PART I

RATIONAL DECISION MAKING

1

The Single-Person Decision Problem

Imagine yourself in the morning, all dressed up and ready to have breakfast. You might be lucky enough to live in a nice undergraduate dormitory with access to an impressive cafeteria, in which case you have a large variety of foods from which to choose. Or you might be a less-fortunate graduate student, whose studio cupboard offers the dull options of two half-empty cereal boxes. Either way you face the same problem: what should you have for breakfast?

This trivial yet ubiquitous situation is an example of a *decision problem.* Decision problems confront us daily, as individuals and as groups (such as firms and other organizations). Examples include a division manager in a firm choosing whether or not to embark on a new research and development project; a congressional representative deciding whether or not to vote for a bill; an undergraduate student deciding on a major; a baseball pitcher contemplating what kind of pitch to deliver; or a lost group of hikers confused about which direction to take. The list is endless.

Some decision problems are trivial, such as choosing your breakfast. For example, if Apple Jacks and Bran Flakes are the only cereals in your cupboard, and if you hate Bran Flakes (they belong to your roommate), then your decision is obvious: eat the Apple Jacks. In contrast, a manager's choice of whether or not to embark on a risky research and development project or a lawmaker's decision on a bill are more complex decision problems.

This chapter develops a *language* that will be useful in laying out rigorous foundations to support many of the ideas underlying strategic interaction in games. The language will be formal, having the benefit of being able to represent a host of different problems and provide a set of tools that will lend structure to the way in which we think about decision problems. The formalities are a vehicle that will help make ideas precise and clear, yet in no way will they overwhelm our ability and intent to keep the more practical aspect of our problems at the forefront of the analysis.

In developing this formal language, we will be forced to specify a set of assumptions about the behavior of decision makers or players. These assumptions will, at times, seem both acceptable and innocuous. At other times, however, the assumptions will be almost offensive in that they will require a significant leap of faith. Still, as the analysis unfolds, we will see the conclusions that derive from the assumptions that we make, and we will come to appreciate how sensitive the conclusions are to these assumptions.

As with any theoretical framework, the value of our conclusions will be only as good as the sensibility of our assumptions. There is a famous saying in computer science—"garbage in, garbage out"—meaning that if invalid data are entered into a system, the resulting output will also be invalid. Although originally applied to computer software, this statement holds true more generally, being applicable, for example, to decision-making theories like the one developed herein. Hence we will at times challenge our assumptions with facts and question the validity of our analysis. Nevertheless we will argue in favor of the framework developed here as a useful benchmark.

1.1 Actions, Outcomes, and Preferences

Consider the examples described earlier: choosing a breakfast, deciding about a research project, or voting on a bill. These problems all share a similar structure: an individual, or player, faces a situation in which he has to choose one of several alternatives. Each choice will result in some outcome, and the consequences of that outcome will be borne by the player himself (and sometimes other players too).

For the player to approach this problem in an intelligent way, he must be aware of three fundamental features of the problem: What are his possible choices? What is the result of each of those choices? How will each result affect his well-being? Understanding these three aspects of a problem will help the player choose his best action. This simple observation offers us a first working definition that will apply to *any decision problem:*

The Decision Problem A **decision problem** consists of three features:

1. **Actions** are all the alternatives from which the player can choose.
2. **Outcomes** are the possible consequences that can result from any of the actions.
3. **Preferences** describe how the player ranks the set of possible outcomes, from most desired to least desired. The **preference relation** $\succsim$ describes the player's preferences, and the notation $x \succsim y$ means "x is at least as good as y."

To make things simple, let's begin with our rather trivial decision problem of choosing between Apple Jacks and Bran Flakes. We can define the set of actions as $A = \{a, b\}$, where a denotes the choice of Apple Jacks and b denotes the choice of Bran Flakes.[1] In this simple example our actions are practically synonymous with the outcomes, yet to make the distinction clear we will denote the set of outcomes by $X = \{x, y\}$, where x denotes eating Apple Jacks (the consequence of *choosing* Apple Jacks) and y denotes eating Bran Flakes.

1. More on the concept of a set and the appropriate notation can be found in Section 19.1 of the mathematical appendix.

1.1.1 Preference Relations

Turning to the less familiar notion of a **preference relation,** imagine that you prefer eating Apple Jacks to Bran Flakes. Then we will write $x \succsim y$, which should be read as "x is at least as good as y." If instead you prefer Bran Flakes, then we will write $y \succsim x$, which should be read as "y is at least as good as x." Thus our preference relation is just a shorthand way to express the player's ranking of the possible outcomes.

We follow the common tradition in economics and decision theory by expressing preferences as a "weak" ranking. That is, the statement "x is at least as good as y" is consistent with x being *better* than y or *equally as good as* y. To distinguish between these two scenarios we will use the **strict preference relation,** $x \succ y$, for "x is strictly better than y," and the **indifference relation,** $x \sim y$, for "x and y are equally good."

It need not be the case that actions are synonymous with outcome, as in the case of choosing your breakfast cereal. For example, imagine that you are in a bar with a drunken friend. Your actions can be to let him drive home or to order him a cab. The outcome of letting him drive is a certain accident (he's *really* drunk), and the outcome of ordering him a cab is arriving safely at home. Hence for this decision problem your actions are physically different from the outcomes.

In these examples the action set is *finite,* but in some cases one might have infinitely many actions from which to choose. Furthermore there may be infinitely many outcomes that can result from the actions chosen. A simple example can be illustrated by me offering you a two-gallon bottle of water to quench your thirst. You can choose how much to drink and return the remainder to me. In this case your action set can be described as the interval $A = [0, 2]$: you can choose any action a as long as it belongs to the interval $[0, 2]$, which we can write in two ways: $0 \le a \le 2$ or $a \in [0, 2]$.[2] If we equate outcomes with actions in this example then $X = [0, 2]$ as well. Finally it need not be the case that more is better. If you are thirsty then drinking a pint may be better than drinking nothing. However, drinking a gallon may cause you to have a stomachache, and you may therefore prefer a pint to a gallon.

Before proceeding with a useful way to represent a player's preferences over various outcomes, it is important to stress that we will make two important assumptions about the player's ability to think through the decision problem.[3] First, we require the player to be able to rank *any two outcomes* from the set of outcomes. To put this more formally:

The Completeness Axiom The preference relation $\succsim$ is **complete:** any two outcomes $x, y \in X$ can be ranked by the preference relation, so that either $x \succsim y$ or $y \succsim x$.

