are the stuff of science, and there is nothing more fundamental for a political scientist than an ability to be precise in concept formation and the statement of expected relationships. Thinking abstractly in terms of constants and variables is a first step in developing clear theories and testable hypotheses.

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[2]Definitions of concepts are, quite properly, contested in all areas of academia, and gender is no exception. Though it is not a debate that generates a great deal of interest among students of participation or party identification, it will be rather easy for you to find literature in other fields debating the value of defining gender as a binary variable.

[3]By measure we mean an operational indicator of a concept. For example, the concept gender might be measured with a survey question. The survey data provide a measure of the concept.

Table 1.1: Common Sets

Notation	Meaning
$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers
$\mathbb{R}$	Real (rational and irrational) numbers
$\mathbb{C}$	Complex numbers
Subscript: $\mathbb{N}_+$	Positive (negative) values of the set
Superscript: $\mathbb{N}^d$	Dimensionality (number of dimensions)

1.2 SETS

This leads us naturally into a discussion of sets. For our purposes,[4] a *set* is just a collection of elements. One can think of them as groups whose members have something in common that is important to the person who has grouped them together. The most common sets we utilize are those that contain all possible values of a variable. You undoubtedly have seen these types of sets before, as all numbers belong to them. For example, the counting numbers $(0, 1, 2, \ldots,$ where $\ldots$ signifies that this progression goes on indefinitely) belong to the set of natural numbers.[5] The set of all natural numbers is denoted $\mathbb{N}$, and any variable n that is a natural number must come from this set. If we add negative numbers to the set of natural numbers, i.e., $\ldots, -3, -2, -1$, then we get the set of all integers, denoted $\mathbb{Z}$. All numbers that can be expressed as a ratio of two integers are called rational numbers, and the set of these is denoted $\mathbb{Q}$. This set is larger than the set of integers (though both are infinite!) but is still missing some important irrational numbers such as π and e. The set of all rational and irrational numbers together is known as the real numbers and is denoted $\mathbb{R}$.[6]

Political scientists are interested in general relationships among concepts. Sets prove fundamental to this in two ways. We have already discussed the association between concepts and variables. As the values of each variable, and so of each concept, are drawn from a set, each such set demarcates the range of possible values a variable can take. Some variables in political science have ranges of values equal to all possible numbers of a particular type, typically either integers, for a variable such as net migration, or real numbers, for a

[4] These purposes, you will recall, are to build intuition rather than to be exact. We play somewhat loosely with ordered sets in what follows, and ignore things like Russell's paradox.

[5] Some define the natural numbers without the zero. We are not precise enough in this book to make this distinction important.

[6] You may have occasion to use complex numbers, denoted $\mathbb{C}$. These have two components, a real and an imaginary part, and can be written $a + bi$, where a and b are both real numbers and $i = \sqrt{-1}$. These are beyond the scope of this book, though amply covered by classes in complex analysis.

variable such as GDP. More typically, variables draw their values from some subset of possible numbers, and we say the variable x is an element of a subset of $\mathbb{R}$. For example, population is typically an element of $\mathbb{Z}_+$, the set of all positive integers, which is a subset of all integers. (A + subscript typically signifies positive numbers, and a − negative.) The size and qualities of the subset can be informative. We saw this earlier for the gender variable: depending on the empirical setting, the sets of all possible values were either $\{\text{Male}, \text{Female}\}$ or $\{\text{Male}\}$.[7] The type of set from which a variable's values are drawn can also guide our theorizing. Researchers who develop a formal model, game theoretic or otherwise, must explicitly note the range of their variables, and we can use set notation to describe whether they are discrete or continuous variables, for example. A variable is discrete if each one of its possible values can be associated with a single integer. We might assign a 1 for a female and 2 for male, for instance. Continuous variables are those whose values cannot each be assigned a single integer.[8] We typically assume that continuous variables are drawn from a subset of the real numbers, though this is not necessary.

A *solution set* is the set of all solutions to some equation, and may be discrete or continuous. For example, the set of solutions to the equation $x^2 - 5x + 6 = 0$ is $\{2, 3\}$, a discrete set. We term a *sample space* a set that contains all of the values that a variable can take in the context of statistical inference. When discussing individuals' actions in game theory, we instead use the term *strategy space* for the same concept. For example, if a player in a one-shot game[9] can either (C)ooperate with a partner for some joint goal or (D)efect to achieve personal goals, then the strategy space for that player is $\{C, D\}$. This will make sense in context, as you study game theory.

Note that each of these is termed a *space* rather than a set. This is not a typo; spaces are usually sets with some structure. For our purposes the most common structure we will encounter is a metric—a measure of distance between the elements of the set. Sets like $\mathbb{Z}$ and $\mathbb{R}$ have natural metrics. These examples of sets form one-dimensional spaces: the elements in them differ along a single axis. Sets may also contain multidimensional elements. For example, a set might contain a number of points in three-dimensional space. In this case, each element can be written (x, y, z), and the set from which these elements are drawn is written $\mathbb{R}^3$. More generally, the superscript indicates the dimensionality of the space. We will frequently use the d-dimensional space $\mathbb{R}^d$ in this book. When $d = 3$, this is called Euclidean space. Another common multidimensional element is an ordered pair, written (a, b). Unlike elements of $\mathbb{R}^3$, in which each

[7] As explained below, curly brackets indicate that the set is discrete. Continuous sets are demarcated by parentheses and square brackets.

[8] Formally, a discrete variable draws values from a countable set, while a continuous variable draws from an uncountable set. We define countability shortly.

[9] A one-shot game is one that is played only once, rather than repeatedly. You will encounter unfamiliar terms in the reading you do in graduate school. It is important to get in the habit of referencing a good dictionary (online or printed) and looking up terms. A search on a site like Google is often a useful way to find definitions of terms that are not found in dictionaries.

of x, y, and z is a real number, each member of an ordered pair may be quite different. For example, an ordered pair might be (orange, lunch), indicating that one often eats an orange at lunch. Ordered pairs, or more generally ordered n-tuples, which are ordered pairs with n elements, are often formed via Cartesian products. We describe these in the next section, but they function along the lines of "take one element from the set of all fruit and connect it to the set of all meals."

Political scientists also think about sets informally (i.e., nonmathematically) on a regular basis. We may take as an example the article by Sniderman, Hagendoorn, and Prior (2004). The authors were interested in the source of the majority public's opposition to immigrant minorities and studied survey data to evaluate several hypotheses. The objects they studied were individual people, and each variable over which they collected data can be represented as a set. For example, they developed measures of people's perceptions of threat with respect to "individual safety," "individual economic well-being," "collective safety," and "collective economic well-being." They surveyed 2,007 people, and thus had four sets, each of which contained 2,007 elements: each individual's value for each measure.[10] In this formulation sets contain not the possible values a variable might take, but rather the realized values that many variables do take, where each variable is one person's perception of one threat. Thus, sets here provide us with a formal way to think about membership in categories or groups.

