LECTURE 2

Utility

The Concept of Utility Representation

Think of examples of preferences. In the case of a small number of alternatives, we often describe a preference relation as a list arranged from best to worst. In some cases, the alternatives are grouped into a small number of categories and we describe the preferences on X by specifying the preferences on the set of categories. But, in my experience, most of the examples that come to mind are similar to: "I prefer the taller basketball player," "I prefer the more expensive present," "I prefer a teacher who gives higher grades," "I prefer the person who weighs less."

Common to all these examples is that they can naturally be specified by a statement of the form "$x \succsim y$ if $V(x) \geq V(y)$" (or $V(x) \leq V(y)$), where $V : X \to \Re$ is a function that attaches a real number to each element in the set of alternatives X. For example, the preferences stated by "I prefer the taller basketball player" can be expressed formally by: X is the set of all conceivable basketball players, and $V(x)$ is the height of player x.

Note that the statement $x \succsim y$ if $V(x) \geq V(y)$ always defines a preference relation since the relation $\geq$ on $\Re$ satisfies completeness and transitivity.

Even when the description of a preference relation does not involve a numerical evaluation, we are interested in an equivalent numerical representation. We say that *the function* $U : X \to \Re$ *represents the preference* $\succsim$ if for all x and $y \in X$, $x \succsim y$ if and only if $U(x) \geq U(y)$. If the function U represents the preference relation $\succsim$, we refer to it as a *utility function* and we say that $\succsim$ has a *utility representation*.

It is possible to avoid the notion of a utility representation and to "do economics" with the notion of preferences. Nevertheless, we usually use utility functions rather than preferences as a means of describing an economic agent's attitude toward alternatives, probably because we find it more convenient to talk about the maximization of a numerical function than of a preference relation.

Note that when defining a preference relation using a utility function, the function has an intuitive meaning that carries with it additional information. In contrast, when the utility function is formed in order to represent an existing preference relation, the utility function has no meaning other than that of representing a preference relation. Absolute numbers are meaningless in the latter case; only relative order has meaning. Indeed, if a preference relation has a utility representation, then it has an infinite number of such representations, as the following simple claim shows:

Claim:

If U represents $\succsim$, then for any strictly increasing function $f : \Re \to \Re$, the function $V(x) = f(U(x))$ represents $\succsim$ as well.

Proof:

$a \succsim b$
iff $U(a) \geq U(b)$ (since U represents $\succsim$)
iff $f(U(a)) \geq f(U(b))$ (since f is strictly increasing)
iff $V(a) \geq V(b)$.

Existence of a Utility Representation

If any preference relation could be represented by a utility function, then it would "grant a license" to use utility functions rather than preference relations with no loss of generality. Utility theory investigates the possibility of using a numerical function to represent a preference relation and the possibility of numerical representations carrying additional meanings (such as, a is preferred to b more than c is preferred to d).

We will now examine the basic question of "utility theory": Under what assumptions do utility representations exist?

Our first observation is quite trivial. When the set X is finite, there is always a utility representation. The detailed proof is presented here mainly to get into the habit of analytical precision. We start with a lemma regarding the existence of minimal elements (an element $a \in X$ is *minimal* if $a \precsim x$ for any $x \in X$).

Lemma:

In any finite set $A \subseteq X$ there is a minimal element (similarly, there is also a maximal element).

Proof:

By induction on the size of A. If A is a singleton, then by completeness its only element is minimal.

For the inductive step, let A be of cardinality $n+1$ and let $x \in A$. The set $A-\{x\}$ is of cardinality n and by the inductive assumption has a minimal element denoted by y. If $x \succsim y$, then y is minimal in A. If $y \succsim x$, then by transitivity $z \succsim x$ for all $z \in A-\{x\}$ and thus x is minimal.

Claim:

If $\succsim$ is a preference relation on a finite set X, then $\succsim$ has a utility representation with values being natural numbers.