At some level the completeness axiom is quite innocuous. If I show you two foods, you should be able to rank them according to how much you like them (including being indifferent if they are equally tasty and nutritious). If I offer you two cars, you should be able to rank them according to how much you enjoy driving them, their safety

2. The notation symbol $\in$ means "belongs to." Hence "$x, y \in X$" means "elements x and y belong to the set X." If you are unfamiliar with sets and these kinds of descriptions please refer to Section 19.1 of the mathematical appendix.

3. These assumptions are referred to as "axioms," following the language used in the seminal book by von Neumann and Morgenstern (1944) that laid many of the foundations for both decision theory and game theory.

specifications, and so forth. If I offer you two investment portfolios, you should be able to rank them according to the extent to which you are willing to balance risk and return. In other words, the completeness axiom *does not let you be indecisive between any two outcomes*.[4]

The second assumption we make guarantees that a player can rank *all* of the outcomes. To do this we introduce a rather mild consistency condition called *transitivity:*

The Transitivity Axiom The preference relation $\succsim$ is **transitive:** for any three outcomes $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

Faced with several outcomes, completeness guarantees that any two can be ranked, and transitivity guarantees that there will be no contradictions in the ranking, which could create an indecisive cycle. To observe a violation of the transitivity axiom, consider a player who strictly prefers Apple Jacks to Bran Flakes, $a \succ b$, Bran Flakes to Cheerios, $b \succ c$, and Cheerios to Apple Jacks, $c \succ a$. When faced with any two boxes of cereal, say $A = \{a, b\}$, he has no problem choosing his preferred cereal a. What happens, however, when he is presented with all three alternatives, $A = \{a, b, c\}$? The poor guy will be unable to decide which of the three to choose, because for any given box of cereal, there is another box that he prefers. Therefore, by requiring that the player have complete and transitive preferences, we basically guarantee that among *any set of outcomes,* he will always have at least one *best outcome* that is as good as or better than any other outcome in that set.

To foreshadow what will be our premise for decision making, a preference relation that is complete and transitive is called a **rational preference relation.** We will be concerned only with players who have such rational preferences, for without such preferences we can offer neither predictive nor prescriptive insights.

Remark As noted by the Marquis de Condorcet in 1785, it is possible to have a group of rational individual players who, when put together to make decisions as a group, will become an "irrational" group. For example, imagine three roommates, called players 1, 2, and 3, who have to choose one box of cereal for their apartment kitchen. Player 1's preferences are given by $a \succ_1 c \succ_1 b$, player 2's are given by $c \succ_2 b \succ_2 a$, and player 3's are given by $b \succ_3 a \succ_3 c$. Imagine that our three players make choices in a democratic way and use majority voting to reach a decision. What will be the resulting preferences of the group, $\succ_G$? When faced with the pair a and c, players 1 and 3 will vote for Apple Jacks, hence $a \succ_G c$. When faced with the pair c and b, players 1 and 2 will vote for Cheerios, hence $c \succ_G b$. When faced with the pair a and b, players 2 and 3 will vote for Bran Flakes, hence $b \succ_G a$. As a result, our three rational players will not be able to reach a conclusive decision using the group preferences that result from majority voting! This type of group indecisiveness resulting from majority voting is often referred to as the *Condorcet Paradox.* Because we will not be analyzing group decisions, it is not something we will confront, but it is useful to be mindful of such phenomena, in which imposing individual rationality does not imply "group rationality."

4. In other words, this axiom prohibits the kind of problem referred to as "Buridan's ass." One version describes a situation in which an ass is placed between two identical stacks of hay, assuming that the ass will always go to whichever stack is closer. However, since the stacks are both the same distance from the ass, it will not be able to choose between them and will die of hunger.

1.1.2 Payoff Functions

When we restrict attention to players with rational preferences, not only do we get players who behave in a consistent and appealing way, but as an added bonus we can replace the preference relation with a much friendlier, and more operational, apparatus. Consider the following simple example. Imagine that you open a lemonade stand on your neighborhood corner. You have three possible actions: choose low-quality lemons (l), which imply a cost of \$10 and a revenue from sales of \$15; choose medium-quality lemons (m), which imply a cost of \$15 and a revenue from sales of \$25; or choose high-quality lemons (h), which imply a cost of \$28 and a revenue from sales of \$35. Thus the action set is $A = \{l, m, h\}$, and the outcome set is given by net profits and is $X = \{5, 10, 7\}$, where the action l yields a profit of \$5, the action m yields a profit of \$10, and the action h yields a profit of \$7. Assuming that obtaining higher profits is strictly better, we have $10 \succ 7 \succ 5$. Hence you should choose alternative m and make a profit of \$10.

Notice that we took a rather obvious profit-maximizing problem and fit it into our framework for a decision problem. We derived the preference relation that is consistent with maximizing profit, the objective of any for-profit business. Arguably it would be more natural and probably easier to comprehend the problem if we looked at the actions and their associated profits. In particular we can define the **profit function** in the obvious way: every action $a \in A$ yields a profit $\pi(a)$. Then, instead of considering a preference relation over profit outcomes, we can just look at the profit from each action directly and choose an action that maximizes profits. In other words, we can *use the profit function to evaluate actions and outcomes.*

As this simple example demonstrates, a profit function is a more direct way for a player to rank his actions. The question then is, can we find similar ways to approach decision problems that are not about profits? It turns out that we can do exactly that if we have players with rational preferences, and to do that we define a payoff function.[5]

Definition 1.1 A **payoff function** $u : X \to \mathbb{R}$ represents the preference relation $\succsim$ if for any pair $x, y \in X$, $u(x) \geq u(y)$ if and only if $x \succsim y$.

To put the definition into words, we say that the preference relation $\succsim$ is represented by the payoff function $u : X \to \mathbb{R}$ that assigns to each outcome in X a real number, if and only if the function assigns a higher value to higher-ranked outcomes.

It is important to notice that representing preferences with payoff functions is convenient, but that payoff values by themselves have no meaning whatsoever. Payoff is an *ordinal* construct: it is used to order the alternatives from most to least desirable. For example, if I like Apple Jacks more than Bran Flakes, then I can construct the payoff function $u(\cdot)$ so that $u(a) = 5$ and $u(b) = 3$. I can also use a different payoff function $\widetilde{u}(\cdot)$ that represents the same preferences as follows: $\widetilde{u}(a) = 100$ and $\widetilde{u}(b) = -237$. Just as Fahrenheit and Celsius are two different ways to describe hotter and colder temperatures, there are many ways to represent preferences with payoff functions.

