LECTURE 6

Choice over Budget Sets and the Dual Problem

Indirect Preferences

As an introduction to the first topic in this lecture, let us go back to the general choice function concept discussed in Lecture 3. Having in mind a preference relation \( \succsim \) on a set \( X \), the decision maker may want to construct a preference relation over the set \( D \), the domain of his choice function. When assessing a choice problem in \( D \), the decision maker may then ask himself which alternative he would choose if he had to choose from that set. The “rational” decision maker will prefer a set \( A \) over a set \( B \) if the alternative he intends to choose from \( A \) is preferable to that which he intends to choose from \( B \). This leads us to the definition of \( \succsim^* \), the indirect preferences induced from \( \succsim \):

\[
A \succsim^* B \text{ if } C_{\succsim}(A) \succsim C_{\succsim}(B).
\]

The definition of indirect preferences ignores some considerations that might be taken into account when comparing choice sets. Excluded are considerations such as, “I prefer \( A - \{b\} \) to \( A \) even though I intend to choose \( a \) in any case since I am afraid to make a mistake and choose \( b \)” or “I will choose \( a \) from \( A \) whether \( b \) is available or not. However, since I don’t want to have to reject \( b \), I prefer \( A - \{b\} \) to \( A \).”

Of course, if \( u \) represents \( \succsim \) and the choice function is well defined, \( v(A) = u(C(A)) \) represents \( \succsim^* \). We will refer to \( v \) as the indirect utility function.

Finally, note that sometimes (depending on the set \( D \)) one can reconstruct the choice function \( C_{\succsim}(A) \) from the indirect preferences \( \succsim^* \). For example, if \( a \in A \) and \( A \succsim^* A - \{a\} \), then \( C_{\succsim}(A) = a \).
The Consumer’s Indirect Utility Function

Let us return to the consumer who chooses bundles from budget sets. He might be interested in formulating indirect preferences when choosing a market to live in or when assessing the effect of tax reforms (which cause changes in prices or wealth) on his welfare. Since a budget set is characterized by the \( K + 1 \) parameters \((p, w)\), the above approach leads to the definition of the indirect preferences \( \succeq^{*} \) on the set \( \mathbb{R}^{K+1} \) as \((p, w) \succeq^{*} (p', w')\) if \( x(p, w) \succeq x(p', w')\). Interpreting \( p \) in the standard manner, as prices prevailing in the market, defining indirect preferences in this way precludes considerations such as, “I prefer to live in an area where alcohol is very expensive even though I drink a lot”.

The following are basic properties of the indirect preferences \( \succeq^{*} \), induced from the preferences \( \succeq \) on the bundle space. The first is an “invariance to presentation” property, which follows from the definition of indirect preferences independently of the properties of the consumer’s preferences. The other three properties depend on the following characteristics of the consumer’s preferences: monotonicity (using the partial orderings on the bundle space), continuity (using the topological structure), and convexity (using the algebraic structure).

1. \((\lambda p, \lambda w) \succeq (p, w)\) (this follows from \(x(\lambda p, \lambda w) = x(p, w)\)).
2. \( \succeq^{*} \) is nonincreasing in \( p \) and increasing in \( w \) (reducing the scope of the choice is never beneficial, and additional wealth makes it possible to consume bundles containing more of all commodities).
3. If the preference relation \( \succeq \) is continuous, then so is \( \succeq^{*} \), and there is a continuous function \( v \) representing \( \succeq^{*} \). (The function \( x(p, w) \) is continuous. Let \( u \) be a continuous function representing \( \succeq \); then \( u(x(p, w)) \) is a continuous utility representation of \( \succeq^{*} \) and thus \( \succeq^{*} \) is continuous.)
4. If \((p^1, w^1) \succeq^{*} (p^2, w^2)\), then \((p^1, w^1) \succeq^{*} (\lambda p^1 + (1 - \lambda)p^2, \lambda w^1 + (1 - \lambda)w^2)\) for all \( 1 \geq \lambda \geq 0 \). (See fig. 6.1.) (Thus, if \( v \) represents \( \succeq^{*} \), then it is quasi-convex, that is, the set \( \{(p, w) | v(p, w) \leq v(p^*, w^*)\} \) is convex). To see this, let \( z \) be the best bundle in the budget set \( B(\lambda p^1 + (1 - \lambda)p^2, \lambda w^1 + (1 - \lambda)w^2)\). Then \( (\lambda p^1 + (1 - \lambda)p^2)z \leq \lambda w^1 + (1 - \lambda)w^2 \) and therefore \( p^1z \leq w^1 \) or \( p^2z \leq w^2 \). Thus \( z \in B(p^1, w^1) \) or \( z \in B(p^2, w^2) \) and then \( x(p^1, w^1) \succeq z \).
or \( x(p^2, w^2) \succeq z \). From \( x(p^2, w^2) \succeq x(p^1, w^1) \) it follows that \( x(p^1, w^1) \succeq z \).