Proof:

We will construct a sequence of sets inductively. Let X_1 be the subset of elements that are minimal in X. By the above lemma, X_1 is not empty. Assume we have constructed the sets $X_1, \ldots, X_k$. If $X = X_1 \cup X_2 \cup \ldots \cup X_k$ we are done. If not, define X_{k+1} to be the set of minimal elements in $X - X_1 - X_2 - \cdots - X_k$. By the lemma $X_{k+1} \neq \emptyset$. Since X is finite we must be done after at most $|X|$ steps. Define $U(x) = k$ if $x \in X_k$. Thus, $U(x)$ is the step number at which x is "eliminated." To verify that U represents $\succsim$, let $a \succsim b$. Then $b \notin X - X_1 - X_2 - \cdots - X_{U(a)}$ and thus $U(a) \geq U(b)$.

Without any further assumptions on the preferences, the existence of a utility representation is guaranteed when the set X is countable (recall that X is countable and infinite if there is a one-to-one function from the natural numbers to X, namely, it is possible to specify an enumeration of all its members $\{x_n\}_{n=1,2,\dots}$).

Claim:

If X is countable, then any preference relation on X has a utility representation with a range $(-1, 1)$.

Proof:

Let $\{x_n\}$ be an enumeration of all elements in X. We will construct the utility function inductively. Set $U(x_1) = 0$. Assume that you have completed the definition of the values $U(x_1), \dots, U(x_{n-1})$ so that $x_k \succsim x_l$ iff $U(x_k) \geq U(x_l)$. If x_n is indifferent to x_k for some $k < n$, then assign $U(x_n) = U(x_k)$. If not, by transitivity, all numbers in the set $\{U(x_k) |\ x_k \prec x_n\} \cup \{-1\}$ are below all numbers in the set $\{U(x_k) |\ x_n \prec x_k\} \cup \{1\}$. Choose $U(x_n)$ to be between the two sets. This guarantees that for any $k < n$ we have $x_n \succsim x_k$ iff $U(x_n) \geq U(x_k)$. Thus, the function we defined on $\{x_1, \dots, x_n\}$ represents the preference on those elements.

To complete the proof that U represents $\succsim$, take any two elements, x and $y \in X$. For some k and l we have $x = x_k$ and $y = x_l$. The above applied to $n = \max\{k, l\}$ yields $x_k \succsim x_l$ iff $U(x_k) \geq U(x_l)$.

Lexicographic Preferences

Lexicographic preferences are the outcome of applying the following procedure for determining the ranking of any two elements in a set X. The individual has in mind a sequence of criteria that could be used to compare pairs of elements in X. The criteria are applied in a fixed order until a criterion is reached that succeeds in distinguishing between the two elements, in that it determines the preferred alternative. Formally, let $(\succsim_k)_{k=1,\dots,K}$ be a K-tuple of orderings over the set X. The lexicographic ordering induced by those orderings is defined by $x \succsim_L y$ if (1) there is k^* such that for all $k < k^*$ we have $x \sim_k y$ and $x \succ_{k*} y$ or (2) $x \sim_k y$ for all k. Verify that $\succsim_L$ is a preference relation.

Example:

Let X be the unit square, i.e., $X = [0,1] \times [0,1]$. Let $x \succsim_k y$ if $x_k \geq y_k$. The lexicographic ordering $\succsim_L$ induced from $\succsim_1$ and $\succsim_2$ is: $(a_1, a_2) \succsim_L (b_1, b_2)$ if $a_1 > b_1$ or both $a_1 = b_1$ and $a_2 \geq b_2$. (Thus, in this example, the left component is the primary criterion while the right component is the secondary criterion.)

We will now show that the preferences $\succsim_L$ do *not* have a utility representation. The lack of a utility representation excludes lexicographic preferences from the scope of standard economic models in spite of the fact that they constitute a simple and commonly used procedure for preference formation.

Claim:

The preference relation $\succsim_L$ on $[0,1] \times [0,1]$, which is induced from the relations $x \succsim_k y$ if $x_k \geq y_k$ $(k = 1, 2)$, does not have a utility representation.

Proof:

Assume by contradiction that the function $u : X \to \Re$ represents $\succsim_L$. For any $a \in [0,1]$, $(a,1) \succ_L (a,0)$ we thus have $u(a,1) > u(a,0)$. Let $q(a)$ be a rational number in the nonempty interval $I_a = (u(a,0), u(a,1))$. The function q is a function from X into the set of rational numbers. It is a one-to-one function since if $b > a$ then $(b,0) \succ_L (a,1)$ and therefore $u(b,0) > u(a,1)$. It follows that the intervals I_a and I_b are disjoint and thus $q(a) \neq q(b)$. But the cardinality of the rational numbers is lower than that of the continuum, a contradiction.