Using payoff functions instead of preferences will allow us to operationalize a theory of how decision makers with rational preferences ought to behave, and how they often will behave. They will choose actions that maximize a payoff function that

5. Recall that a function relates each of its inputs to exactly one output. For more on this see Section 19.2 of the mathematical appendix.

represents their preferences. One last question we need to ask is whether we know for sure that this method will work: is it true that players will surely have a payoff function representing their preferences? One case is easy and worth going through briefly. In what follows, we provide a formal proposition and a formal, yet fairly easy to follow, proof.

Proposition 1.1 *If the set of outcomes X is finite then any rational preference relation over X can be represented by a payoff function.*

Proof The proof is by construction. Because the preference relation is complete and transitive, we can find a least-preferred outcome $\underline{x} \in X$ such that all other outcomes $y \in X$ are at least as good as $\underline{x}$, that is, $y \succsim \underline{x}$ for all other $y \in X$. Now define the "worst outcome equivalence set," denoted X_1, to include $\underline{x}$ and any other outcome for which the player is indifferent between it and $\underline{x}$. Then, from the remaining elements of $X \backslash X_1$,[6] define the "second worst outcome equivalence set," X_2, and continue in this fashion until the "best outcome equivalence set," X_n, is created. Because X is finite and $\succsim$ is rational, such a finite collection of n equivalence sets exists. Now consider n arbitrary values $u_n > u_{n-1} > \cdots > u_2 > u_1$, and assign payoffs according to the function defined by: for any $x \in X_k$, $u(x) = u_k$. This payoff function represents $\succsim$. Hence we have proved that such a function exists. ■

This proposition is useful: for many realistic situations, we can create payoff functions that work in a similar way as profit functions, giving the player a useful tool to see which actions are best and which ought to be avoided. We will not explore this issue further, but payoff representations exist in many other cases that include infinitely many outcomes. The treatment of such cases is beyond the scope of this textbook, but you are welcome to explore one of the many texts that offer a more complete treatment of the topic, which is referred to under the title "representation theorems." (See, e.g., Kreps [1990a, pp. 18–37, and 1988] for an in-depth treatment of this topic.)

As we have seen so far, the formal structure of a decision problem offers a coherent framework for analysis. For decades, however, teachers, students, and practitioners have instead used the intuitive and graphically simple tool of *decision trees*.

Imagine that, in addition to Apple Jacks (a) and Bran Flakes (b), your breakfast options include a muffin (m) and a scone (s). Your preferences are given as $s \succ a \succ m \succ b$. (Recall that we now consider preferences over outcomes as directly over actions.) Consider the following payoff representation: $v(s) = 4$, $v(a) = 3$, $v(m) = 2$, and $v(b) = 1$. We can write down the corresponding decision tree, which is depicted in Figure 1.1.

To read this simple decision tree, notice that the player resides at the "root" of the tree on the left, and that the tree then branches off, each branch representing a possible action. In the example of choosing breakfast, each action results in a final payoff, and these payoffs are written to correspond to each of the action branches. Our rational decision maker will look down the tree, consider the payoff from each branch, and choose the branch with the highest payoff.

The node at which the player has to make a choice is called a **decision node.** The nodes at the end of the tree where payoffs are attached are called **terminal nodes.** As

6. The notation $A \backslash B$ means "the elements that are in A but are not in B," or sometimes "the set A less the set B."

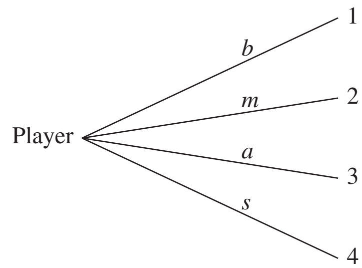

FIGURE 1.1 A simple breakfast decision tree.

the next chapter demonstrates, the structure of a decision tree will become slightly more involved and useful to capture more complex decision problems. We will return to similar trees in Chapter 7, where we consider the strategic interaction between many possible players, which is the main focus of this book.

1.2 The Rational Choice Paradigm

We now introduce *Homo economicus* or "economic man." *Homo economicus* is "rational" in that he chooses actions that maximize his well-being as defined by his payoff function over the resulting outcomes.[7] The assumption that the player is rational lies at the foundation of what is known as the **rational choice paradigm.** Rational choice theory asserts that when a decision maker is choosing between potential actions he will be guided by rationality to choose his best action. This can be assumed to be true for individual human behavior, as well as for the behavior of other entities, such as corporations, committees, or nation-states.

It is important to note, however, that by adopting the paradigm of rational choice theory we are imposing some implicit assumptions, which we now make explicit.

Rational Choice Assumptions The player fully understands the decision problem by knowing:

1. all possible actions, A;
2. all possible outcomes, X;
3. exactly how each action affects which outcome will materialize; and
4. his rational preferences (payoffs) over outcomes.

Perhaps at a first glance this set of assumptions may seem a bit demanding, and further contemplation may make you feel that it is impossible to satisfy for most decision problems. Still, it is a benchmark for a world in which decision problems are completely understood by the player, in which case he can approach the problems in a systematic and structured way. If we let go of any of these four knowledge

7. A naive application of the *Homo economicus* model assumes that our player knows what is best for his long-term well-being and can be relied upon to always make the right decision for himself. We take this naive approach throughout the book, though we will sometimes question how appropriate this approach is.

requirements then we cannot impose the notion of rational choice. If (1) is unknown then the player may be unaware of his best course of action. If (2) or (3) are unknown then he may not correctly foresee the actual consequences of his actions. Finally if (4) is unknown then he may incorrectly perceive the effect of his choice's consequence on his well-being.

To operationalize this paradigm of rationality we must choose among *actions,* yet we have defined preferences—and payoffs—over *outcomes* and not actions. It would be useful, therefore, if we could define preferences—and payoffs—over actions instead of outcomes. In the simple examples of choosing a cereal or how much water to drink, actions and outcomes were synonymous, yet this need not always be the case. Consider the situation of letting your friend drive drunk, in which the actions and outcomes are not the same. Still each action led to one and only one outcome: letting him drive leads to an accident, and getting him a cab leads to safe arrival. Hence, even though preferences and payoff were defined over outcomes, this *one-to-one correspondence,* or function, between actions and outcomes means that we can consider the preferences and payoffs to be over actions, and we can use this correspondence between actions and outcomes to define the payoff over actions as follows: if $x(a)$ is the outcome resulting from action a, then the payoff from action a is given by $v(a) = u(x(a))$, the payoff from $x(a)$. We will therefore use the notation $v(a)$ to represent the payoff from action a.[8] Now we can precisely define a rational player as follows:

Definition 1.2 A player facing a decision problem with a payoff function $v(\cdot)$ over actions is rational if he chooses an action $a \in A$ that maximizes his payoff. That is, $a^* \in A$ is chosen if and only if $v(a^*) \geq v(a)$ for all $a \in A$.