**Example:**

In the single commodity case, each \( \succ^{*} \)-indifference curve is a ray. Assuming monotonicity of \( \succ \), the slope of an indifference curve through \((p_1, w)\) is \( x_1(p_1, w) = w/p_1 \).

**Roy’s Equality**

We will now look at a method of deriving the consumer demand function from indirect preferences. The basic idea is that starting from a budget set \((p^*, w^*)\), any change of \( \varepsilon \) in the price of commodity \( k \) combined with a change of \( \varepsilon x_k(p^*, w^*) \) in wealth cannot be undesirable. Thus, when indirect preferences are differentiable, the tangent to the indifference curve of the indirect preferences through \((p^*, w^*)\) gives the demand for that budget set.
Choice over Budget Sets and the Dual Problem

Claim:

Assume that the demand function satisfies Walras's law. Let $H = \{(p, w) | (x(p^*, w^*), -1)(p, w) = 0\}$ for some $(p^*, w^*)$. The hyperplane $H$ is tangent to the $\succsim^*$ indifference curve through $(p^*, w^*)$.

Proof:

Of course $(p^*, w^*) \in H$. For any $(p, w) \in H$, the bundle $x(p^*, w^*) \in B(p, w)$. Hence $x(p, w) \succsim x(p^*, w^*)$, and thus $(p, w) \succsim^* (p^*, w^*)$.

In the case in which $\succsim^*$ is represented by differentiable $v$,

$$H = \{(p, w) | (\partial v/\partial p_1(p^*, w^*), ..., \partial v/\partial p_K(p^*, w^*), \partial v/\partial w(p^*, w^*))(p - p^*, w - w^*) = 0\}.$$ From the above claim and since $w^* = p^*x(p^*, w^*)$ we have also

$$H = \{(p, w) | (x(p^*, w^*), -1)(p - p^*, w - w^*) = 0\}.$$ Therefore, the vector

$$(\partial v/\partial p_1(p^*, w^*), ..., \partial v/\partial p_K(p^*, w^*), \partial v/\partial w(p^*, w^*))$$ is proportional to the vector

$$(x_1(p^*, w^*), ..., x_K(p^*, w^*), -1),$$ and thus,

$$-[\partial v/\partial p_k(p^*, w^*)]/[\partial v/\partial w(p^*, w^*)] = x_k(p^*, w^*).$$

Dual Problems

In normal discourse, we consider the following two statements to be equivalent:

1. The maximal distance a turtle can travel in 1 day is 1 km.
2. The minimal time it takes a turtle to travel 1 km is 1 day.

This equivalence actually relies on two “hidden” assumptions:

a. For (1) to imply (2) we need to assume the turtle travels a positive distance in any period of time. Contrast this with the case in which the turtle’s speed is 2 km/day but, after half a day,
it must rest for half a day. Then, the maximal distance it can travel in 1 day is 1 km but it can travel this distance in only half a day.

b. For (2) to imply (1) we need to assume the turtle cannot “jump” a positive distance in zero time. Contrast this with the case in which the turtle’s speed is 1 km/day but after a day of traveling it can “jump” 1 km. Thus, it needs 1 day to travel 1 km but within 1 day it can travel 2 km.

The assumptions that in any positive interval of time the turtle can travel a positive distance and that the turtle cannot “jump” are sufficient for the equivalence of (1) and (2). Let \( M(t) \) be the maximal distance the turtle can travel in time \( t \). Assume that the function \( M \) is strictly increasing and continuous. Then, the statement, “The maximal distance a turtle can travel in \( t^* \) is \( x^* \)” is equivalent to the statement, “The minimal time it takes a turtle to travel \( x^* \) is \( t^* \).”

If the maximal distance that the turtle can pass within \( t^* \) is \( x^* \), and if it covers the distance \( x^* \) in \( t < t^* \) then, by the strict monotonicity of \( M \), the turtle would cover a distance larger than \( x^* \) in \( t^* \), a contradiction.

If it takes \( t^* \) for to the turtle to cover the distance \( x^* \) and if it passes the distance \( x > x^* \) in \( t^* \), then by the continuity of \( M \) at some \( t < t^* \) the turtle will already be beyond the distance \( x^* \), a contradiction.