Given the importance of both ways of thinking about sets, we will take some time now to discuss their properties. A set can be finite or infinite, countable or uncountable, bounded or unbounded. All these terms mean what we would expect them to mean. The number of elements in a **finite set** is finite; that is, there are only so many elements in the set, and no more. In contrast, there is no limit to the number of elements in an **infinite set**. For example, the set $\mathbb{Z}$ is infinite, but the subset containing all integers from one to ten is finite. A **countable set** is one whose elements can be counted, i.e., each one can be associated with a natural number (or an integer). An **uncountable set** does not have this property. Both $\mathbb{Z}$ and the set of numbers from one to ten are countable, whereas the set of all real numbers between zero and one is not. A **bounded set** has finite size (but may have infinite elements), while an **unbounded set** does not. Intuitively, a bounded set can be encased in some finite shape (usually a ball), whereas an unbounded set cannot. We say a set has a lower bound if there is a number, u, such that every element in the set is no smaller than it, and an upper bound if there is a number, v, such that every element in the set is no bigger than it. These bounds need not be in the set themselves, and there may be many of them. The greatest lower bound is the largest such lower bound, and the least upper bound is the smallest such upper bound.

Sets contain **elements**, so we need some way to indicate that a given element

[10]One could also view this as four sets of ordered pairs, with each pair containing a variable name and a person's perceptions, or one set of ordered 5-tuples, each with a person's name and her responses to each question, in order.

is a member of a particular set. A "funky E" serves this purpose: $x \in A$ states that "x is an element of the set A" or "x is in A." You will find this symbol used when the author restricts the values of a variable to a specific range: $x \in \{1, 2, 3\}$ or $x \in [0, 1]$. This means that x can take the value 1, 2, or 3 or x can be any real number from 0 to 1, inclusive. It is also convenient to use this notation to identify the range of, say, a dichotomous dependent variable in a statistical analysis: $y \in \{0, 1\}$. This means that *y either can take a value of 0 or a value of 1.* So the "funky E" is an important symbol with which to become familiar. Conversely, when something is not in a set, we use the symbol $\notin$, as in $x \notin A$. This means that, for the examples in the previous paragraph, x does not take the values 1, 2, or 3 or is not between 0 and 1. As you may have guessed from our usage, curly brackets like $\{\}$ are used to denote discrete sets, e.g., $\{A, B, C\}$. Continuous sets use square brackets or parentheses depending on whether they are closed or open (terms we define in Chapter 4), e.g., $[0, 1]$ or $(0, 1)$, which are the sets of all real numbers between 0 and 1, inclusive and exclusive, respectively.

Much as sets contain elements, they also can contain, and be contained by, other sets. The expression $A \subset B$ (read "A is a **proper subset** of B") implies that set B contains all the elements in A, plus at least one more. More formally, $A \subset B$ if all x that are elements in A are also elements in B (i.e., if $x \in A$, then $x \in B$). $A \subseteq B$ (read "A is a **subset** of B"), in contrast, allows A and B to be the same. We say that A is a proper subset of B in the first case but not in the second. So the set of voters is a subset of the set of eligible voters, and is most likely a proper subset, since we rarely experience full turnout. We also occasionally say that a set that contains another set is a superset of the smaller one, but this terminology is less common. The **cardinality of a set** is the number of elements in that set. Note that proper subsets have smaller cardinalities than their supersets, finite sets have finite cardinalities, and infinite sets have infinite cardinalities.

A **singleton** is a set with only one element and so a cardinality of one. The power set of A is the set of all subsets of A, and has a cardinality of $2^{|A|}$, where $|A|$ is the cardinality of A. Power sets come up reasonably often in political science by virtue of our attention to bargaining and coalition formation. When one considers all possible coalitions or alliances, one is really considering all possible subsets of the overall set of individuals or nations. Power sets of infinite sets are always uncountable, but are not usually seen in political science applications. The **empty set** (or **null set**) is the set with nothing in it and is written $\emptyset$. The **universal set** is the set that contains all elements. This latter concept is particularly common in probability.

Finally, sets can be ordered or unordered. The **ordered set** $\{a, b, c\}$ differs from $\{c, a, b\}$, but the **unordered set** $\{a, b, c\}$ is the same as $\{c, a, b\}$. That is, when sets are ordered, the order of the elements is important. Political scientists primarily work with ordered sets. For example, all *datasets* are ordered sets. Consider again the study by Sniderman et al. (2004). We sketched four of the sets they used in their study; the order in which the elements of those sets is maintained is critically important. That is, the first element in each set must refer to the first person who was surveyed, the second element must refer to the second person, and the 1,232nd element must refer to the 1,232nd person surveyed, etc. All data analyses use ordered sets. Similarly, all equilibrium strategy sets in game theory are ordered according to player. However, this does not mean all sets used in political science are ordered. For example, the set of all strategies one might play may or may not be ordered.

1.2.1 Why Should I Care?

Sets are useful to political scientists for two reasons: (1) one needs to understand sets before one can understand relations and functions (covered in this chapter and Chapter 3), and (2) sets are used widely in formal theory and in the presentation of some areas of statistics (e.g., probability theory is often developed using set theory). They provide us with a more specific method for doing the type of categorization that political scientists are always doing. They also provide us with a conceptual tool that is useful for developing other important ideas. So a basic familiarity with sets is important for further study.

For example, game theory is concerned with determining what two or more actors should choose to do, given their goals (expressed via their utility) and their beliefs about the likelihood of different outcomes given the choices they might make and their beliefs about the expected behavior of the other actor(s). Sets play a central role in game theory. The choices available to each actor form a set. The best responses of an actor to another actor's behavior form a set. All possible states of the world form a set. And so on.

Those of you who are unfamiliar with game theory will find this brief discussion less than illuminating, but do not be concerned. Our point is not to explain sets of actions, best response sets, or information sets—each is covered in game theory courses and texts—but rather to underscore why it is important to have a functional grasp of elementary set theory if one wants to study formal models. Finally, we note that Riker's (1962) celebrated game theoretic model of political coalition formation makes extensive use of set theory to develop what he calls the size principle (see Appendix I, pp. 247–78, of his book). That is, of course, but one of scores of examples we might have selected.[11]

1.3 OPERATORS

We now have formalizations of concepts (variables) and ways to order and group these variables (sets), but as yet nothing to do with them. Operators, the topic of this section, are active mathematical constructs that, as their name implies, operate on sets and elements of sets. Some operators on variables have been familiar since early childhood: addition ($+$), subtraction ($-$), multiplication ($*$ or $\times$ or $\cdot$ or just placing two variables adjacent to each other as in xy),

[11]Readers interested in surveys of formal models in political science that are targeted at students might find Shepsle and Bonchek (1997) and Gelbach (2013) useful.

and division ($\div$ or $/$). We assume you know how to perform these operations. Exponentiation, or raising x to the power a (x^a), is likely also familiar, as is taking an nth root ($\sqrt[n]{x}$), and perhaps finding a factorial (!) as well.