We now have a formal definition of *Homo economicus:* a player who has rational preferences and is rational in that he understands all the aspects of his decision problem and always chooses an option that yields him the highest payoff from the set of possible actions.

So far we have seen some simple examples with finite action sets. Consider instead an example with a continuous action space, which requires some calculus. Imagine that you're at a party and are considering engaging in social drinking. Given your physique, you'd prefer some wine, both for taste and for the relaxed feeling it gives you, but too much will make you sick. There is a one-liter bottle of wine, so your action set is $A = [0, 1]$, where $a \in A$ is how much you choose to drink. Your preferences are represented by the following payoff function over actions: $v(a) = 2a - 4a^2$, which is depicted in Figure 1.2. As you can see, some wine is better than no wine (0.1 liter gives you some positive payoff, while drinking nothing gives you zero), but drinking a whole bottle will be worse than not drinking at all ($v(1) = -2$). How much should you drink? Your maximization problem is

$$\max_{a\in[0,1]} \; 2a - 4a^2.$$

Taking the derivative of this function and equating it to zero to find the solution, we obtain that $2 - 8a = 0$, or $a = 0.25$, which is a bit more than two normal glasses of

8. To be precise, let $x : A \to X$ be the function that maps actions into outcomes, and let the payoff function over outcomes be $u : X \to \mathbb{R}$. Define the payoff over actions as the composite function $v = u \circ x : A \to \mathbb{R}$, where $v(a) = u(x(a))$.

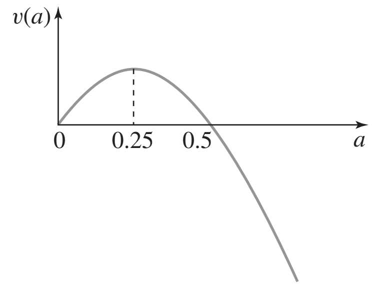

FIGURE 1.2 The payoff from drinking wine.

wine.[9] Thus, by considering how much wine to drink as a decision problem, you were able to find your optimal action.

1.3 Summary

- A simple decision problem has three components: actions, outcomes, and preferences over outcomes.
- A rational player has complete and transitive preferences over outcomes and hence can always identify a best alternative from among his possible actions. These preferences can be represented by a payoff (or profit) function over outcomes and the corresponding payoffs over actions.
- A rational player chooses the action that gives him the highest possible payoff from the possible set of actions at his disposal. Hence by maximizing his payoff function over his set of alternative actions, a rational player will choose his optimal decision.
- A decision tree is a simple graphic representation for decision problems.

1.4 Exercises

1.1 **Your Decision:** Think of a simple decision you face regularly and formalize it as a decision problem, carefully listing the actions and outcomes without the preference relation. Then assign payoffs to the outcomes and draw the decision tree.

1.2 **Going to the Movies:** There are two movie theaters in your neighborhood: Cineclass, which is located one mile from your home, and Cineblast, located three miles from your home. Each is showing three films. Cineclass is showing *Casablanca, Gone with the Wind,* and *Dr. Strangelove,* while Cineblast is showing *The Matrix, Blade Runner,* and *Aliens.* Your problem is to decide which movie to go to.

9. To be precise, we must also make sure that first, the second derivative is negative for the solution $a = 0.25$ to be a local maximum, and second, the value of $v(a)$ is not greater at the two boundaries $a = 0$ and $a = 1$. For more on maximizing the value of a function, see Section 19.3 of the mathematical appendix.

a. Draw a decision tree that represents this problem without assigning payoff values.
b. Imagine that you don't care about distance and that your preferences for movies are alphabetic (i.e., you like *Aliens* the most and *The Matrix* the least). Using payoff values 1 through 6 complete the decision tree you drew in part (1). Which option would you choose?
c. Now imagine that your car is in the shop and that the cost of walking each mile is equal to one unit of payoff. Update the payoffs in the decision tree. Would your choice change?

1.3 **Fruit or Candy:** A banana costs \$0.50 and a piece of candy costs \$0.25 at the local cafeteria. You have \$1.25 in your pocket and you value money. The money-equivalent value (payoff) you get from eating your first banana is \$1.20, and that of each additional banana is half the previous one (the second banana gives you a value of \$0.60, the third \$0.30, and so on). Similarly the payoff you get from eating your first piece of candy is \$0.40, and that of each additional piece is half the previous one (\$0.20, \$0.10, and so on). Your value from eating bananas is not affected by how many pieces of candy you eat and vice versa.

a. What is the set of possible actions you can take given your budget of \$1.25?
b. Draw the decision tree that is associated with this decision problem.
c. Should you spend all your money at the cafeteria? Justify your answer with a rational choice argument.
d. Now imagine that the price of a piece of candy increases to \$0.30. How many possible actions do you have? Does your answer to (c) change?

1.4 **Alcohol Consumption:** Recall the example in which you needed to choose how much to drink. Imagine that your payoff function is given by $\theta a - 4a^2$, where θ is a parameter that depends on your physique. Every person may have a different value of θ, and it is known that in the population (1) the smallest θ is 0.2; (2) the largest θ is 6; and (3) larger people have higher θs than smaller people.

a. Can you find an amount that no person should drink?
b. How much should you drink if your $\theta = 1$? If $\theta = 4$?
c. Show that in general smaller people should drink less than larger people.
d. Should any person drink more than one 1-liter bottle of wine?

