LECTURE 10

Social Choice

Aggregation of Orderings

When a rational decision maker forms a preference relation, it is often on the basis of more primitive relations. For example, the choice of a PC may depend on considerations such as "size of memory," "ranking by PC magazine," and "price." Each of these considerations expresses a preference relation on the set of PCs. In this lecture we look at some of the logical properties and problems that arise in the formation of preferences on the basis of more primitive preference relations.

Although the aggregation of preference relations can be thought of in a context of a single individual's decision making, the classic context in which preference aggregation is discussed is "social choice," where the "will of the people" is thought of as an aggregation of the preference relations held by members of society.

The foundations of social choice theory lie in the "Paradox of Voting." Let $X = \{a, b, c\}$ be a set of alternatives. Consider a society that consists of three members called 1, 2, and 3. Their rankings of X are $a \succ_1 b \succ_1 c$; $b \succ_2 c \succ_2 a$, and $c \succ_3 a \succ_3 b$. A natural criterion for the determination of collective opinion on the basis of individuals' preference relations is the *majority rule*. According to the majority rule, $a \succ b$, $b \succ c$, and $c \succ a$, which conflicts with the transitivity of the social ordering. Note that although the majority rule does not induce a transitive social relation for *all* profiles of individuals' preference relations, transitivity is guaranteed if we restrict ourselves to a smaller domain of profiles (see problem 3 in the problem set).

The interest in social choice in economics is motivated by the recognition that explicit methods for the aggregation of preference relations are essential for doing any *welfare economics*. The theory is also related to the design of *voting systems*, which are methods for determining social action on the basis of individuals' preferences.

The Basic Model

A basic model of social choice consists of the following:

- X: a set of social *alternatives*.
- N: a finite set of *individuals* (denote the number of elements in N by n).
- $\succ_i$: individual i's linear ordering on X (a linear ordering is a preference relation with no indifferences, i.e., for no $x \neq y$, $x \sim_i y$).
- *Profile*: An n-tuple of orderings $(\succ_1, \ldots, \succ_n)$ interpreted as a certain "state of society."
- *SWF* (*Social Welfare Function*): A function that assigns a single (social) preference (*not* necessarily a linear ordering) to every profile.

Note that

1. The assumption that the domain of an SWF includes only strict preferences is made only for simplicity of presentation.
2. An SWF attaches a preference relation to *every* possible profile and not just to a single profile.
3. The SWF aggregation of preference relations is required to produce a complete preference relation. An alternative concept, called Social Choice Function, attaches a social alternative, interpreted as the society's choice, to every profile of preference relations.
4. An SWF aggregates only ordinal preference relations. The framework does not allow us to make a statement such as "the society prefers a to b since agent 1 prefers b to a but agent 2 prefers a to b much more."
5. In this model we cannot express a consideration of the type "I prefer what society prefers."
6. The elements in X are social alternatives. Thus, an individual's preferences may exhibit considerations of fairness and concern about other individuals' well-being.

Examples

Let us consider some examples of aggregation procedures. We will often use $\succsim$ as a short form of $F(\succ_1, \ldots, \succ_n)$.

1. $F(\succ_1, \ldots, \succ_n) = \succsim^*$ for some ordering $\succsim^*$. (This is a degenerate SWF that does not account for the individuals' preferences.)
2. Define $x \to z$ if a majority of individuals prefer x to z. Order the alternatives by the number of "victories" they score, that is, $x \succsim y$ if $|\{z|x \to z\}| \geq |\{z|y \to z\}|$.
3. For $X = \{a, b\}$, $a \succsim b$ unless 2/3 of the individuals prefer b to a.
4. "*The anti-dictator*": There is an individual i so that x is preferred to y if and only if $y \succ_i x$.
5. Define $d(\succ; \succ_1, \ldots, \succ_n)$ as the number of (x, y, i) for which $x \succ_i y$ and $y \succ x$. The function d can be interpreted as the sum of the distances between the preference relation $\succ$ and the n preference relations of the individuals. Choose $F(\succ_1, \ldots, \succ_n)$ to be an ordering that minimizes $d(\succ; \succ_1, \ldots, \succ_n)$ (ties are broken arbitrarily).
6. Let $F(\succ_1, \ldots, \succ_n)$ be the ordering that is the most common among $(\succ_1, \ldots, \succ_n)$ (with ties broken in some predetermined way).