Continuity of Preferences

In economics we often take the set X to be an infinite subset of a Euclidean space. The following is a condition that will guarantee the existence of a utility representation in such a case. The basic intuition, captured by the notion of a continuous preference relation,

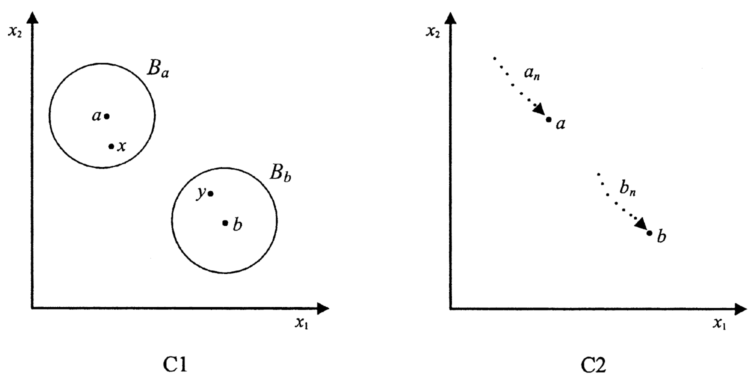

Figure 2.1
Two definitions of continuity of preferences.

is that if a is preferred to b, then "small" deviations from a or from b will not reverse the ordering.

Definition C1:

A preference relation $\succsim$ on X is *continuous* if whenever $a \succ b$ (namely, it is not true that $b \succsim a$), there are neighborhoods (balls) B_a and B_b around a and b, respectively, such that for all $x \in B_a$ and $y \in B_b$, $x \succ y$ (namely, it is not true that $y \succsim x$). (See fig. 2.1.)

Definition C2:

A preference relation $\succsim$ on X is *continuous* if the graph of $\succsim$ (that is, the set $\{(x, y)|x \succsim y\} \subseteq X \times X$) is a closed set (with the product topology); that is, if $\{(a_n, b_n)\}$ is a sequence of pairs of elements in X satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$, then $a \succsim b$. (See fig. 2.1.)

Claim:

The preference relation $\succsim$ on X satisfies C1 if and only if it satisfies C2.

Proof:

Assume that $\succsim$ on X is continuous according to C1. Let $\{(a_n, b_n)\}$ be a sequence of pairs satisfying $a_n \succsim b_n$ for all n and $a_n \to a$ and $b_n \to b$. If it is not true that $a \succsim b$ (that is, $b \succ a$), then there exist two balls B_a and B_b around a and b, respectively, such that for all $y \in B_b$ and $x \in B_a$, $y \succ x$. There is an N large enough such that for all $n > N$, both $b_n \in B_b$ and $a_n \in B_a$. Therefore, for all $n > N$, we have $b_n \succ a_n$, which is a contradiction.

Assume that $\succsim$ is continuous according to C2. Let $a \succ b$. Denote by $B(x, r)$ the set of all elements in X distanced less than r from x. Assume by contradiction that for all n there exist $a_n \in B(a, 1/n)$ and $b_n \in B(b, 1/n)$ such that $b_n \succsim a_n$. The sequence (b_n, a_n) converges to (b, a); by the second definition (b, a) is within the graph of $\succsim$, that is, $b \succsim a$, which is a contradiction.

Remarks

1. If $\succsim$ on X is represented by a *continuous* function U, then $\succsim$ is continuous. To see this, note that if $a \succ b$ then $U(a) > U(b)$. Let $\varepsilon = (U(a) - U(b))/2$. By the continuity of U, there is a $\delta > 0$ such that for all x distanced less than δ from a, $U(x) > U(a) - \varepsilon$, and for all y distanced less than δ from b, $U(y) < U(b) + \varepsilon$. Thus, for x and y within the balls of radius δ around a and b, respectively, $x \succ y$.

2. The lexicographic preferences which were used in the counterexample to the existence of a utility representation are not continuous. This is because $(1, 1) \succ (1, 0)$, but in any ball around $(1, 1)$ there are points inferior to $(1, 0)$.

3. Note that the second definition of continuity can be applied to any binary relation over a topological space, not just to a preference relation. For example, the relation $=$ on the real numbers ($\Re^1$) is continuous while the relation $\neq$ is not.