1.5 **Buying a Car:** You plan on buying a used car. You have \$12,000, and you are not eligible for any loans. The prices of available cars on the lot are given as follows:

Make, model, and year	Price
Toyota Corolla 2002	\$9,350
Toyota Camry 2001	10,500
Buick LeSabre 2001	8,825
Honda Civic 2000	9,215
Subaru Impreza 2000	9,690

For *any given year,* you prefer a Camry to an Impreza, an Impreza to a Corolla, a Corolla to a Civic, and a Civic to a LeSabre. For *any given year,* you are willing to pay up to \$999 to move from any given car to the next preferred one. For example, if the price of a Corolla is \$$z$, then you are willing to buy it rather than a Civic if the Civic costs more than \$$(z - 999)$, but you would prefer to buy the Civic if it costs less than this amount. Similarly you prefer the Civic at \$$z$ to a Corolla that costs more than \$$(z + 1000)$, but you prefer the Corolla if it costs less. For *any given car,* you are willing to move to a model a year older if it is cheaper by at least \$500. For example, if the price of a 2003 Civic is \$$z$, then you are willing to buy it rather than a 2002 Civic if the 2002 Civic costs more than \$$(z - 500)$, but you would prefer to buy the 2002 Civic if it costs less than this amount.

a. What is your set of possible alternatives?
b. What is your preference relation between the alternatives in (a) above?
c. Draw a decision tree and assign payoffs to the terminal nodes associated with the possible alternatives. What would you choose?
d. Can you draw a decision tree with different payoffs that represents the same problem?

1.6 **Fruit Trees:** You have room for up to two fruit-bearing trees in your garden. The fruit trees that can grow in your garden are either apple, orange, or pear. The cost of maintenance is \$100 for an apple tree, \$70 for an orange tree, and \$120 for a pear tree. Your food bill will be reduced by \$130 for each apple tree you plant, by \$145 for each pear tree you plant, and by \$90 for each orange tree you plant. You care only about your total expenditure in making any planting decisions.

a. What is the set of possible actions and related outcomes?
b. What is the payoff of each action/outcome?
c. Draw the associated decision tree. What will a rational player choose?
d. Now imagine that the reduction in your food bill is half for the second tree of the same kind. (You like variety.) That is, the first apple tree still reduces your food bill by \$130, but if you plant two apple trees your food bill will be reduced by $\$130 + \$65 = \$195$, and similarly for pear and orange trees. What will a rational player choose now?

1.7 **City Parks:** A city's mayor has to decide how much money to spend on parks and recreation. City codes restrict this spending to no more than 5% of the budget, and the yearly budget of the city is \$20,000,000. The mayor wants to please his constituents, who have diminishing returns from parks. The money-equivalent benefit from spending \$$c$ on parks is $v(c) = \sqrt{400c} - \frac{1}{80}c$.

a. What is the action set for the city's mayor?
b. How much should the mayor spend?
c. The movie *An Inconvenient Truth* has shifted public opinion, and now people are more willing to pay for parks. The new preferences of the people are given by $v(c) = \sqrt{1600c} - \frac{1}{80}c$. What now is the action set for the mayor, and how much spending should he choose to cater to his constituents?

2

Introducing Uncertainty and Time

Now that we have a coherent and precise language to describe decision problems, we move on to be more realistic about the complexity of many such problems. The cereal example was fine to illustrate a simple decision problem and to get used to our formal language, but it is certainly not very interesting.

Consider a division manager who has to decide on whether a research and development (R&D) project is worthwhile. What will happen if he does not go ahead with it? Maybe over time his main product will become obsolete and outdated, and the profitability of his division will no longer be sustainable. Then again, maybe profits will still continue to flow in. What happens of he does go ahead with the project? It may lead to vast improvements in the product line and offer the prospect of sustained growth. Or perhaps the research will fail and no new products will emerge, leaving behind only a hefty bill for the expensive R&D endeavor. In other words, both actions have uncertainty over what outcomes will materialize, implying that the choice of a best action is not as obvious as in the cereal example.

How should the player approach this more complex problem? As you can imagine, using language like "maybe this will happen, or maybe that will happen" is not very useful for a rational player who is trying to put some structure on his decision problem. We must introduce a method through which the player can compare uncertain consequences in a meaningful way. For this approach, we will use the concept of stochastic (random) outcomes and probabilities, and we will describe a framework within which payoffs are defined over random outcomes.

2.1 Risk, Nature, and Random Outcomes

Put yourself in the shoes of our division manager who is deciding whether or not to embark on the R&D project. Denote his actions as g for going ahead or s for keeping the status quo, so that $A = \{g, s\}$. To make the problem as simple as possible, imagine that there are only two final outcomes: his product line is successful, which is equivalent to a profit of 10 (choose your denomination), or his product line is obsolete, which is equivalent to a profit of 0, so that $X = \{0, 10\}$. However, as already explained, there is no one-to-one correspondence here between actions and outcomes. Instead there is uncertainty about which outcome will prevail, and the uncertainty is tied to the choice made by the player, the division manager.

In order to capture this uncertainty in a precise way, we will use the well-understood notion of randomness, or risk, as described by a random variable. Use of random variables is the common way to precisely and consistently describe random prospects in mathematics and statistics. We will not use the most formal mathematical representation of a random variable but instead present it in its most useful depiction for the problems we will address. Section 19.4 of the mathematical appendix has a short introduction to random variables that you can refer to if this notion is completely new to you. Be sure to make yourself familiar with the concept: it will accompany us closely throughout this book.

2.1.1 Finite Outcomes and Simple Lotteries

Continuing with the R&D example, imagine that a successful product line is more likely to be created if the player chooses to go ahead with the R&D project, while it is less likely to be created if he does not. More precisely, the odds are 3 to 1 that success happens if g is chosen, while the odds are only 50-50 if s is chosen. Using the language of probabilities, we have the following description of outcomes following actions: If the player chooses g then the probability of a payoff of 10 is 0.75 and the probability of a payoff of 0 is 0.25. If, however, the player chooses s then the probability of a payoff of 10 is 0.5, as is the probability of a payoff of 0.

We can therefore think of the player as if he is choosing between two **lotteries.** A lottery is exactly described by a random payoff. For example, the state lottery offers each player either several million dollars or zero, and the likelihood of getting zero is extremely high. In our example, the choice of g is like choosing a lottery that pays zero with probability 0.25 and pays 10 with probability 0.75. The choice of s is like choosing a lottery that pays either zero or 10, each with an equal probability of 0.5.

It is useful to think of these lotteries as choices of another player that we will call "Nature." The probabilities of outcomes that Nature chooses depend on the actions chosen by our decision-making player. In other words, Nature chooses a probability distribution over the outcomes, and the probability distribution is conditional on the action chosen by our decision-making player.