Other useful basic operators include summation ($\sum_i x_i$), which dictates that all the x_i indexed by i should be added, and product ($\prod_i x_i$), which dictates that all the x_i be multiplied. These operators are common in empirical work, where each i corresponds to a data point (or observation). Here are a couple of examples:

$$\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3,$$

and

$$\prod_{i=1}^{3} x_i = x_1 \times x_2 \times x_3.$$

Because they are just shorthand ways of writing multiple sums or products, each of these operators obeys all the rules of addition and multiplication that we lay out in the next chapter. So, for example, $\sum_{i=1}^{n} x_i^2$ does not generally equal $\left(\sum_{i=1}^{n} x_i\right)^2$ for the same reason that $(2^2+3^2) = 13$ does not equal $(2+3)^2 = 25$.[12] Other operators and their properties will be introduced as needed throughout the book. We present a collection of notation below in section 1.6 of this chapter.

You may be less familiar with operators on sets, though they are no less fundamental. We consider six here: *differences*, *complements*, *intersections*, *unions*, *partitions*, and *Cartesian products*. The **difference** between two sets A and B, denoted $A \backslash B$ (read "A difference B"), is the set containing all the elements of A that are not also in B: $x \in A \backslash B$ if $x \in A$ but $x \notin B$. This set comes up a great deal in game theory when one is trying to exclude individual players or strategies from consideration. The **complement** of a set, denoted A' or A^c, is the set that contains the elements that are not contained in A: $x \in A^c$ if x is not an element of A.[13] Continuing the example from above, the complement of the set of registered voters is the set of all people who are not registered voters.

Venn diagrams can be used to depict set relationships. Figure 1.1 illustrates the concepts of set difference and set complement. The shaded part of the left diagram is the set Registered Voters $\backslash$ Registered Democrats, which is read "Registered Voters difference Registered Democrats." Or, in other words, all registered voters who are *not* registered Democrats. The shaded part of the right diagram illustrates the set Registered Voters^c, which is "the complement

[12] Summations and products can also be repeated; this is known as a double (or triple, etc.) summation or product. If x_{ij} is indexed by i and j, then we could write $\sum_i \sum_j x_{ij}$ or $\prod_i \prod_j x_{ij}$. Multiple summations may be useful, for example, when employing discrete distributions in more than one dimension, or when considering more than one random variable in game theory.

[13] One can also think of the complement of a set A as the difference between the universal set and A.

of Registered Voters." Or, in other words, people who are not registered voters, since the universal set in this case is the set of All People. Both diagrams illustrate the concept of a subset: the set Registered Voters is a (proper) subset of the set All People, and the set Registered Democrats is a (proper) subset of the set Registered Voters. And both diagrams illustrate another concept: the sets Registered Voters and Registered Voters^c are collectively exhaustive, in that together they constitute the set All People, which is the universal set in this case. In general, a group of sets is **collectively exhaustive** if together the sets constitute the universal set.[14]

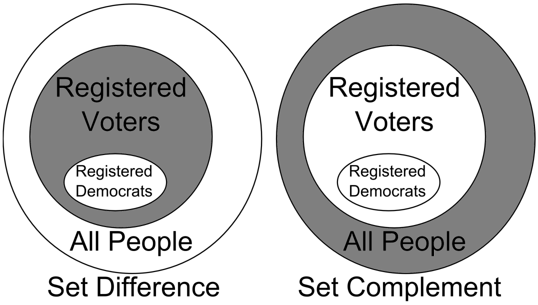

Figure 1.1: Set Difference and Complement

The **intersection** of two sets A and B, denoted $A \cap B$ (read "A intersection B"), is the set of elements common to both sets. In other words, $x \in A \cap B$ if $x \in A$ and $x \in B$. Thus, if set A consists of elected Democrats in the state of Florida and set B consists of legislators in the Florida House of Representatives, then the intersection of A and B is the set containing all Democratic House members in Florida.

The **union** of two sets is written $A \cup B$ (read "A union B") and is the set of all elements contained in either set. In other words, $x \in A \cup B$ if $x \in A$ or $x \in B$. Note that any x in both sets is also in their union. Continuing the example from above, the union of A and B is the set composed of all elected Democrats in Florida *and* all House members in Florida. Figure 1.2 shows the intersection of the sets House Members and Elected Democrats in the shaded part on the left, and their union in the shaded part on the right. The diagram on the left also illustrates the concept of mutually exclusive sets. **Mutually exclusive** sets are sets with an intersection equal to the empty set, i.e., sets with no elements in their intersection. In the diagram on the left, the two unshaded portions of the sets House Members and Elected Democrats are mutually exclusive sets. In fact, any two sets are mutually exclusive once their intersection has been removed, since they then must have an intersection that is empty.

A **partition** is a bit more complex: it is the collection of subsets whose union forms the set. The more elements a set has, the greater the number of partitions

[14]Strictly speaking, their union must equal the universal set. We discuss unions next.

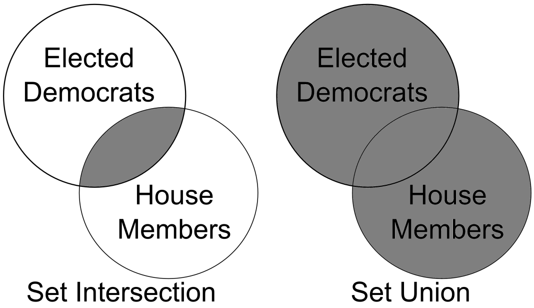

Figure 1.2: Set Intersection and Union

one can create. Let's consider the following example, the set of candidates for the 2004 US presidential election who received national press coverage:[15] $A = \{\text{Bush}, \text{Kerry}, \text{Nader}\}$. We can partition A into three subsets: {Bush}, {Kerry}, {Nader}; or we can partition it into two subsets: {Bush, Nader}, {Kerry}; or {Kerry, Nader}, {Bush}; or {Bush, Kerry}, {Nader}. Finally, the set itself is a partition: {Bush, Kerry, Nader}.

A **Cartesian product** is more complex still. Consider two sets A and B, and let $a \in A$ and $b \in B$. Then the Cartesian product $A \times B$ is the set consisting of all possible ordered pairs (a, b), where $a \in A$ and $b \in B$. For example, if $A =$ {Female, Male} and $B =$ {Income over \$50k, Income under \$50k}, then the Cartesian product is the set of cardinality four consisting of all possible ordered pairs: $A \times B =$ {(Female, Income over \$50k), (Female, Income under \$50k), (Male, Income over \$50k), (Male, Income under \$50k)}. Note that the type of element (ordered pairs) in the product is different from the elements of the constituent sets. Cartesian products are commonly used to form larger spaces from smaller constituents, and appear commonly in both statistics and game theory. We can extend the concept of ordered pairs to ordered n-tuples in this manner, and each element in the n-tuple represents a *dimension*. So x is one-dimensional, (x, y) is two-dimensional, (x, y, z) is three-dimensional, and so on. Common examples of such usage would be $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, which is three-dimensional space, and $S = S_1 \times S_2 \times \ldots \times S_n$, which is a strategy space formed from the individual strategy spaces of each of the n players in a game.