Axioms

Once again we use the axiomatization methodology. We suggest a set of axioms on social welfare functions and study their implications. Let F be an SWF.

Condition Par (Pareto):

For all $x, y \in X$ and for every profile $(\succ_i)_{i \in N}$, if $x \succ_i y$ for all i then $x \succ y$.

The Pareto axiom requires that if all individuals prefer one alternative over the other, then the social preferences agree with the individuals'.

Condition IIA (Independence of Irrelevant Alternatives):

For any pair $x, y \in X$ and any two profiles $(\succ_i)_{i \in N}$ and $(\succ'_i)_{i \in N}$ if for all i, $x \succ_i y$ iff $x \succ'_i y$, then $x \succsim y$ iff $x \succsim' y$.

The IIA condition requires that if two profiles agree on the relative rankings of two particular alternatives, then the social preferences attached to the two profiles also agree in their relative ranking of the two alternatives.

Notice that IIA allows an SWF to apply one criterion when comparing a to b and another when comparing c to d. For example, the simple social preference between a and b can be determined according to majority rule while that between c and d requires a 2/3 majority.

Condition IIA is sufficient for Arrow's theorem. However, for the sake of simplifying the proof in this presentation, we will make do with a stronger requirement:

Condition I^* (Independence of Irrelevant Alternatives + Neutrality):

For all $a, b, c, d \in X$, and for any profiles $\succ$ and $\succ'$ if for all i, $a \succ_i b$ iff $c \succ'_i d$, then $a \succsim b$ iff $c \succsim' d$.

In other words, in addition to what is required by *IIA*, condition I^* requires that the criterion that determines the social preference between a and b be applied to *any* pair of alternatives.

Arrow's Impossibility Theorem

Theorem (Arrow):

If $|X| \geq 3$, then any SWF F that satisfies conditions *Par* and I^* is dictatorial, that is, there is some i^* such that $F(\succ_1, \ldots, \succ_n) \equiv \succ_{i^*}$.

We can break the theorem's assumptions into four: *Par*, I^*, *Transitivity* (of the social ordering), and $|X| \geq 3$. Before we move on to the proof, let us show that the assumptions are *independent*. Namely, for each of the four assumptions, we give an example of a nondictatorial SWF, demonstrating the theorem would not hold if that assumption were omitted.

- *Par*: An anti-dictator SWF satisfies I^* but not *Par*.
- I^*: Consider the Borda Rule. Let $w(1) > w(2) > \ldots > w(|X|)$ be a fixed profile of weights. Say that i assigns to x the score

$w(k)$ if x appears in the k'th place in $\succ_i$. Attach to x the sum of the weights assigned to x by the n individuals and rank the alternatives by those sums. The Borda rule is an SWF satisfying *Par* but not I^*.
- *Transitivity of the Social Order:* The majority rule satisfies all assumptions but can induce a relation which is not transitive.
- $|X| \geq 3$: For $|X| = 2$ the majority rule satisfies *Par* and I^* and induces (a trivial) transitive relation.

Proof of Arrow's Impossibility Theorem

Let F be an SWF that satisfies *Par* and I^*. Hereinafter, we denote the relation $F(\succ_1, \ldots, \succ_n)$ by $\succsim$.

Given the SWF we say that

- a coalition G is *decisive* if for all x,y, [for all $i \in G$, $x \succ_i y$] implies [$x \succ y$], and
- a coalition G is *almost decisive* if for all x,y, [for all $i \in G$, $x \succ_i y$ and for all $j \notin G$ $y \succ_j x$] implies [$x \succ y$].

Note that if G is decisive it is almost decisive since the "almost decisiveness" refers only to the subset of profiles where all members of G prefer x to y and all members of $N - G$ prefer y to x.

Field Expansion Lemma:

If G is almost decisive, then G is decisive.

Proof:

We have to show that for any x,y and for any profile $(\succ'_i)_{i \in N}$ for which $x \succ'_i y$ for all $i \in G$, the preference $F(\succ'_1, \ldots, \succ'_n)$ determines x to be superior to y. By I^* it is sufficient to show that for one pair of social alternatives a and b, and for one profile $(\succ_i)_{i \in N}$ that agrees with the profile $(\succ'_i)_{i \in N}$ on the pair $\{a, b\}$, the preference $F(\succ_1, \ldots, \succ_n)$ determines a to be preferred to b.