We can utilize a decision tree to describe the player's decision problem that includes uncertainty. The R&D example is described in Figure 2.1. First the player takes an action, either g or s. Then, conditional on the action chosen by the player, Nature (denoted by N) will choose a probability distribution over the outcomes 10 and 0. The branches of the player are denoted by his actions, and the branches of Nature's

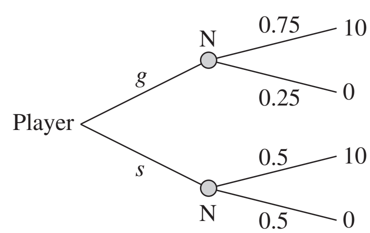

FIGURE 2.1 The R&D decision problem.

choices are denoted by their corresponding probabilities, which are conditional on the choice made by the player.

We now introduce a definition that generalizes the kind of randomness that was demonstrated by the R&D example. Consider a decision problem with n possible outcomes, $X = \{x_1, x_2, \ldots, x_n\}$.

Definition 2.1 A simple lottery over outcomes $X = \{x_1, x_2, \ldots, x_n\}$ is defined as a probability distribution $p = (p(x_1), p(x_2), \ldots, p(x_n))$, where $p(x_k) \geq 0$ is the probability that x_k occurs and $\sum_{k=1}^{n} p(x_k) = 1$.

By the definition of a probability distribution over elements in X, the probability of each outcome cannot be a negative number, and the sum of all probabilities over all outcomes must add up to 1. In our R&D example, following a choice of g, the lottery that Nature chooses is $p(10) = 0.75$ and $p(0) = 0.25$. Similarly, following a choice of s, the lottery that Nature chooses is $p(10) = p(0) = 0.5$.

Remark To be precise, the lottery that Nature chooses is conditional on the action taken by the player. Hence, given an action $a \in A$, the conditional probability that $x_k \in X$ occurs is given by $p(x_k|a)$, where $p(x_k|a) \geq 0$, and $\sum_{k=1}^{n} p(x_k|a) = 1$ for all $a \in A$.

Note that our trivial decision problem of choosing a cereal can be considered as a decision problem in which the probability over outcomes after any choice is equal to 1 for some outcome and 0 for all other outcomes. We call such a lottery a **degenerate lottery.** You can now see that decision problems with no randomness are just a very special case of those with randomness. Thus we have enriched our language to include more complex decision problems while encompassing everything we have developed earlier.

2.1.2 Simple versus Compound Lotteries

Arguably a player should care only about the probabilities of the various final outcomes that are a consequence of his actions. It seems that the exact way in which randomness unfolds over time should not be consequential to a player's well-being, but that only distributions over final outcomes should matter.

To understand this concept better, imagine that we make the R&D decision problem a bit more complicated. As before, if the player chooses not to embark on the R&D project (s) then the product line is successful with probability 0.5. If he chooses to go ahead with R&D (g) then two further stages will unfold. First, it will be determined whether the R&D effort was successful or not. Second, the outcome of the R&D phase will determine the likelihood of the product line's success. If the R&D effort is a failure then the success of the product is as likely as if no R&D had been performed; that is, the product line succeeds with probability 0.5. If the R&D effort is a success, however, then the probability of a successful product line jumps to 0.9. To complete the data for this example, we assume that R&D succeeds with probability 0.625 and fails with probability 0.375.

In this modified version of our R&D problem we have Nature moving once after the choice s and twice in a row after the choice g: once through the outcome of the R&D phase and then through the determination of the product line's success. This new decision problem is depicted in Figure 2.2.

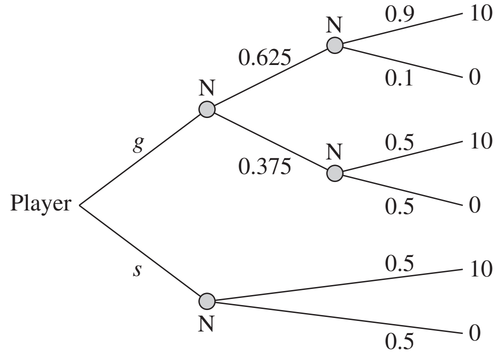

FIGURE 2.2 The modified R&D decision problem.

It seems like the two decision problems in Figures 2.1 and 2.2 are of different natures (no pun intended). Then again, let's consider what a decision problem ought to be about: actions, distributions over outcomes, and preferences. It is apparent that the player's choice of s in both Figure 2.1 and Figure 2.2 leads to the *same distribution* over outcomes. What about the choice of g? In Figure 2.2 this is followed by two random stages. However, the outcomes are still either 10 or 0. What are the probabilities of each outcome?

There are two ways that 10 can be obtained after the choice of g: First, with probability 0.625 the R&D project succeeds, and then with probability 0.9 the payoff 10 will be obtained. Hence the probability of "R&D success followed by 10" is equal to $0.625 \times 0.9 = 0.5625$. Second, with probability 0.375 the R&D project fails, and then with probability 0.5 the payoff 10 will be obtained. Hence the probability of "R&D failure followed by 10" is equal to $0.375 \times 0.5 = 0.1875$. Thus if the player chooses g then the probability of obtaining 10 is just the sum of the probabilities of these two *exclusive events,* which equals $0.5625 + 0.1875 = 0.75$. It follows that if the player chooses g then the probability of obtaining a payoff of 0 is 0.25, the complement of the probability of obtaining 10 (you should check this).

What then is the difference between the two decision problems? The first, simpler, R&D problem has a simple lottery following the choice of g. The second, more complex, problem has a *simple lottery over simple lotteries* following the choice of g. We call such lotteries over lotteries **compound lotteries.** Despite this difference, we impose on the player a rather natural sense of rationality. In his eyes the two decision problems are the same: he has the same set of actions, each one resulting in the same probability distributions over final outcomes. This innocuous assumption will make it easier for the player to evaluate and compare the benefits from different lotteries over outcomes.

2.1.3 Lotteries over Continuous Outcomes

Before moving on to describe how the player will evaluate lotteries over outcomes, we will go a step further to describe random variables, or lotteries, over continuous-outcome sets. To start, consider the following example. You are growing 10 tomato vines in your backyard, and your crop, measured in pounds, will depend on two inputs. The first is how much you water your garden per day and the second is the weather. Your action set can be any amount of water up to 50 gallons (50 gallons will completely flood your backyard), so that $A \in [0, 50]$, and your outcome set can be any amount of crop that 10 vines can yield, which is surely no more than 100 pounds, hence $X = [0, 100]$. Temperatures vary daily, and they vary continuously. This implies that your final yield, given any amount of water, will also vary continuously.