1.3.1 Why Should I Care?

Operators on variables are essential; without them we could not even add two numbers. Operators on sets are equally essential, as they allow us to manipulate sets and form spaces that better capture our theories, including complex inter-

[15]In August 2004 Project Vote Smart listed over ninety candidates for president of the United States, but working with the full set would be unwieldy, so we restrict attention to the subset of candidates who received national press coverage (`http://www.vote-smart.org/election_president_party.php?party_name=All`).

actions. They are also necessary for properly specifying functions of all sorts, as we shall see in Chapter 3.

1.4 RELATIONS

Now we have variables, conceptually informed groups of variables, and ways to manipulate them via operators, but we still lack ways to compare concepts and discern relationships between them. This is where relations enter. A mathematical **relation** allows one to compare constants, variables, or expressions of these (or, if you prefer, concepts). Binary relations (i.e., the relation between two constants/variables/expressions or concepts) are easiest to consider, so we will restrict the discussion to the two variable case, but the idea can be generalized to an n-ary relation. Similarly, we can define orders on sets, but these admit many possibilities and are less commonly observed in political science, so we will eschew this topic as well.

A binary **relation** can be represented as an ordered pair. So, if $a \in A$ is greater than $b \in A$, we can write the relation as (a, b). When constants or variables are drawn from the integers or real numbers, though, we have more familiar notation. Integers and real numbers have natural associated orders: three is greater than two is greater than one, and so on. When one is certain of the value of a concept, as one is with a constant, then we can write $3 > 2$, $1 < 4$, and $2.5 = 2.5$. The symbols $>$, $<$, and $=$ form the familiar relations of arithmetic. When one is less sure of the values of a concept, as one is with a variable, then we also have the relations $\geq$ and $\leq$, as in $x \geq z$. Algebra, reviewed in the next chapter, deals with the manipulation of these sorts of relations.

The concept of relations is more general than these orders, however. A relation exists between two sets (or concepts) when knowing one element provides information about the other element. So, for example, in networks the relation could be "linked," while in game theory it might be "like as well as." We will explore this latter idea more in Chapter 3. While relations can be specified quite generally,[16] typically we will only be concerned with a few types of relation. Inequalities are one, and preference relations, discussed in Chapter 3, are another. The most common relation we'll use, though, is a function, which is the topic of Chapter 3. In this context we want to know the mapping between sets A and B. In other words, we want to know how the function transforms an element of A into an element of B. In this case we call A the **domain** and B the **range**. Relations (and so functions) can have various properties, some of which we discuss in Chapter 3.

[16] A relation is a mathematical object that takes as input two sets A and B (called its domain in this context) and returns a subset of $A \times B$ (called its graph in this context).

1.4.1 Why Should I Care?

Relations are important because they help us describe the mapping of values across concepts. Relations such as "greater than" and "equal to" are critical to descriptive claims about the world as well as to making theoretical claims. Further, functions—a specific type of relation—are very common in both theoretical and empirical work in political science.

1.5 LEVEL OF MEASUREMENT

We now have most of the building blocks we need to describe relationships between concepts. These in turn allow us to distinguish among different levels of measurement: **nominal**, **ordinal**, **interval**, and **ratio**. Note that though levels of measurement tend to be associated with variables, they are equally applicable and important to conceptualization.[17] We briefly discuss each level of measurement in turn.

1.5.1 Differences of Kind

In some theories all we require of our concepts is that they distinguish one type from another. That is, some concepts are about differences of kind, but not differences of degree. Concepts that identify different types but do not order them on any scale are nominal, and they require only nominal level measurement of their indicators.[18]

Nominal level measurement does not establish mathematical relationships among the values. In other words, it does not make sense to assert that a case with a nominal value of 3 is greater than one with a nominal value of 1, or that two cases with a nominal value of 2 are equal. The symbols $<$, $\leq$, $=$, $\geq$, and $>$ have no meaning for variables measured at the nominal level. Gender is a good example of a nominal level variable. When entering data for a measure of gender into a computer a researcher might assign the values of 0 and 1 (or 1 and 2) to female and male, respectively. But she might also have assigned the values -64 and $3,241$. Or she might have assigned the values 1 and 0 (or 2 and 1) to female and male, respectively. The point is that higher values do not convey any meaning: the numerical values are placeholders that indicate a difference, but the numerical values do not tell us anything meaningful.

1.5.2 Differences of Degree

At other times we are interested in differences of degree. Whether one case has more, is stronger, etc., is important to us as we define concepts and then think

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[17] Students interested in an extended discussion will find Cohen and Nagel (1934, pp. 223–44) useful.

[18] The four levels of measurement—nominal, ordinal, interval, and ratio—were proposed by Stevens (1946).

about ways to measure those concepts. In such cases, nominal level concepts and measures are inadequate for they do not imply mathematical relationships among the values.

Ordinal level measurement, on the other hand, does imply mathematical relationships among the values. More specifically, the symbols $<$, $\leq$, $=$, $\geq$, and $>$ have meaning for ordinal level concepts (variables). However, the distance between any two values does not measure a constant quantity across the values the variable might take. For example, a voting scholar might be interested in people's self-placement on an ideological scale. He might put together a survey that includes a question asking people to mark themselves as far left, moderate left, middle of the road, moderate right, far right. Such a concept makes "greater than," "less than," and "equal to" distinctions. For example, we can say that moderate left is further to the left on the scale than middle of the road. And when we assign numerical values we do not have the same freedom as with a nominal measure. That is, once we have assigned two values, we are constrained on others. For example, if we assign "middle of the road" the value 3 and "far left" the value 1, then we must assign "moderate left" a value greater than 1 and less than 3. If this were a nominal level variable, then we would not be so constrained and could assign any value we wish. But ordinal variables must use numerical values that retain the order of the concept's values because the order matters in the sense that it conveys meaning. So concepts with an ordinal level of measurement have ordered values that indicate "more than" and "less than."

The next level of measurement is **interval**. This requires that the distance between values be constant over the range of values. This property is important because it makes addition and subtraction meaningful. One cannot meaningfully add or subtract variables with nominal or ordinal values because the operation does not make sense. To see that this is so, consider that we can assign any values to a binary nominal variable: 0, 1; 1, 2; or -64 and $3,241$. We cannot meaningfully add or subtract the values of such a variable because the values do not have meaning as numerical values. Ordinal measures, on the other hand, have meaning up to "greater than" and "less than" operations, but they also cannot be added or subtracted. If one considers the example above, we might assign the numerical values 1, 2, 3, 4, and 5 to the ideology scale, or we might assign the numerical values -3, 2, 7, 44, and 1,324. Any set of numerical values that retains the order of the concept's values is valid. The distances in the first numerical value set are constant (they are each one unit apart), but the distances in the second set vary. As such, and because both sets of values are valid, the addition and subtraction of ordinal measures do not have meaning.

Interval level measures, on the other hand, have meaningful distances between values: the intervals between numbers are constant across the range of values. Put differently, a change of $\pm x$ on the scale is the same distance regardless of where one is on the scale.