Let c be a third alternative. Let $(\succ_i)_{i\in N}$ be a profile satisfying $a \succ_i b$ iff $a \succ'_i b$ for all i, and for all $i \in G$, $a \succ_i c \succ_i b \succ_i x$ for every $x \in X - \{a, b, c\}$ and for all $j \notin G$, $c \succ_j y \succ_j x$ for every $y \in \{a, b\}$ and for every $x \in X - \{a, b, c\}$.

Since G is almost decisive, $a \succ c$. By *Par*, $c \succ b$, therefore, $a \succ b$ by transitivity.

Group Contraction Lemma:

If G is decisive and $|G| \geq 2$, then there exists $G' \subset G$ such that G' is decisive.

Proof:

Let $G = G_1 \cup G_2$, where G_1 and G_2 are nonempty and $G_1 \cap G_2 = \emptyset$. By the Field Expansion Lemma it is enough to show that G_1 or G_2 is almost decisive.

Take three alternatives a, b, and c and a profile of preference relations $(\succ_i)_{i\in N}$ satisfying

- for all $i \in G_1$, $c \succ_i a \succ_i b$, and
- for all $i \in G_2$, $a \succ_i b \succ_i c$, and
- for all other i, $b \succ_i c \succ_i a$.

If G_1 is not almost decisive, then there are x and y and a profile $(\succ'_i)_{i\in N}$ such that $x \succ'_i y$ for all $i \in G_1$ and $y \succ'_i x$ for all $i \notin G_1$, such that $F(\succ'_1, \ldots, \succ'_n)$ determines y to be at least as preferable as x. Therefore, by I^*, $b \succsim c$.

Similarly, if G_2 is not almost decisive, then $c \succsim a$. Thus, by transitivity $b \succsim a$, but since G is decisive, $a \succ b$, a contradiction. Thus, G_1 or G_2 is almost decisive.

Proof of the Theorem:

By *Par*, the set N is decisive. By the Group Contraction Lemma, every decisive set that includes more than one member has a proper subset that is decisive. Thus, there is a set $\{i^*\}$ that is decisive, namely, $F(\succ_1, \ldots, \succ_n) \equiv \succ_{i^*}$.

Related Issues

Arrow's theorem was the starting point for a huge literature. We mention three other impossibility results.

1. *Monotonicity* is another axiom that has been widely discussed in the literature. Consider a "change" in a profile so that an alternative a, which individual i ranked below b, is now ranked by i above b. Monotonicity requires that there is no alternative c such that this change deteriorates the ranking of a vs. c. Muller and Satterthwaite (1977)'s theorem shows that the only SWF's satisfying *Par* and monotonicity are dictatorships.

2. An SWF specifies a preference relation for every profile. A *social choice function* attaches an alternative to every profile. The most striking theorem proved in this framework is the Gibbard-Satterthwaite theorem. It states that any social choice function C satisfying the condition that it is never worthwhile for an individual to mispresent his preferences, namely, it is never that $C(\succ_1, \ldots, \succ_i'$ $, \ldots, \succ_n) \succ_i C(\succ_1, \ldots, \succ_i, \ldots, \succ_n)$, is a dictatorship.

3. Another related concept is the following.

Let $Ch(\succ_1, \ldots, \succ_n)$ be a function that assigns a choice function to every profile of orderings on X. We say that Ch satisfies *unanimity* if for every $(\succ_1, \ldots, \succ_n)$ and for any $x, y \in A$, if $y \succ_i x$ for all i then, $x \notin Ch(\succ_1, \ldots, \succ_n)(A)$.

We say that Ch is *invariant to the procedure* if, for every profile $(\succ_1$ $, \ldots, \succ_n)$ and for every choice set A, the following two "approaches" lead to the same outcome:

a. Partition A into two sets A' and A''. Choose an element from A' and an element from A'' and then choose one element from the two choices.
b. Choose an element from the unpartitioned set A.

Dutta, Jackson, and Le Breton (2001) show that only dictatorships satisfy both unanimity and invariance to the procedure.

Bibliographic Notes

Recommended readings: Kreps 1990, chapter 5; Mas-Colell et al. 1995, chapter 21.

This lecture focuses mainly on Arrow's Impossibility Theorem, one of the most famous results in economics, proved by Arrow in his Ph.D. dissertation and published in 1951 (see the classic book Arrow 1963). Social choice theory is beautifully introduced in Sen (1970). The proof brought here is one of many for Arrow's Impossibility Theorem (see Kelly 1988). Reny (2001) provides an elementary proof that demonstrates the strong logical link between Arrow's theorem and the Gibbard-Satterthwaite theorem.