In this case we will describe the uncertainty not with a discrete probability, as we did for the R&D case, but instead with a **cumulative distribution function** (CDF) defined as follows:[1]

Definition 2.2 A simple lottery over an interval $X = [\underline{x}, \overline{x}]$ is given by a cumulative distribution function $F : X \to [0, 1]$, where $F(\widehat{x}) = \Pr\{x \le \widehat{x}\}$ is the probability that the outcome is less than or equal to $\widehat{x}$.

For those of you who have seen continuous random variables, this is not new. If you have not, Section 19.4 of the mathematical appendix may fill in some of the gaps.[2] The basic idea is simple. Because we have infinitely many possible outcomes, it is somewhat meaningless to talk about the probability of growing a certain exact weight of tomatoes. In fact it is correct to say that the probability of producing any particular predefined weight is zero. However, it is meaningful to talk about the probability of being below a certain weight x, which is given by the CDF $F(x)$, or similarly the probability of being above a certain weight x, which is given by the complement $1 - F(x)$.

Remark Just as in the case of finite outcomes, we wish to consider the case in which the distribution over outcomes is conditional on the action taken. Hence, to be precise, we need to use the notation $F(x|a)$.

Now that we have concluded with a description of what randomness is, we can move along to see how our decision-making player evaluates random outcomes.

2.2 Evaluating Random Outcomes

From now on we will consider the choice of an action $a \in A$ as the *choice of a lottery* over the outcomes in X. If the decision problem does not involve any randomness, then these lotteries are degenerate. This implies that we can stick to our notation of defining a decision problem by the three components of actions, outcomes, and preferences. The novelty is that each action is a lottery over outcomes.

The next natural question is: how will a player faced with the R&D problem in Figure 2.1 choose between his options of going forward or staying the course? Upon reflection, you may have already reached a conclusion. Despite the fact that his different choices lead to different lotteries, it seems that the two lotteries that follow g and s are easy to compare. Both have the same set of outcomes, a profit of 10 or a profit of 0. The choice g has a higher chance at getting the profit of 10, and hence we would expect anyone in their right mind to choose g. This implicitly assumes, however, that there are no costs to launching the R&D project.

1. The definition considers the outcome set to be a finite interval $X = [\underline{x}, \overline{x}]$. We can use the same definition for any subset of the real numbers, including the real line $(-\infty, \infty)$. An example of a lottery over the real line is the normal "bell-shape" distribution.

2. You are encouraged to learn this material since it will be useful, but one can continue through most of Parts I–III of this book without this knowledge.

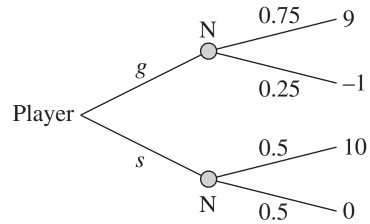

FIGURE 2.3 The R&D problem with costs.

Let's consider a less obvious revision of the R&D problem, and imagine that there is a real cost of pursuing the R&D project equivalent to 1. Hence the outcome of success yields a profit of 9 instead of 10, and the outcome of failure yields a profit of -1 instead of 0. This new problem is depicted in Figure 2.3. Now the comparison is not as obvious: is it better to have a coin toss between 10 and 0, or to have a good shot at 9, with some risk of losing 1?

2.2.1 Expected Payoff: The Finite Case

To our advantage, there is a well-developed methodology for evaluating how much a lottery is worth for a player, how different lotteries compare to each other, and how lotteries compare to "sure" payoffs (degenerate lotteries). This methodology, called "expected utility theory," was first developed by John von Neumann and Oskar Morgenstern (1944), two of the founding fathers of game theory, and explored further by Leonard Savage (1951). It turns out that there are some important assumptions that make this method of evaluation valid. (The foundations that validate expected payoff theory are beyond the scope of this text, and are rather technical in nature.)[3]

The intuitive idea is about averages. It is common for us to think of our actions as sometimes putting us ahead and sometimes dealing us a blow. But if *on average* things turn out on the positive side, then we view our actions as pretty good because the gains will more than make up for the losses. We want to take this idea, with its intuitive appeal, and use it in a precise way to tackle a single decision problem. To do this we introduce the following definition:

Definition 2.3 Let $u(x)$ be the player's payoff function over outcomes in $X = \{x_1, x_2, \ldots, x_n\}$, and let $p = (p_1, p_2, \ldots, p_n)$ be a lottery over X such that $p_k = \Pr\{x = x_k\}$. Then we define the player's **expected payoff from the lottery** p as

$$E[u(x)|p] = \sum_{k=1}^{n} p_k u(x_k) \; = p_1 u(x_1) + p_2 u(x_2) + \cdots + p_n u(x_n).$$

The idea of an expected payoff is naturally related to the intuitive idea of averages: if we interpret a lottery as a list of "weights" on payoff values, so that numbers that appear with higher probability have more weight, then the expected payoff of a lottery is nothing other than the weighted average of payoffs for each realization of the lottery.

3. The key idea was introduced by von Neumann and Morgenstern (1944) and is based on the "Independence Axiom." A nice treatment of the subject appears in Kreps (1990a, Chapter 3).

That is, payoffs that are more likely to occur receive higher weight while payoffs that are less likely to occur receive lower weight.

Using the definition of expected payoff we can revisit the R&D problem in Figure 2.3. First assume that the payoff to the player is equal to his profit, so that $u(x) = x$. By choosing g, the expected payoff to the player is

$$v(g) = E[u(x)|g] = 0.75 \times 9 + 0.25 \times (-1) = 6.5.$$

In contrast, by choosing s his expected payoff is

$$v(s) = E[u(x)|s] = 0.5 \times 10 + 0.5 \times 0 = 5.$$

Hence his best action using expected profits as a measure of preferences over actions is to choose g. You should be able to see easily that in the original R&D game in Figure 2.1 the expected payoff from s is still 5, while the expected payoff from g is 7.5, so that g was also his best choice, as we intuitively argued earlier.

Notice that we continue to use our notation $v(a)$ to define the expected payoff of an action given the distribution over outcomes that the action causes. This is a convention that we will use throughout this book, because the object of our analysis is what a player should do, and this notation implies that his ranking should be over his actions.

