LECTURE 7

Production

The Producer: The Basic Model

We will turn now to a very brief discussion of the basic concepts in classic producer theory. As this involves only a few new abstract ideas, we make do with a short introduction to the concepts and implicit assumptions.

Usually we view the firm as a special type of rational decision maker. Recall that when discussing the consumer we imposed a strong structure on the choice sets but few constraints on the preferences. In contrast, classic producer theory assigns the producer a highly structured target function but fewer constraints on the choice sets.

Let $1, \ldots, K$ be commodities. The producer's choice will be made from subsets of the "grand set," which will be taken to be a K-dimensional Euclidean space. A vector z in this space is interpreted as a production combination; positive components in z are interpreted as outputs and negative components as inputs.

Producer's Preferences

It is assumed that the goal of the *producer (firm)* is to maximize profits. The competitive producer faces a vector of prices $p = (p_k)_{k=1,\ldots,K}$ (for inputs and outputs). If he chooses z, his profits (revenues minus costs) will be $pz = \sum_{k=1}^{k} p_k \cdot z_k$. In other words, it is assumed that his preferences over any set of possible production combinations are represented by the utility function pz.

Technology

A producer's choice set is called a *technology* and specifies the production constraints.

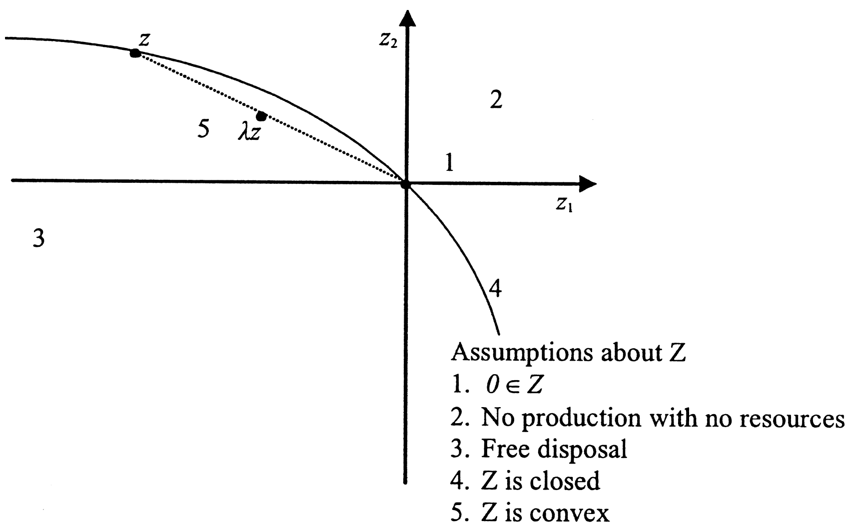

Figure 7.1

The following restrictions are usually placed on the technology space (fig. 7.1):

1. $0 \in Z$ (this is interpreted to mean that the producer can remain "idle").
2. There is no $z \in Z \cap \Re^K_+$ besides the vector 0 (no production with no resources).
3. *Free disposal*: if $z \in Z$ and $z' \leq z$, then $z' \in Z$ (nothing prevents the producer from being inefficient in the sense that it uses more resources than necessary to produce the same amount of commodities).
4. Z is a closed set.
5. Z is a convex set. (This assumption embodies decreasing marginal productivity. Together with the assumption that $0 \in Z$, it implies *nonincreasing returns to scale*: if $z \in Z$, then for every $\lambda < 1$, $\lambda z \in Z$.)

The Production Function

Consider the case in which commodity K is produced from commodities $1, \ldots, K-1$, that is, for all $z \in Z$, $z_K \geq 0$ and for all $k \neq K$, $z_k \leq 0$. In this case, another intuitive way of specifying the technological constraints on the producer is by a *production function* which specifies, for any positive vector of inputs $v \in R_+^{K-1}$, the maximum amount of commodity K that can be produced.

If we start from technology Z, we can derive the production function by defining

$$f(v) = max\{x | \ (-v, x) \in Z\}.$$

Alternatively, if we start from the production function f, we can derive the "technology" by defining $Z(f) = \{(-w, x) | \ x \leq y$ and $w \geq v$ for some $y = f(v)\}$. If the function f satisfies the assumptions of $f(0) = 0$, continuity, and concavity, then $Z(f)$ satisfies the above assumptions.

The Supply Function

We will now discuss the producer's behavior. The producer's problem is defined as $max_{z \in Z} pz$.

The existence of a unique solution for the producer problem requires some additional assumptions such as that Z be *bounded from above* (that is, there is some bound B such that $B \geq z_k$ for any $z \in Z$) and that Z be *strictly convex* (that is, if z and z' are in Z, then the combination $\lambda z + (1 - \lambda)z'$ is an internal point in Z for any $1 > \lambda > 0$).