Interval levels measures may be **discrete** or **continuous**. Discrete variables with interval level measurement are integers (or natural numbers). For example, a common survey item is the feeling thermometer, which asks respondents to identify the strength of their feelings toward a politician on a scale of 0 to 100, where 0 represents extremely cold and 100 represents extremely hot (e.g., Cain, 1978; Abramowitz, 1980). Most researchers submit that the respondent recognizes that a shift of +10 points is the same anywhere on the scale.[19] That is, the distance from 0 to 5 is equivalent to the distance from 26 to 31, from 47 to 52, from 83 to 88, etc. To the extent that this is so, the measure is interval. One can meaningfully add and subtract interval level measures.

Ratio level variables are interval level variables that have a meaningful zero value. The feeling thermometer variable has a zero value, but it does not represent the absence of feeling. Instead, it represents a very strong feeling: intense dislike. So zero is not a meaningful point on the scale. As such, while we can conduct meaningful addition and subtraction operations with such variables, we cannot conduct meaningful multiplication and division operations.

The label "ratio level" comes from the fact that the same ratio at two points on the scale conveys the same meaning. This is not terribly intuitive, so let us explain. On an interval level scale any distance x between two points has the same meaning, regardless of where we are on the scale. Ratio level measurement also has this property, but it has a constant ratio property that interval level measurement lacks: the ratio of two points on the scale conveys the same meaning regardless of where one is on the scale. A good example of a ratio level scale is a public budget. Imagine that a municipal government spends four times as much on public safety as it does on public health. This is a ratio of 4:1.[20] Thus, if the city spends $4.8 million on public safety, it must spend $1.2 million on public health. Similarly, if it spends $2 million on public safety, it must spend $0.5 million on public health. Ratios can only convey meaning (i.e., measure a constant ratio) when the scale over which they are measured has a 0 value that indicates the absence (i.e., none of) whatever is being measured.

To return to the feeling thermometer example, if the value 0 represents intense negative affect (i.e., dislike), 50 indicates an absence of affect (i.e., indifference), and 100 represents intense positive affect, then 0 is not an absence of affect. Thus, it is an interval level scale, not a ratio level scale, and we cannot conclude that the first member of two pairs of respondents with scores of 20 and 10, and 50 and 25, respectively, each have twice as much affect for a candidate as the second member of each pair. However, we could rescale the feeling thermometer to make it centered on zero, perhaps assigning the value of -50 to intense negative affect, 0 to the absence of affect (or indifference), and 50 to intense positive affect. Doing so would transform the level of measurement from interval to ratio.[21]

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[19] Note that the respondents' (implicit) beliefs about the scale of the item are important in survey research.

[20] We discuss ratios in more detail in the first section of Chapter 2. You may want to skip ahead to there if you are unfamiliar with ratios.

[21] You may be thinking that this is a trivial transformation that is not consequential, but this is not the case. To see why, try the following. Arbitrarily select a ratio—perhaps 3:1—and select two pairs of points on the transformed feeling thermometer (the one with the proper

There are lots of examples of discrete ratio level variables in political science. Political scientists are often interested in the number of events that occur, and an event count has a meaningful constant distance between values and a meaningful zero point. Thus, they are ratio variables. Examples of event counts that have been used in political science include the number of seats a party holds in parliament, the number of vetoes issued by an executive, the number of unanimous decisions by a court, and the number of wars in which a country has participated.

Thus far we have restricted our attention to discrete variables. Continuous variables have an interval or ratio level of measurement, depending on whether the value 0 represents the absence of the concept. The vast majority of (empirical) concepts that political scientists have either created or borrowed from other disciplines are discrete, but some examples of continuous measures of interest to political scientists are income and GDP.[22]

You have likely noticed that each level of measurement subsumes the levels below it. That is, ordinal level measurement is also nominal, and an interval measure has ordinal and nominal properties. This suggests that whenever we have a concept at a high level of measurement we can reconceptualize and remeasure it at a lower level of measurement should we have cause to do so.

Some people mistakenly view the hierarchy of the levels of measurement as a means to judge the heuristic value of concepts. This is an error. Concepts can be evaluated on their clarity (vague concepts have little heuristic value), and one can make normative judgments about concepts (e.g., freedom, peace, order), but all sufficiently clear concepts are merely inputs to specific theories, and theories, not their concepts, should be judged. A proper discussion of this issue is beyond the scope of this book, but it is important to recognize that a nominal conceptualization may yield insights that a ratio conceptualization would miss and vice versa. Put differently, it would be an error to judge the levels of measurement as an ordinal scale with respect to their value to causal theory: it is nominal.

1.5.3 Why Should I Care?

Recognizing whether one is thinking about differences of kind (nominal) or degree (ordinal, interval, or ratio) is critical. If one is thinking about differences of degree, then how precise are those differences? Without a firm grasp on levels of measurement one cannot be precise about one's concepts, much less one's measures of one's concepts.

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ratio scale where −50 is intense dislike, 0 is indifference, and 50 is strong positive affect) that have that ratio. Now transform the scale to the actual feeling thermometer (the one with the range from 0 to 100). Recalculate the ratios. They are different, right? The two scales do not produce the same ratio levels, and that means that one of them preserves ratios and the other does not. The one with the meaningful zero is the only scale that produces meaningful ratios. For a more detailed explanation, see Stevens (1946).

[22] If one rounds either to dollars, thousands of dollars, etc., then the values are integers (or natural numbers) and the measure is discrete.

1.6 NOTATION

Here we list, and in some cases briefly describe, common notation. This section is one you will likely refer to from time to time, but not everything might be clear now. Also, as a reference section it is heavier on the math and lighter on the intuition. It is important to read it once now, but if you find yourself unclear on some notation later, please refer back to this section. To make reference easier, we begin with the summary Table 1.2.

Operators take many forms, and are commonly used. We have already discussed some: $+$, $-$, $\times$, $/$, x^n, $\sqrt[n]{x}$, $\sum$, $\prod$, !. Some of these have multiple ways to represent them, others mean multiple things depending on context. For example, there are several ways to represent multiplication: $a \times b \times c = a * b * c = a \cdot b \cdot c = abc$. Of course, as we have seen, $\times$ can also mean a Cartesian product when applied to sets. Both $/$ and $\div$ mean divide; the mod operator, written $8 \bmod 3$, means divide the first number by the second, and report the remainder: $8 \bmod 3 = 2$.

One can also use the product operator, $\prod$, to represent the product of a, b, and c: $\prod_a^c$.

> One reads that as *the product of a through c.*

More typically, the product operator is used by indexing a variable (this is accomplished by adding a subscript: x_i) and writing: $\prod_{i=k}^{l} x_i$.

> One reads that as *the product of* x_i *over the range from* $i = k$ *through* $i = l$.

When the product operator is used in an equation that is set apart from the text, it looks like this:

$$\prod_{i=k}^{l} x_i = x_k \times \ldots \times x_l.$$

The "..." here signals the reader to assume all interim values are included in the product. When used at the end of a list, e.g., $1, 2, 3, \ldots$, "..." signifies that the list (or product or sum) goes on indefinitely. In these cases you may also see ∞ as an end to the sequence instead, e.g., $1, 2, 3, \ldots, \infty$; ∞ is the symbol for infinity. In other words, ... means continue the progression until told to stop.