Problem Set 10

Problem 1. (*Moderately difficult.* Based on May 1952.)
Assume that the set of social alternatives, X, includes only two alternatives. Define a social welfare function to be a function that attaches a preference to any profile of preferences (allow indifference for the SWF and the individuals' preference relations). Consider the following axioms:

- *Anonymity* If σ is a permutation of N and if $p = \{ \succsim_i \}_{i \in N}$ and $p' = \{ \succsim'_i \}_{i \in N}$ are two profiles of preferences on X so that $\succsim'_{\sigma(i)} = \succsim_i$, then $\succsim (p) = \succsim (p')$.
- *Neutrality* For any preference $\succsim_i$ define $(- \succsim_i)$ as the preference satisfying $x(- \succsim_i)y$ iff $y \succsim_i x$. Then $\succsim (\{- \succsim_i \}_{i \in N}) = - \succsim (\{ \succsim_i \}_{i \in N})$.
- *Positive Responsiveness* If the profile $\{ \succsim'_i \}_{i \in N}$ is identical to $\{ \succsim_i \}_{i \in N}$ with the exception that for one individual j either ($x \sim_j y$ and $x \succ'_j y$) or ($y \succ_j x$ and $x \sim'_j y$) and if $x \succsim y$ then $x \succ' y$.

a. Interpret the axioms.
b. Does anonymity imply non-dictatorship?
c. Show that the majority rule satisfies all axioms.
d. Prove May's theorem by which the majority rule is the only SWF satisfying the above axioms.
e. Are the above three axioms independent?

Problem 2. (*Moderately difficult*)
N individuals choose a single object from among a set X. We are interested in functions that aggregate the individuals' recommendations (*not preferences*, just recommendations!) into a social decision (i.e., $F : X^N \to X$).

Discuss the following axioms:

- *Par*: If all individuals recommend x^* then the society chooses x^*.
- I: If the same individuals support an alternative $x \in X$ in two profiles of recommendations, then x is chosen in one profile if and only if it chosen in the other.

a. Show that if X includes at least three elements, then the only aggregation method that satisfies P and I is a dictatorship.
b. Show the necessity of the three conditions P, I, and $|X| \geq 3$ for this conclusion.

Problem 3. (*Easy*)
Assume that the set of alternatives, X, is the interval $[0, 1]$ and that each individual's preference is *single-peaked*, i.e., for each i there is an alternative a_i^* such that if $a_i^* \geq b > c$ or $c > b \geq a_i^*$, then $b \succ_i c$.

a. Provide an interpretation of single-peaked preferences.
b. Show that for any odd n, if we restrict the domain of preferences to single-peaked preferences, then the majority rule induces a "well-behaved" SWF.

Problem 4. (*Moderately difficult.* Based on Kasher and Rubinstein 1997.)
Who is an economist? Departments of economics are often sharply divided over this question. Investigate the approach according to which the determination of who is an economist is treated as an aggregation of the views held by department members on this question.

Let $N = \{1, \ldots, n\}$ be a group of individuals ($n \geq 3$). Each $i \in N$ "submits" a set E_i, a *proper* subset of N, which is interpreted as the set of "real economists" in his view. An aggregation method F is a function that assigns a *proper* subset of N to each profile $(E_i)_{i=1,\ldots,n}$ of proper subsets of N. $F(E_1, \ldots, E_n)$ is interpreted as the set of all members of N who are considered by the group to be economists. (Note that we require that all opinions be proper subsets of N.)

Consider the following axioms on F:

- *Consensus* If $j \in E_i$ for all $i \in N$, then $j \in F(E_1, \ldots, E_n)$.
- *Independence* If $(E_1, \ldots, E_n)$ and $(G_1, \ldots, G_n)$ are two profiles of views so that for all $i \in N$, [$j \in E_i$ if and only if $j \in F(G_1, \ldots, G_n)$].

a. Interpret the two axioms.
b. Find one aggregation method that satisfies Consensus but not Independence and one that satisfies Independence but not Consensus.
c. (*Difficult*) Provide a proof similar to that of Arrows' Impossibility Theorem of the claim that the only aggregation methods that satisfy the above two axioms are those for which there is a member i^* such that $F(E_1, \ldots, E_n) \equiv E_{i^*}$.