

# Contents

|          |   |           |
|----------|---|-----------|
| <b>1</b> | <b>Introduction</b>                                   | <b>1</b>  |
| 1.1      | Math versus Physics                                   | 1         |
| 1.2      | What This Book Is About                               | 2         |
| 1.3      | A Physical versus a Mathematical Solution: An Example | 6         |
| 1.4      | Acknowledgments                                       | 8         |
| <b>2</b> | <b>The Pythagorean Theorem</b>                        | <b>9</b>  |
| 2.1      | Introduction  | 9         |
| 2.2      | The “Fish Tank” Proof of the Pythagorean Theorem      | 9         |
| 2.3      | Converting a Physical Argument into a Rigorous Proof  | 12        |
| 2.4      | The Fundamental Theorem of Calculus                   | 14        |
| 2.5      | The Determinant by Sweeping                           | 15        |
| 2.6      | The Pythagorean Theorem by Rotation                   | 16        |
| 2.7      | Still Water Runs Deep                                 | 17        |
| 2.8      | A Three-Dimensional Pythagorean Theorem               | 19        |
| 2.9      | A Surprising Equilibrium                              | 21        |
| 2.10     | Pythagorean Theorem by Springs                        | 22        |
| 2.11     | More Geometry with Springs                            | 23        |
| 2.12     | A Kinetic Energy Proof: Pythagoras on Ice             | 24        |
| 2.13     | Pythagoras and Einstein?                              | 25        |
| <b>3</b> | <b>Minima and Maxima</b>                              | <b>27</b> |
| 3.1      | The Optical Property of Ellipses                      | 28        |
| 3.2      | More about the Optical Property                       | 31        |
| 3.3      | Linear Regression (The Best Fit) via Springs          | 31        |
| 3.4      | The Polygon of Least Area                             | 34        |
| 3.5      | The Pyramid of Least Volume                           | 36        |
| 3.6      | A Theorem on Centroids                                | 39        |
| 3.7      | An Isoperimetric Problem                              | 40        |
| 3.8      | The Cheapest Can                                      | 44        |
| 3.9      | The Cheapest Pot                                      | 47        |

|          |  |           |
|----------|--|-----------|
| 3.10     | The Best Spot in a Drive-In Theater  | 48        |
| 3.11     | The Inscribed Angle  | 51        |
| 3.12     | Fermat's Principle and Snell's Law   | 52        |
| 3.13     | Saving a Drowning Victim by Fermat's Principle                                 | 57        |
| 3.14     | The Least Sum of Squares to a Point  | 59        |
| 3.15     | Why Does a Triangle Balance on the Point of<br>Intersection of the Medians?    | 60        |
| 3.16     | The Least Sum of Distances to Four Points in Space                             | 61        |
| 3.17     | Shortest Distance to the Sides of an Angle                                     | 63        |
| 3.18     | The Shortest Segment through a Point   | 64        |
| 3.19     | Maneuvering a Ladder   | 65        |
| 3.20     | The Most Capacious Paper Cup   | 67        |
| 3.21     | Minimal-Perimeter Triangles  | 69        |
| 3.22     | An Ellipse in the Corner   | 72        |
| 3.23     | Problems   | 74        |
| <b>4</b> | <b>Inequalities by Electric Shorting</b>                                       | <b>76</b> |
| 4.1      | Introduction   | 76        |
| 4.2      | The Arithmetic Mean Is Greater than the Geometric<br>Mean by Throwing a Switch | 78        |
| 4.3      | Arithmetic Mean $\geq$ Harmonic Mean for $n$ Numbers                           | 80        |
| 4.4      | Does Any Short Decrease Resistance?  | 81        |
| 4.5      | Problems   | 83        |
| <b>5</b> | <b>Center of Mass: Proofs and Solutions</b>                                    | <b>84</b> |
| 5.1      | Introduction   | 84        |
| 5.2      | Center of Mass of a Semicircle by Conservation of<br>Energy                    | 85        |
| 5.3      | Center of Mass of a Half-Disk (Half-Pizza)                                     | 87        |
| 5.4      | Center of Mass of a Hanging Chain  | 88        |
| 5.5      | Pappus's Centroid Theorems   | 89        |
| 5.6      | Ceva's Theorem   | 92        |
| 5.7      | Three Applications of Ceva's Theorem   | 94        |
| 5.8      | Problems   | 96        |
| <b>6</b> | <b>Geometry and Motion</b>   | <b>99</b> |
| 6.1      | Area between the Tracks of a Bike  | 99        |
| 6.2      | An Equal-Volumes Theorem   | 101       |
| 6.3      | How Much Gold Is in a Wedding Ring?  | 102       |
| 6.4      | The Fastest Descent  | 104       |