The Dual Consumer Problem

Let \( u \) be a utility function that is continuous and monotonic. Applying the duality idea to the consumer problem, we compare the following pair of maximization problems:

**The prime problem** \( P(p, w^*) \)

Find a bundle maximizing utility given an expense level \( w^* \), that is,

\[
\max_{x} \{ u(x) \mid px \leq w^* \}.
\]
The dual problem \( D(p, u^*) \)

Find a bundle minimizing the expenses needed to obtain a level of utility \( u^* \), that is,

\[
\min_x \{px | u(x) \geq u^*\}.
\]

Claim:

1. If \( x^* \) is the solution to the problem \( P(p, w^*) \), then it is also the solution to the dual problem \( D(p, u(x^*)) \).
2. If \( x^* \) is a solution to the dual problem \( D(p, u^*) \), then it is also the solution to the problem \( P(p, px^*) \).

Proof:

1. If \( x^* \) is not a solution to the dual problem \( D(p, u(x^*)) \), then there exists a strictly cheaper bundle \( x \) for which \( u(x) \geq u(x^*) \). For some positive vector \( \epsilon \) (that is, \( \epsilon_k > 0 \) for all \( k \)), it still holds that \( p(x + \epsilon) < px^* \leq w \). By monotonicity \( u(x + \epsilon) > u(x) \geq u(x^*) \), contradicting the assumption that \( x^* \) is a solution to \( P(p, w^*) \).
2. If \( x^* \) is not a solution to the problem \( P(p, w^*) \), then there exists an \( x \) such that \( px \leq px^* \) and \( u(x) > u(x^*) \geq u^* \). By continuity, for some nonnegative vector \( \epsilon \neq 0 \), \( x - \epsilon \) is a bundle such that \( u(x - \epsilon) > u^* \) and \( p(x - \epsilon) < px^* \), contradicting the assumption that \( x^* \) is a solution to \( D(p, u^*) \).

The Hicksian Demand Function

Assume that the dual problem \( D(p, u) \) has a unique solution. This is the case, for example, if \( u \) represents strictly convex continuous preferences. The Hicksian demand function \( h(p, u) \) is the solution to \( D(p, u) \). This concept is analogous to the demand function in the prime problem.
Here are some properties of the Hicksian demand function:

1. \( h(\lambda p, u) = h(p, u) \). If \( x \) is a solution to the problem \( D(p, u) \), it is also a solution to the problem \( D(\lambda p, u) \). The function \( \lambda px \) is a positive linear transformation of \( px \); thus, the problem \( \min_x \{\lambda px| u(x) \geq u\} \) has the same solution as the problem \( \min_x \{px| u(x) \geq u\} \).

2. \( h_k(p, u) \) is nonincreasing in \( p_k \). Note that for every \( p'ph(p, u) \leq ph(p', u) \) since \( h(p', u) \) also satisfies the constraint of achieving a utility level of at least \( u \) and \( h(p, u) \) is the cheapest bundle satisfying the constraint. Similarly, \( p'h(p', u) \leq p'h(p, u) \). Thus,
   \[
   (p - p')(h(p, u) - h(p', u)) = p[h(p, u) - h(p', u)]
   + p'[h(p', u) - h(p, u)] \leq 0.
   \]
   When \( p - p' = (0, \ldots, \epsilon, \ldots, 0) \) we get that \( h_k(p, u) - h_k(p', u) \leq 0 \). Thus, increasing the price of commodity \( k \) has a nonpositive effect on Hicksian demand.

3. \( h(p, u) \) is continuous in \( p \) (verify!).

Define \( e(p, u) = ph(p, u) \) to be the expenditure function. This concept is analogous to the indirect utility function in the prime problem. Here are some properties of the expenditure function:

1. \( e(\lambda p, u) = \lambda e(p, u) \) (it follows from \( h(\lambda p, u) = h(p, u) \)).
2. \( e(p, u) \) is non-decreasing in \( p_k \) and strictly increasing in \( u \).
3. \( e(p, u) \) is continuous in \( p \) (this follows from the continuity of \( h(p, u) \)).
4. \( e(p, u) \) is concave in \( p \) (not only in \( p_k \)). To prove this, let \( x = h(\lambda p^1 + (1 - \lambda)p^2, u^*) \). Since \( u(x) = u^* \), \( e(p^1, u^*) \leq p^1x \); thus \( e(\lambda p^1 + (1 - \lambda)p^2, u^*) = (\lambda p^1 + (1 - \lambda)p^2)x \geq \lambda e(p^1, u^*) + (1 - \lambda)e(p^2, u^*) \).

**Claim (the Dual Roy’s Equality):**

The hyperplane \( H = \{(p, e)| e = ph(p^*, u^*)\} \) is tangent to the graph of the function \( e = e(p, u^*) \) at point \( p^* \).
Proof:

Since $ph(p^*, u^*) \geq ph(p, u^*)$ for all price vectors $p$, the hyperplane $H$ lies on one side of the graph of the function $e = ph(p, u^*)$ and intersects the graph at the point $(p^*, e(p^*, u^*))$.