Debreu's Theorem

Debreu's theorem, which states that continuous preferences have a *continuous* utility representation, is one of the classic results in economic theory. For a complete proof of Debreu's theorem see Debreu 1954, 1960. Here we prove only that continuity guarantees the existence of a utility representation.

Lemma:

If $\succsim$ is a continuous preference relation on a convex set $X \subseteq \Re^n$, and if $x \succ y$, then there exists z in X such that $x \succ z \succ y$.

Proof:

Assume not. Construct a sequence of points on the interval that connects the points x and y in the following way. First define $x_0 = x$ and $y_0 = y$. In the inductive step we have two points, x_t and y_t, on the line that connects x and y, such that $x_t \succsim x$ and $y \succsim y_t$. Consider the middle point between x_t and y_t and denote it by m. According to the assumption, either $m \succsim x$ or $y \succsim m$. In the former case define $x_{t+1} = m$ and $y_{t+1} = y_t$, and in the latter case define $x_{t+1} = x_t$ and $y_{t+1} = m$. The sequences $\{x_t\}$ and $\{y_t\}$ are converging, and they must converge to the same point z since the distance between x_t and y_t converges to zero. By the continuity of $\succsim$ we have $z \succsim x$ and $y \succsim z$ and thus, by transitivity, $y \succsim x$, contradicting the assumption that $x \succ y$.

Comment on the Proof:

Another proof could be given for the more general case, in which the assumption that the set X is convex is replaced by the assumption that it is a connected subset of $\Re^n$. Remember that a connected set cannot be covered by two disjoint open sets. If there is no z such that $x \succ z \succ y$, then X is the union of two disjoint sets $\{a | a \succ y\}$ and $\{a | x \succ a\}$, which are open by the continuity of the preference relation.

Recall that a set $Y \subseteq X$ is *dense* in X if in every open subset of X there is an element in Y. For example, the set $Y = \{x \in \Re^n | \ x_k$ is a rational number for $k = 1, .., n\}$ is a countable dense set in $\Re^n$.

Proposition:

Assume that X is a convex subset of $\Re^n$ that has a countable dense subset Y. If $\succsim$ is a continuous preference relation, then $\succsim$ has a (continuous) utility representation.

Proof:

By a previous claim we know that there exists a function $v : Y \to [-1, 1]$, which is a utility representation of the preference relation $\succsim$ restricted to Y. For every $x \in X$, define $U(x) = sup\{v(z)|z \in Y$ and $x \succ z\}$. Define $U(x) = -1$ if there is no $z \in Y$ such that $x \succ z$, which means that x is the minimal element in X. (Note that for $z \in Y$ it could be that $U(z) < v(z)$.)

If $x \sim y$, then $x \succ z$ iff $y \succ z$. Thus, the sets on which the supremum is taken are the same and $U(x) = U(y)$.

If $x \succ y$, then by the lemma there exists z in X such that $x \succ z \succ y$. By the continuity of the preferences $\succsim$ there is a ball around z such that all the elements in that ball are inferior to x and superior to y. Since Y is dense, there exists $z_1 \in Y$ such that $x \succ z_1 \succ y$. Similarly, there exists $z_2 \in Y$ such that $z_1 \succ z_2 \succ y$. Finally,

$U(x) \geq v(z_1)$ (by the definition of U and $x \succ z_1$),
$v(z_1) > v(z_2)$ (since v represents $\succsim$ on Y and $z_1 \succ z_2$), and
$v(z_2) \geq U(y)$ (by the definition of U and $z_2 \succ y$).

Bibliographic Notes

Recommended readings: Kreps 1990, 30–32; Mas-Colell et al. 1995, chapter 3, C.

Fishburn (1970) covers the material in this lecture very well. The example of lexicographic preferences originated in Debreu (1959) (see also Debreu 1960, in particular Chapter 2, which is available online at http://cowles.econ.yale.edu/P/cp/p00b/p0097.pdf.)