When the producer's problem has a unique solution, we denote it by $z(p)$. We refer to the function $z(p)$ as the *supply function*. Note that it specifies both the producer's supply of outputs and its demand for inputs. The *profit function* $\pi(p) = max_{z \in Z} pz$ is analogous to the indirect utility function in the consumer model.

Recall that when discussing the consumer, we specified the preferences and we described his behavior as making a choice from a budget set that had been determined by prices. The consumer's behavior (demand) specified the dependence of his consumption on prices. In the case of the producer, we specify the technology and we describe his behavior as maximizing a profit function which is determined by prices. The producer's behavior (supply) specifies the dependence of output and the consumption of inputs on prices.

In the case of the producer, preferences are linear and the constraint is a convex set, whereas in the consumer model the constraint is a linear inequality and preferences are convex. The structure (continuity and convexity) is imposed on the producer's choice set

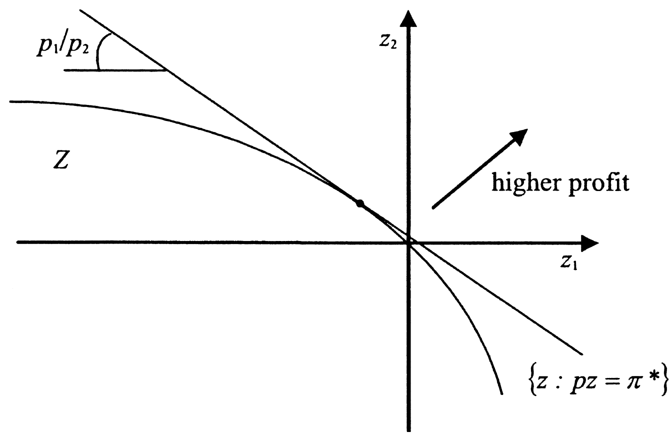

Figure 7.2
Profit maximization.

and on the consumer's preferences. Thus, the producer's problem is similar to that of the consumer's *dual* problem. (See fig. 7.2.)

Properties of the Supply and Profit Functions

Let us turn to some of the properties of the supply and profit functions. The properties and their proofs are analogous to the properties and proofs in the discussion of the consumer's dual problem.

Supply Function

1. $z(\lambda p) = z(p)$. (The producer's preferences are induced by the price vector p and are identical to those induced by the price vector λp.)
2. z is continuous.
3. Assume the supply function is well defined. If $z(p) \neq z(p')$, we have $(p - p')[z(p) - z(p')] = p[z(p) - z(p')] + p'[z(p') - z(p)] > 0$. In particular, if (only) the kth price increases, z_k increases; that is, if k is an output ($z_k > 0$), the supply of k increases; and if k is an input ($z_k < 0$), the demand for k decreases. Note that this result, called *the law of supply*, applies to the standard supply

function (unlike the law of demand, which was applied to the compensated demand function).

Profit Function

1. $\pi(\lambda p) = \lambda\pi(p)$. (Follows from $z(\lambda p) = z(p)$.)
2. π is continuous. (Follows from the continuity of the supply function.)
3. π is convex. (For all p,p' and λ, if z maximizes profits with $\lambda p + (1-\lambda)p'$ then $\pi(\lambda p + (1-\lambda)p') = \lambda pz + (1-\lambda)p'z \le \lambda\pi(p) + (1-\lambda)\pi(p')$.)
4. *Hotelling's lemma*: For any vector p^*, $\pi(p) \ge pz(p^*)$ for all p. Therefore, the hyperplane $\{(p,\pi) \mid \pi = pz(p^*)\}$ is tangent to $\{(p,\pi) \mid \pi = \pi(p)\}$, the graph of function π at the point (p^*, $\pi(p^*)$). If π is differentiable, then $d\pi/dp_k(p^*) = z_k(p^*)$.
5. From Hotelling's lemma it follows that if π is differentiable, then $dz_j/dp_k(p^*) = dz_k/dp_j(p^*)$.

The Cost Function

If we are only interested in the firm's behavior in the output market (but not in the input markets), it is sufficient to specify the costs associated with the production of any combination of outputs as opposed to the details of the production function. Thus, for a producer of the commodities $L+1,\ldots,K$, we define $c(p,y)$ to be the minimal cost associated with the production of the combination $y \in \Re_+^{K-L}$ given the price vector $p \in \Re_{++}^L$ of the input commodities $1,\ldots,L$. If the model's primitive is a technology Z, we have $c(p,y) = min_a\{pa| \ (-a,y) \in Z\}$. (See fig. 7.3.)