The summation operator, $\sum$, can be used to represent the addition of several numbers. For example, if we want to add together all members of a set indexed by i, then we can write: $\sum_i$. One reads that as *the sum over* i. You will also see summation represented over a range of values, say from value k through value l: $\sum_{i=k}^{l} x_i$.

> One reads that as *the sum of* x_i *over the range from* $i = k$ *through* $i = l$.

Table 1.2: Summary of Symbols and Notation

Symbol	Meaning
$+$	Addition
$-$	Subtraction
$*$ or $\times$ or $\cdot$	Multiplication
$/$ or $\div$	Division
$\pm$	Plus or minus
x^n	Exponentiation ("to the nth power")
$\sqrt[n]{x}$	Radical or nth root
$!$	Factorial
∞	Infinity
$\sum_{i=k}^{l} x_i$	Sum of x_i from index $i = k$ to $i = l$
$\prod_{i=k}^{l} x_i$	Product of x_i from index $i = k$ to $i = l$
$\ldots$	Continued progression
$\frac{d}{dx}$	Total derivative with respect to x
$\frac{\partial}{\partial x}$	Partial derivative with respect to x
$\int dx$	Integral over x
$\cup$	Set union
$\cap$	Set intersection
$\times$	Cartesian product of sets
$\setminus$	Set difference
A^c	Complement of set A
$\emptyset$	Empty (or null) set
$\in$	Set membership
$\notin$	Not member of set
$\mid$ or $:$ or $\ni$	Such that
$\subset$	Proper subset
$\subseteq$	Subset
$<$	Less than
$\leq$	Less than or equal to
$=$	Equal to
$>$	Greater than
$\geq$	Greater than or equal to
$\neq$	Not equal to
$\equiv$	Equivalent to or Defined as
$f()$ or $f(\cdot)$	Function
$\{\ \}$	Delimiter for discrete set
$(\ )$	Delimiter for open set
$[\ ]$	Delimiter for closed set
$\forall$	For all (or for every or for each)
$\exists$	There exists
$\Rightarrow$	Implies
$\Leftrightarrow$	If and only if
$\neg C$ or $\sim C$	Negation (not C)

Set apart from the text in an equation, the summation operator looks like this:

$$\sum_{i=k}^{l} x_i = x_k + \ldots + x_l.$$

The exponential operator, x^n (read "x to the nth power," or "x-squared" when $n = 2$ and "x-cubed" when $n = 3$), represents the power to which we raise the variable, x. The root operator, $\sqrt[n]{x}$ (read "the nth root of x," or "the square root of x" when $n = 2$ or "the cube root of x" when $n = 3$), represents the root of x.

Factorial notation is used to indicate the product of a specific sequence of numbers. Thus, $n! = n \times (n-1) \times (n-2) \ldots \times 2 \times 1$. So $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$, and $10! = 10 \times 9 \times \ldots 3 \times 2 \times 1 = 3,628,800$. This notation is especially useful for calculating probabilities.

You may not be familiar with some of the operators used in calculus. The derivative of x with respect to t is represented by the operator $\frac{dx}{dt}$. The operator ∂ indicates the partial derivative, and $\int$ indicates the integral. These will be the focus of Parts II and V of this book.

Though it's not an operator, one more symbol is useful to mention here: $\pm$. Read as "plus or minus," this symbol implies that one cannot be sure of the sign of what comes next. For example, $\sqrt{4} = \pm 2$, because squaring either 2 or -2 would produce 4.

Sets, as we have seen, have a good deal of associated notation. There are the set operators $\cap$, $\cup$, $\times$, and $\backslash$, plus the complement of A (A^c or A'). There are also the empty set $\emptyset$, set membership $\in$, set nonmembership $\notin$, proper subset $\subset$, and subset $\subseteq$. To these we add $|$, : , or $\ni$, which are each read as "such that." These are typically used in the definition of a set. For example, we define the set $A = \{x \in B | x \leq 3\}$, read as "the set of all x in B such that x is less than or equal to 3." In other words, the $|$ indicates the condition that defines the set. It serves the same purpose in conditional probabilities ($P(A|B)$), as we will see in Part III of the book. Sets also make use of delimiters, described below.

Relations include $<$, $\leq$, $=$, $\geq$, $>$. They also include $\neq$, which means "not equal to," and $\equiv$, which means "exactly equivalent to" or, often, "defined as." Relations between variables or constants typically have a left-hand side, to the left of the relation symbol, and a right-hand side, to the right of the relation symbol. These are often abbreviated as LHS and RHS, respectively. Functions are typically written as $f()$ or $f(\cdot)$, both of which imply that f is a function of one or more variables and constants. The "$\cdot$" here is a placeholder for a variable or constant; do not confuse it with its occasional use as a multiplication symbol, which occurs only when there are things to multiply.

Delimiters are used to indicate groups. Sometimes the groups are used to identify the order of the operations that are to be performed: $(x + x^2)(x - z)$. One performs the operations inside the innermost parentheses first and then moves outward. Square braces and parentheses are also used to identify closed and open sets, respectively. The open set (x_1, x_n) excludes the endpoint values

Table 1.3: Greek Letters

Upper-case	Lower-case	English	Upper-case	Lower-case	English
A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ϵ	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ	sigma
H	η	eta	T	τ	tau
Θ	θ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

x_1 and x_n, whereas the closed set $[x_1, x_n]$ includes the endpoint values x_1 and x_n. Curly braces are used to denote set definitions, as above, or discrete sets: $\{x_1, x_2, \ldots, x_n\}$. Parentheses are also often used for ordered pairs or n-tuples, as we have seen; for example, $(2, 3, 1)$. They are also often used in vectors, which have a similar meaning. Both parentheses and square braces are used interchangeably to indicate the boundaries of matrices. We will discuss both vectors and matrices in Part IV of the book.

Proofs, the topic of the next section, have their own notation, which may pop up in other sections as well. The symbol $\forall$ means "for all," so $\forall x \in A$ means the associated statement applies for all x in the set A. The symbol $\exists$ means "there exists," typically used in the context of $\exists$ some $x \in A$ such that $x < 3$. The symbol $\Rightarrow$ is read as "implies" and is used as $C \Rightarrow D$, which means that whenever statement C is true, D is too. One can also use the reverse, $C \Leftarrow D$, which means that C is true if D is true. The symbol $\Leftrightarrow$ means that both implications are true and is read as "if and only if," so $C \Leftrightarrow D$ means that C is true if D is true, and only if D is true. In other words, C and D are equivalent statements. The symbol $\neg$ denotes negation, so $\neg C$ means statement C is not true. You will also sometimes see $\sim C$ used to mean C is not true.

People sometimes use **Greek letters** to represent variables, particularly in formal theory; they are often used to represent constants (aka parameters) in statistical analysis. Table 1.3 lists the Greek alphabet. If you have never encountered the Greek alphabet you may want to make a copy of this page, cut out the table, and tape it to the wall where you study for this and other courses that use math. Or just save it to your preferred portable electronic device.

1.6.1 Why Should I Care?

Notation that you cannot read is a serious stumbling block to understanding!