Bibliographic Notes


Roy and Hicks are the sources for most of the material in this lecture. Specifically, the concept of the indirect utility function is due to Roy (1942); the concept of the expenditure function is due to Hicks (1946); and the concepts of consumer surplus used in Problem 6 are due to Hicks (1939). See also McKenzie (1957). For a full representation of the duality idea, see, for example, Varian (1984) and Diewert (1982).
Problem Set 6

Problem 1. (Easy)
In a world with two commodities, consider a consumer's preferences that are represented by the utility function \( u(x_1, x_2) = \min\{x_1, x_2\} \).

a. Calculate the consumer's demand function.
b. Verify that the preferences satisfy convexity.
c. Calculate the indirect utility function \( v(p, w) \).
d. Verify Roy's Equality.
e. Calculate the expenditure function \( e(p, u) \) and verify Dual Roy's Equality.

Problem 2. (Moderate)
Imagine that you are reading a paper in which the author uses the indirect utility function \( v(p_1, p_2, w) = w/p_1 + w/p_2 \). You suspect that the author's conclusions in the paper are the outcome of the "fact" that the function \( v \) is inconsistent with the model of the rational consumer. Take the following steps to make sure that this is not the case:

a. Use Roy’s Equality to derive the demand function.
b. Show that if demand is derived from a smooth utility function, then the indifference curve at the point \((x_1, x_2)\) has the slope \(-\sqrt{x_2}/\sqrt{x_1}\).
c. Construct a utility function with the property that the ratio of the partial derivatives at the bundle \((x_1, x_2)\) is \( \sqrt{x_2}/\sqrt{x_1} \).
d. Calculate the indirect utility function derived from this utility function. Do you arrive at the original \( v(p_1, p_2, w) \)? If not, can the original indirect utility function still be derived from another utility function satisfying the property in (c).

Problem 3. (Moderate)
A consumer with wealth \( w \) is interested in purchasing only one unit of one of the items included in a (finite) set \( A \). All items are indivisible. The consumer does not derive any "utility" from leftover wealth. The consumer evaluates commodity \( x \in A \) by the number \( V_x \) (where the value of not purchasing any of the goods is 0). The price of commodity \( x \in A \) is \( p_x > 0 \).

a. Formulate the consumer's problem.
b. Check the properties of the indirect utility function (homogeneity of
degree zero, monotonicity, continuity and quasi-convexity).
c. Calculate the indirect utility function for the case in which \( A = \{a, b\} \)
and \( V_a > V_b > 0 \)

**Problem 4. (Moderate)**
Show that if the utility function is continuous, then so is the Hicksian de-
mand function \( h(p, u) \).

**Problem 5. (Moderate)**
A commodity \( k \) is *Giffen* if the demand for the \( k \)-th good, \( x_k(p, w) \), is in-
creasing in \( p_k \). A commodity \( k \) is *inferior* if the demand for the commodity
decreases with wealth. Show that if a commodity \( k \) is Giffen in some neigh-
borhood of \((p, w)\), then \( k \) is inferior.

**Problem 6. (Moderate)**
One way to compare budget sets is by using the relation \( \succeq^* \) as defined in
the text. According to this approach, the comparison between \((p, w)\) and
\((p', w)\) is made by comparing two numbers \( u(x(p, w)) \) and \( u(x(p', w)) \), where
\( u \) is a utility function defined on the space of the bundles.

Following are two other approaches for making such comparisons using
“concrete terms.”
Define:

\[
CV(p, p', w) = w - e(p', u) = e(p, u) - e(p', u)
\]

where \( u = u(x(p, w)) \).

This is the answer to the question: What is the change in wealth that
would be equivalent, from the perspective of \((p, w)\), to the change in price
vectors from \( p \) to \( p' \)?

Define:

\[
EV(p, p', w) = e(p, u') - w = e(p, u) - e(p', u')
\]

where \( u' = u(x(p', w)) \).

This is the answer to the question: What is the change in wealth that
would be equivalent, from the perspective of \((p', w)\), to the change in price
vectors from \( p \) to \( p' \)?

Now, answer the following questions regarding a consumer in a two-
commodity world with a utility function \( u \):

a. For the case \( u(x_1, x_2) = x_1 + x_2 \), calculate the two “consumer surplus”
measures.
b. Explain why the two measures may give different values for some other utility functions.

c. Explain why the two measures are identical if the individual has quasi-linear preferences in the second commodity and in a domain where the two commodities are consumed in positive quantities.

d. Assume that the price of the second commodity is fixed and that the price vectors differ only in the price of the first commodity. What is the relation of the two measures to the “area below the demand function” (which is a standard third definition of consumer surplus)?