Discussion

In the conventional economic approach we allow the consumer "general" preferences but restrict producer goals to profit maximization. Thus, a consumer who consumes commodities in order to destroy his health is within the scope of our discussion, while a producer who cares about the welfare of his workers or has in mind a target other than profit maximization is not. This is odd since there

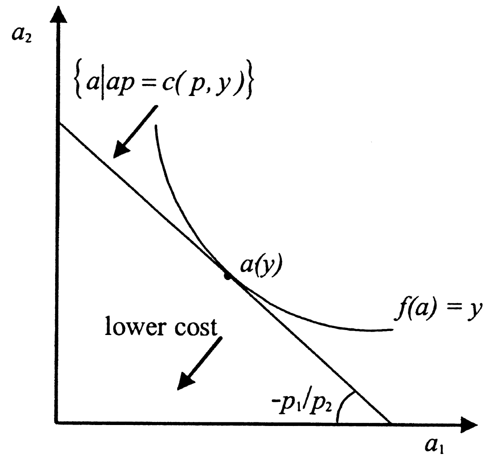

Figure 7.3
Cost Minimization.

are various empirically plausible alternative targets for a producer. For example, it seems that the goal of some producers is to increase production subject to not incurring a loss. Some firms are managed so as to increase the managers' salaries with less regard for the level of profits.

I sometimes wonder why this difference exists between the generality of consumer preferences and the narrowness of the producer objectives. It might be that this is simply the result of mathematical convenience. I don't think this is a result of an ideological conspiracy. But, by making profit maximization the key assumption about producer behavior, do we not run the risk that students will interpret it to be the exclusive normative criterion guiding a firm's actions?

Bibliographic Notes

Recommended readings: Kreps 1990, Chapter 8; Mas-Colell et al. 1995, Chapter 5, A–D,G.

The material in this lecture (apart from the discussion) is standard and can be found in any microeconomics textbook. Debreu (1959) is an excellent source.

Problem Set 7

Problem 1. (*Easy*)
Assume that technology Z and the production function f describe the same producer who produces commodity K using inputs $1, \ldots, K-1$. Show that Z is a convex set if and only if f is a concave function.

Problem 2. (*Boring*)
Here is a very standard exercise (if you have not done it in the past, it may be "fun" to do it "once in a lifetime"): Calculate the supply function $z(p)$ and the profit function $\pi(p)$ for each of the following production functions:

- $f(a) = a_1^\alpha$ for $\alpha \leq 1$.
- $g(a) = \alpha a_1 + \beta a_2$ for $\alpha > 0$ and $\beta > 0$.
- $h(a) = min\{a_1, a_2\}$.
- $i(a) = (a_1^\alpha + a_2^\alpha)^{1/\alpha}$ for $\alpha \leq 1$.

Problem 3. (*Easy*)
Consider a producer who uses L inputs to produce $K - L$ outputs. Show the following:

- $C(\lambda w, y) = \lambda C(w, y)$.
- C is nondecreasing in any input price w_k.
- C is concave in w.
- Shepherd's lemma: If C is differentiable, $dC/dw_k(w, y) = a_k(w, y)$ (the kth input commodity).
- If C is twice continuously differentiable, then for any two commodities j and k $da_k/dw_j(w, y) = da_j/dw_k(w, y)$.

Problem 4. (*Moderately difficult.* Based on Radner (1993).)
It is usually assumed that the cost function C is convex in the output vector. Much of the research on production has been aimed at investigating whether the convexity assumptions can be induced in more detailed models. Convexity often fails when the product is related to the gathering of information or data processing.

Consider, for example, a firm conducting a telephone survey immediately following a TV program. Its goal is to collect information about as many viewers as possible within 4 units of time. The wage paid to each worker is w (even when he is idle). In one unit of time, a worker can talk to one respondent or be involved in the transfer of information to or from exactly one colleague. At the end of the 4 units of time, the collected information must be in the hands of one colleague (who will announce the results).

- What is the firm's product?
- Calculate the cost function and examine its convexity.

Problem 5. (*Moderately difficult*)
Consider a firm producing one commodity using L inputs, which maximizes production subject to the constraint of nonnegative profits. Show some interesting properties of such a firm's behavior.

Problem 6. (*Standard*)
An event that could have happened with probability 0.5 either did or did not occur. A firm has to provide a report of the form "the event occurred" or "the event did not occur." The quality of the report, q (the firm's product), is the probability that the report is correct. The firm employs k experts (input) to prepare the report. Each of them receives an independent signal whether the event occurred or not, which is correct with probability $1 > p > 0.5$.

- Calculate the production function $q = f(k)$ for (at least) $k = 1, 2, 3, \ldots$.
- We say that a "discrete" production function is concave if the sequence of marginal product is nonincreasing. Is the firm's production function concave?

  Assume that the firm needs information in order to make a decision whether to invest amount m that will yield revenue αm if the event occurs, and 0 otherwise; the decision maker chooses k in order to maximize expected profits. Assume that the wage of each worker is w.
- Explain why it is true that if f is concave, the firm chooses k^* so that the k^*th worker is the last one for whom marginal revenue exceeds the cost of a single worker.
- Is this conclusion true in our case?