1.7 PROOFS, OR HOW DO WE KNOW THIS?

As we progress through this book, we will offer up a great many pieces of information as fact, often without explaining how we knew they were true. As noted in the preface to this book, we do this in order to focus on intuition rather than mathematical formalism. However, it is certainly fair to wonder—more than fair, really—how one comes to these conclusions. The answer, as we discuss briefly here, is that they have been proven to be true.

How does this work? Mathematics is not an empirical science; there are no experiments, and no data except insofar as experience shapes the thought of mathematicians. Rather, the progress of math begins with axioms and **assumptions**, which are stated up front with clarity and taken to be true.[23] One then conjectures a **proposition**, which is just a statement that is thought to be true given the assumptions made. From these assumptions, along with any previously proved theorems, one deductively proves, or disproves, the proposition. A proven proposition is often referred to as a **theorem**, unless it is of little interest in and of itself and is intended to be used only as a stepping stone, in which case it is called a **lemma**. A **corollary** is a type of proposition that follows directly from the proof of another proposition and does not require further proof. You will see assumptions and propositions commonly in pure and applied game theory, and lemmas, theorems, and corollaries somewhat less commonly. Propositions, though deductively derived, are often empirically testable given appropriate measures for the variables used in the proposition. In other words, a proposition might state that y is increasing in x_1 and decreasing in x_2. To test this empirically, one needs measures for y, x_1, and x_2. In some scientific fields it is common to distinguish propositions from hypotheses, with the former referring to statements of expected relationships among concepts and the latter referring to expected relationships among variables. In such contexts propositions are more general statements than hypotheses. At present, these distinctions are not widely used among political scientists.

It is not difficult to make assumptions, though learning to specify them clearly and to identify the implicit assumptions you may be making takes practice. Nor is it difficult to state propositions that may be true, though similar caveats apply. The tricky part is in proving the proposition. There is no one way to prove all propositions, though the nature of the proposition can suggest the appropriate alternative. We will consider a few commonly observed methods here, but this is far from a complete accounting.

We begin by considering four statements: A, B, C, D. A statement can be anything, e.g., A could be $x < 3$ or "all red marbles are in the left urn" or "democracies are characterized primarily by elections." Let's assume that A and B are assumptions. We take them to be true at the start of our proof and

[23]Political scientists rarely specify axioms, which tend to be more significant and wide-ranging assumptions than what are called simply assumptions. The following discussion uses terms as they are commonly observed in political science, which may elide mathematical nuance.

will not deduce them in any way from other statements. Of course, if they are not empirically true, then our conclusions may very well be incorrect empirically, but, as you can guess by the repeated use of the word "empirically," this is an empirical question and not a mathematical one. Let's further assume that C is an interim statement—that is, a deduced statement that is not our intended conclusion—and that D is that conclusion. Thus our goal is to derive D from A and B. This is the general goal of mathematical proofs.

More precisely, in this case we are seeking to show that A and $B \Rightarrow D$ (A and B imply D). This is a sufficiency statement: A and B are *sufficient* to produce D. We also can call this an *if* statement: D is true if A and B are true. This is not the only possible implication we could have written (implications are just a type of mathematical statement). We could instead have stated that A and $B \Leftarrow D$ (A and B are implied by D). This is a statement of necessity: A and B are *necessary* to produce D, since every time D is true, so are A and B. We can also call this an *only if* statement: D is true only if A and B are. Take a moment to think about the difference between these two ideas, as it is fairly central to understanding theory in political science, and it is not always obvious how different the statements are.

Ready? There is also a third common implication we could have written, a necessary *and* sufficient statement: A and $B \Leftrightarrow D$. This is also called an *if and only if* statement, as D is true if and only if A and B are true. In other words, A and B are entirely equivalent logically to D, and one can replace one statement with the other at will. This is one way one uses existing theorems to help in new proofs, by replacing statements with other statements proven to be equivalent. (One can also use if or only if propositions on their own in new proofs.)

In addition to using existing theorems, pretty much any mathematical procedure accepted as true can be used in a proof. We'll cover many in this book, but the most basic of these may be the tools of formal **logic**, which has much in common with set theory. Negation of a statement is much the same as the complement of a set. For example, you cannot be both true and not true, nor can you be both in and outside a set. You can also take the equivalent of a union and an intersection of sets for statements; these are called disjunction and conjunction, or, in symbols, *or* ($\vee$) and *and* ($\wedge$), respectively. Note that the *and* symbol looks like the intersection symbol. This is not accidental—*and* means that both statements are true, which is like being in both sets, which is like the intersection of the sets. Likewise, *or* means that at least one statement is true, which is like being in either set, which is like the union between the sets. Let's call a compound statement anything that takes any two simpler statements, such as A and B, and combines them with a logical operator, such as $\neg, \vee$, or $\wedge$. We can therefore write the implication we're trying to prove as $A \wedge B \Rightarrow D$.

De Morgan's laws prove handy for manipulating both sets and logical statements.[24] We'll present these in terms of logical statements, but they are true for sets as well after altering the notation. The best way to remember them is that the negation of a compound statement using *and* or *or* is the compound statement in which the *and* is switched for *or*, or vice versa, and each of the simpler statements is negated. So, for example, $\neg(A \wedge B)$ is $(\neg A) \vee (\neg B)$ and $\neg(A \vee B)$ is $(\neg A) \wedge (\neg B)$. In words, if both statements aren't true, then at least one of them must be false. Similarly, if it's not the case that at least one statement is true, then both statements are false.

We can use logic to obtain several important variants of our implications that might be useful. A negated implication just negates all the statements that are part of the implication. So the negation of our implication becomes $\neg(A \wedge B) \Rightarrow \neg D$, which by De Morgan's law is $(\neg A) \vee (\neg B) \Rightarrow \neg D$. Even when the statement is true, the negation might not be. Having two democracies may mean you're at peace (for the sake of this argument), but letting at least one of them not be a democracy does not automatically imply war.

The **converse** of an implication switches a necessary statement to a sufficient one, or vice versa. Thus the converse of A and $B \Rightarrow D$ is A and $B \Leftarrow D$ or $D \Rightarrow A$ and B. As noted above, just because an implication is true does not mean the converse is true—something may be necessary without being sufficient. However, negating the converse, called taking the contrapositive, *does* always yield a true statement. The **contrapositive** of our implication is $(\neg A) \vee (\neg B) \Leftarrow \neg D$, or, as it's more typically written, $\neg D \Rightarrow (\neg A) \vee (\neg B)$. If a pair (dyad) of democracies never experiences war, then having a war (the opposite of peace) means that at least one of the pair is not a democracy.

Okay, back to our proof. Proofs are sometimes classed into broad groups of direct and indirect proofs. **Direct proofs** use deduction to string together series of true statements, starting with the assumptions and ending with the conclusion. In addition to the construction of a string of arguments, direct proofs commonly observed in formal theory include proof by exhaustion, construction, and induction. Let us see briefly how these work, starting with a **general deductive proof**.