Problem Set 2

Problem 1. (*Easy*)
The purpose of this problem is to make sure that you fully understand the basic concepts of utility representation and continuous preferences.

a. Is the statement "if both U and V represent $\succsim$ then there is a *strictly* monotonic function $f : \Re \to \Re$ such that $V(x) = f(U(x))$" correct?
b. Can a continuous preference be represented by a discontinuous function?
c. Show that in the case of $X = \Re$, the preference relation that is represented by the discontinuous utility function $u(x) = [x]$ (the largest integer n such that $x \geq n$) is not a continuous relation.
d. Show that the two definitions of a continuous preference relation (C1 and C2) are equivalent to

   Definition C3: For any $x \in X$, the upper and lower contours $\{y| \ y \succsim x\}$ and $\{y| \ x \succsim y\}$ are closed sets in X,

   and to

   Definition C4: For any $x \in X$, the sets $\{y| \ y \succ x\}$ and $\{y| \ x \succ y\}$ are open sets in X.

Problem 2. (*Moderate*)
Give an example of preferences over a countable set in which the preferences cannot be represented by a utility function that returns only integers as values.

Problem 3. (*Moderate*)
Consider the sequence of preference relations $(\succsim^n)_{n=1,2,..}$, defined on $\Re^2_+$ where $\succsim^n$ is represented by the utility function $u_n(x_1, x_2) = x_1^n + x_2^n$. We will say that the sequence $\succsim^n$ converges to the preferences $\succsim^*$ if for every x and y, such that $x \succ^* y$, there is an N such that for every $n > N$ we have $x \succ^n y$. Show that the sequence of preference relations $\succsim^n$ converges to the preferences $\succ^*$ which are represented by the function $max\{x_1, x_2\}$.

Problem 4. (*Moderate*)
The following is a typical example of a utility representation theorem:
Let $X = \Re^2_+$. Assume that a preference relation $\succsim$ satisfies the following three properties:
ADD: $(a_1, a_2) \succsim (b_1, b_2)$ implies that $(a_1 + t, a_2 + s) \succsim (b_1 + t, b_2 + s)$ for all t and s.
MON: If $a_1 \geq b_1$ and $a_2 \geq b_2$, then $(a_1, a_2) \succsim (b_1, b_2)$; in addition, if either $a_1 > b_1$ or $a_2 > b_2$, then $(a_1, a_2) \succ (b_1, b_2)$.
CON: Continuity.

a. Show that if $\succsim$ has a linear representation (that is, $\succsim$ are represented by a utility function $u(x_1, x_2) = \alpha x_1 + \beta x_2$ with $\alpha > 0$ and $\beta > 0$), then $\succsim$ satisfies *ADD*, *MON* and *CON*.
b. Suggest circumstances in which ADD makes sense.
c. Show that the three properties are necessary for $\succsim$ to have a linear representation. Namely, show that for any pair of the three properties there is a preference relation that does not satisfy the third property.
d. (*This part is difficult*) Show that if $\succsim$ satisfies the three properties, then it has a linear representation.

Problem 5. (*Moderate*)
Utility is a numerical representation of preferences. One can think about the numerical representation of other abstract concepts. Here, you will try to come up with a possible numerical representation of the concept "approximately the same" (see Luce (1956) and Rubinstein (1988)). For simplicity, let X be the interval $[0, 1]$.

Consider the following six properties of S:

(S-1) For any $a \in X$, aSa.
(S-2) For all $a, b \in X$, if aSb then bSa.
(S-3) Continuity (the graph of the relation S in $X \times X$ is a closed set).
(S-4) Betweenness: If $d \geq c \geq b \geq a$ and dSa then also cSb.
(S-5) For any $a \in X$ there is an interval around a such that xSa for every x in the interval.
(S-6) Denote $M(a) = max\{x|xSa\}$ and $m(a) = min\{x|aSx\}$. Then, M and m are (weakly) increasing functions and are strictly increasing whenever they do not have the values 0 or 1.

a. Do these assumptions capture your intuition about the concept "approximately the same"?
b. Show that the relation S_ε, defined by $aS_\varepsilon b$ if $|b - a| \leq \varepsilon$ (for positive ε), satisfies all assumptions.

c. (*Difficult*) Let S be a binary relation that satisfies the above six properties and let ε be a strictly positive number. Show that there is a strictly increasing and continuous function $H : X \to \Re$ such that aSb if and only if $|H(a) - H(b)| \leq \varepsilon$.

Problem 6. (*Reading*)
Read Kahneman (2000) (it is available at http://arielrubinstein.tau.ac.il/econt/k.pdf) and discuss his distinction between the different types of "psychological utilities."