|           |   |            |
|-----------|---|------------|
| 6.5       | Finding $\frac{d}{dt} \sin t$ and $\frac{d}{dt} \cos t$ by Rotation   | 106        |
| 6.6       | Problems  | 108        |
| <b>7</b>  | <b>Computing Integrals Using Mechanics</b>                            | <b>109</b> |
| 7.1       | Computing $\int_0^1 \frac{x \, dx}{\sqrt{1-x^2}}$ by Lifting a Weight | 109        |
| 7.2       | Computing $\int_0^x \sin t \, dt$ with a Pendulum                     | 111        |
| 7.3       | A Fluid Proof of Green's Theorem                                      | 112        |
| <b>8</b>  | <b>The Euler-Lagrange Equation via Stretched Springs</b>              | <b>115</b> |
| 8.1       | Some Background on the Euler-Lagrange Equation                        | 115        |
| 8.2       | A Mechanical Interpretation of the Euler-Lagrange Equation            | 117        |
| 8.3       | A Derivation of the Euler-Lagrange Equation                           | 118        |
| 8.4       | Energy Conservation by Sliding a Spring                               | 119        |
| <b>9</b>  | <b>Lenses, Telescopes, and Hamiltonian Mechanics</b>                  | <b>120</b> |
| 9.1       | Area-Preserving Mappings of the Plane: Examples                       | 121        |
| 9.2       | Mechanics and Maps  | 121        |
| 9.3       | A (Literally!) Hand-Waving "Proof" of Area Preservation               | 123        |
| 9.4       | The Generating Function   | 124        |
| 9.5       | A Table of Analogies between Mechanics and Analysis                   | 125        |
| 9.6       | "The Uncertainty Principle"   | 126        |
| 9.7       | Area Preservation in Optics   | 126        |
| 9.8       | Telescopes and Area Preservation                                      | 129        |
| 9.9       | Problems  | 131        |
| <b>10</b> | <b>A Bicycle Wheel and the Gauss-Bonnet Theorem</b>                   | <b>133</b> |
| 10.1      | Introduction  | 133        |
| 10.2      | The Dual-Cones Theorem  | 135        |
| 10.3      | The Gauss-Bonnet Formula Formulation and Background                   | 138        |
| 10.4      | The Gauss-Bonnet Formula by Mechanics                                 | 142        |
| 10.5      | A Bicycle Wheel and the Dual Cones                                    | 143        |
| 10.6      | The Area of a Country   | 146        |
| <b>11</b> | <b>Complex Variables Made Simple(r)</b>                               | <b>148</b> |
| 11.1      | Introduction  | 148        |
| 11.2      | How a Complex Number Could Have Been Invented                         | 149        |

|                                      |  |     |
|--------------------------------------|--|-----|
| 11.3                                 | Functions as Ideal Fluid Flows             | 150 |
| 11.4                                 | A Physical Meaning of the Complex Integral | 153 |
| 11.5                                 | The Cauchy Integral Formula via Fluid Flow | 154 |
| 11.6                                 | Heat Flow and Analytic Functions           | 156 |
| 11.7                                 | Riemann Mapping by Heat Flow               | 157 |
| 11.8                                 | Euler's Sum via Fluid Flow                 | 159 |
| <b>Appendix. Physical Background</b> |  | 161 |
| A.1                                  | Springs                                    | 161 |
| A.2                                  | Soap Films                                 | 162 |
| A.3                                  | Compressed Gas                             | 164 |
| A.4                                  | Vacuum                                     | 165 |
| A.5                                  | Torque                                     | 165 |
| A.6                                  | The Equilibrium of a Rigid Body            | 166 |
| A.7                                  | Angular Momentum                           | 167 |
| A.8                                  | The Center of Mass                         | 169 |
| A.9                                  | The Moment of Inertia                      | 170 |
| A.10                                 | Current                                    | 172 |
| A.11                                 | Voltage                                    | 172 |
| A.12                                 | Kirchhoff's Laws                           | 173 |
| A.13                                 | Resistance and Ohm's Law                   | 174 |
| A.14                                 | Resistors in Parallel                      | 174 |
| A.15                                 | Resistors in Series                        | 175 |
| A.16                                 | Power Dissipated in a Resistor             | 176 |
| A.17                                 | Capacitors and Capacitance                 | 176 |
| A.18                                 | The Inductance: Inertia of the Current     | 177 |
| A.19                                 | An Electrical-Plumbing Analogy             | 179 |
| A.20                                 | Problems                                   | 181 |
| <b>Bibliography</b>                  |  | 183 |
| <b>Index</b>                         |  | 185 |