Let A be the statement that $x \in \mathbb{Z}$ is even, and B be the statement that $y \in \mathbb{Z}$ is even, and D, which we're trying to prove, be the statement that the product xy is even. Well, if x and y are even (our assumptions), then they can be written as $x = 2r$ and $y = 2s$ for some $r, s \in \mathbb{Z}$. (Here we've used the definition of even.) In this case, we can write $xy = (2r)(2s) = 4rs$, which is our new statement C. Since $4rs = 2(2rs)$, xy is even (again using the definition of even), thus proving D. Now we know that the product of any two even integers is also even, and we could use this knowledge in further, more complex proofs.

Proof by exhaustion is similar, save that you also break up the problem into exhaustive cases and prove that your statement is true for each case. This comes up often in game theory as there will be different regions of the parameter space that may behave differently and admit different solutions. (The parameter space

[24]See http://en.wikipedia.org/wiki/De_Morgan_laws.

is the space, in the sense of a set with a measure, spanned by the parameters. We will discuss this concept more in Part III of the book.)

Proof by construction is similarly straightforward, and can be useful when trying to show something like existence: if you can construct an example of something, then it exists.

Proof by induction is a bit different and merits its own example. It is generally useful when you would like to prove something about a sequence (we cover sequences in Chapter 4) or a sequence of statements. It consists of three parts. First, one proves the base case, which in this example is the first element in the sequence. Second, one assumes that the statement is true for some n (the inductive hypothesis). Third, one proves that the statement is true for $n+1$ as well (the inductive step). Thus, since the base case is true and one can always go one further in the sequence and have the statements remain true, the entire sequence of statements is true.[25] Let's see how this works with an example: show that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. We basically need to show this is true for each n, but since they occur in sequence, we'll use induction rather than exhaustion (which wouldn't be appropriate, given that the sequence is infinite anyway). First we try the base case, which is for $n = 1$. We can check this: $\sum_{i=1}^{1} i = 1 = \frac{1(2)}{2} = \frac{1(1+1)}{2}$. So the base case is true. Now we assume, somewhat counterintuitively, the statement that we're trying to prove: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Finally, we show it remains true for $n+1$, so we need to prove that $\sum_{i=1}^{n+1} i = \frac{(n+1((n+1)+1)}{2}$, where we've replaced n in the right-hand side of the statement we're trying to prove with $n+1$. The sum in the left-hand side of this is $\sum_{i=1}^{n} i + (n+1)$, where we've just split the sum into two pieces. The first piece equals $\frac{n(n+1)}{2}$ by step two in our proof. So now we have $\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$. This is just what we needed to show, so the $n+1$ inductive step is true, and we've proved the statement.

Indirect proofs, in contrast, tend to show that something must be true because all other possibilities are not. **Proof by counterexample** and **proof by contradiction** both fall into this category. Counterexamples are straightforward. If the statement is that $A \wedge B \Rightarrow D$ and A and B are both true, then a single counterexample of $\neg D$ is sufficient to disprove the proposition. Proof by contradiction has a similar intent, but instead of finding a counterexample one starts by assuming the statement one is trying to prove is actually false, and then showing that this implies a contradiction. This proves the proposition because if it cannot be false, then it must be true. Although it may seem counterintuitive, proof by contradiction is perhaps the most common type of proof, and is usually worth trying first. Proving the contrapositive, since it indirectly

[25]Though this method of proof is called mathematical induction, it's important to note that it is a deductive method of theory building, not an inductive one. That is, it involves making assumptions and deducing conclusions from these, not stating conclusions derived from a series of statements that may only be probabilistically linked to the conclusion, as in inductive reasoning.

also proves the statement, as they are equivalent, is sometimes also considered an indirect proof, though it seems pretty direct to us.

1.8 EXERCISES

1.8.1 Constants and Variables and Levels of Measurement

1. Identify whether each of the following is a constant or a variable:
   a) Party identification of delegates at a political convention.
   b) War participation of the Great Powers.
   c) Voting record of members of Congress relative to the stated position of the president.
   d) Revolutions in France, Russia, China, Iran, and Nicaragua.
   e) An individual voter's vote choice in the 1992 presidential election.
   f) An individual voter's vote choice in the 1960–1992 presidential elections.
   g) Vote choice in the 1992 presidential election.
2. Identify whether each of the following is a variable or a value of a variable:
   a) The Tonkin Gulf Crisis.
   b) Party identification.
   c) Middle income.
   d) Exports as a percentage of GDP.
   e) Republican.
   f) Female.
   g) Veto.
   h) Ethnic fractionalization.
   i) International crisis.
3. Identify whether each of the following indicators is measured at a nominal, ordinal, interval, or ratio level. Note also whether each is a discrete or a continuous measure:
   a) Highest level of education as (1) some high school, (2) high school graduate, (3) some college, (4) college graduate, (5) postgraduate.
   b) Annual income.
   c) State welfare expenditures, measured in millions of dollars.
   d) Vote choice among Bush, Clinton, and Perot.
   e) Absence or presence of a militarized interstate dispute.
   f) Military personnel, measured in 1,000s of persons.
   g) The number of wars in which countries have participated.

1.8.2 Sets, Operators, and Proofs

4. As a brief illustration of one use of set theory, consider the following question: given three parties in a legislature with a supermajority rule required to pass a bill, what is the likely outcome of a given session? We can use set theory and some rational choice assumptions to get a pretty good handle on that question. Assume that no party has enough seats to pass the bill by itself and that all three parties prefer some outcome other than the status quo. For concreteness, let's define two dimensions over which to define policy: guns (i.e., defense spending) and butter (i.e., health, education, and welfare spending). We can now create a two-dimensional space where spending on guns is plotted on the vertical axis and spending on butter is plotted on the horizontal axis. Take out a sheet of paper and draw this. Let the axes range from 0% of the budget, marked where the axes intersect, to 100% of the budget, marked as the maximum value on each axis. Connect the two maximum values with a straight line. You now have a triangle, and the legislature cannot go outside the triangle: the line you just drew represents spending the entire budget on some mix of guns and butter. Let's assume that the legislators want to spend some money on non-guns and non-butter, and thus both parties' most preferred combination of guns and butters is somewhere inside the budget constraint. Pick some point inside the budget constraint and mark it as the status quo. Now select a most preferred combination for each party and mark each as Party 1, Party 2, and Party 3. Finally, pick a fifth point and label it a bill. Make a conjecture on whether the bill will pass or whether the status quo will be sustained. (For now this is just a conjecture, but we'll return to this in the exercises to Chapter 3, so save your answer.)

5. Let $A = \{1, 5, 10\}$ and $B = \{1, 2, \ldots, 10\}$.
   a) Is $A \subset B$, $B \subset A$, both, or neither?
   b) What is $A \cup B$?
   c) What is $A \cap B$?
   d) Partition B into two sets, A and everything else. Call everything else C. What is C?
   e) What is $A \cup C$?
   f) What is $A \cap C$?

6. Write an element of the Cartesian product $[0, 1] \times (1, 2)$.

7. Prove that $\sqrt{2}$ is an irrational number. That is, show that it cannot be written as the ratio of two integers, p and q.

8. Prove that the sum of any two even numbers is even, the sum of any two odd numbers is even, and the sum of any odd number with any even number is odd.