2.2.2 Expected Payoff: The Continuous Case

Consider the case in which the outcomes can be any one of a continuum of values distributed on some interval X. The definition of expected utility will be analogous, as follows:

Definition 2.4 Let $u(x)$ be the player's payoff function over outcomes in the interval $X = [\underline{x}, \overline{x}]$ with a lottery given by the cumulative distribution $F(x)$, with density $f(x)$. Then we define the player's expected payoff as[4]

$$E[u(x)] = \int_{\underline{x}}^{\overline{x}} u(x) f(x) dx.$$

To see an example with continuous actions and outcomes, recall the tomato growing problem in Section 2.1.3, in which your choice is how much water to use in the set $A = [0, 50]$ and the outcome is the weight of your crop that will result in the set $X = [0, 100]$. Imagine that given a choice of water $a \in A$, the distribution over outcomes is uniform over the quantity support $[0, 2a]$. (Alternatively the distribution of x conditional on a is given by $x|a \sim U[0, 2a]$.) For example, if you use 10 gallons of water, the output will be uniformly distributed over the weight interval [0, 20], with the cumulative distribution function given by $F(x|a = 10) = \frac{x}{20}$ for $0 \le x \le 20$,

4. More generally, if there are continuous distributions that do not have a density because $F(\cdot)$ is not differentiable, then the expected utility is given by

$$E[u(x)] = \int_{x \in X} u(x) dF(x).$$

This topic is covered further in Section 19.4 of the mathematical appendix.

and $F(x|a=10)=1$ for all $x>20$. More generally the cumulative distribution function is given by $F(x|a)=\frac{x}{2a}$ for $0\leq x\leq 2a$, and $F(x|a)=1$ for all $x>2a$. The density is given by $f(x|a)=\frac{1}{2a}$ for $0\leq x\leq 2a$, and $f(x|a)=0$ for all $x>2a$. Thus if your payoff from quantity x is given by $u(x)$ then your expected payoff from any choice $a\in A$ is given by

$$v(a)=E[u(x)|a]=\int_0^{2a} u(x)f(x|a)dx=\frac{1}{2a}\int_0^{2a} u(x)dx.$$

Given a specific function to replace $u(\cdot)$ we can compute $v(a)$ for any $a\in[0,50]$. As a concrete example, let $u(x)=18\sqrt{x}$. Then we have

$$v(a)=\frac{1}{2a}\int_0^{2a} 18x^{\frac{1}{2}}dx=\frac{9}{a}\left[\frac{2}{3}x^{\frac{3}{2}}\right|_0^{2a}=\frac{6}{a}(2a)^{\frac{3}{2}}=12\sqrt{2a}.$$

2.2.3 Caveat: It's Not Just the Order Anymore

Recall that when we introduced the idea of payoffs in Section 1.1.2, we argued that any payoff function that preserves the order of outcomes as ranked by the preference relation $\succsim$ will be a valid representation for the preference relation $\succsim$. It turns out that this statement is no longer true when we step into the realm of expected payoff theory as a paradigm for evaluating random outcomes.

Looking back at the R&D problem in Figure 2.3, we took a leap when we equated the player's payoff with profit. This step may seem innocuous: it is pretty reasonable to assume that, other things being equal, a rational player will prefer more money to less. Hence for the player in the R&D problem we have $10\succ 9\succ 0\succ -1$, a preference relation that is indeed captured by our imposed payoff function where $u(x)=x$.

What would happen if payoffs were not equated with profits? Consider a different payoff function to represent these preferences. In fact, consider only a slight modification as follows: $u(10)=10$, $u(9)=9$, $u(0)=0$, and $u(-1)=-8$. The order of outcomes is unchanged, but what happens to the expected payoffs? $E[u(s)]=5$ is unchanged, but now

$$v(g)=E[u(x)|g]=0.75\times 9+0.25\times(-8)=4.75.$$

Thus even though the order of preferences has not changed, the player would now prefer to choose s instead of g, just because of the different payoff number we assigned to the profit outcome of -1.

The reason behind this reversal of choice has important consequences. When we choose to use expected payoff then the *intensity of preferences* matters—something that is beyond the notion of simple order. We can see this from our intuitive description of expected payoff. Recall that we used the intuitive notion of "weights": payoffs that appear with higher probability have more weight in the expected payoff function. But then, if we change the number value of the payoff of some outcome without changing its order in the payoff representation, we are effectively changing its weight in the expected payoff representation.

This argument shows that, unlike payoff over *certain* outcomes, which is meant to represent *ordinal preferences* $\succsim$, the expected payoff representation involves a *cardinal ranking,* in which values matter just as much as order. At some level this implies that we are making assumptions that are not as innocuous about decision making when we extend our rational choice model to include preferences over lotteries and choices among lotteries. Nevertheless we will follow this prescription as a benchmark for putting structure on decision problems with uncertainty. We now briefly explore some implications of the intensity of preferences in evaluating random outcomes.

2.2.4 Risk Attitudes

Any discussion of the evaluation of uncertain outcomes would be incomplete without addressing a player's attitudes toward risk. By treating the value of outcomes as "payoffs" and by invoking the expected payoff criterion to evaluate lotteries, we have effectively circumvented the need to discuss risk, because by assumption all that people care about is their expected payoff.

To illustrate the role of risk attitudes, it will be useful to distinguish between monetary rewards and their associated payoff values. Imagine that a player faces a lottery with three monetary outcomes: $x_1 = \$4$, $x_2 = \$9$, and $x_3 = \$16$ with the associated probabilities p_1, p_2, and p_3. If the player's payoff function over money x is given by some function $u(x)$ then his expected payoff is

$$E[u(x)|p] = \sum_{k=1}^{3} p_k u(x_k) \ = p_1 u(x_1) + p_2 u(x_2) + p_3 u(x_3).$$

Now consider two different lotteries: $p' = (p'_1, p'_2, p'_3) = \left(\frac{7}{12}, 0, \frac{5}{12}\right)$ and $p'' = (p''_1, p''_2, p''_3) = (0, 1, 0)$. That is, the lottery p' randomizes between \$4 and \$16 with probabilities $\frac{7}{12}$ and $\frac{5}{12}$, respectively, while the lottery p'' picks \$9 for sure. Which lottery should the player prefer? The obvious answer will depend on the expected payoff of each lottery. If $\frac{7}{12}u(4) + \frac{5}{12}u(16) > u(9)$, then p' will be preferred to p'', and vice versa. This answer, by itself, tells us nothing about risk, but taken together with the special way in which p' and p'' relate to each other, it tells us a lot about the player's risk attitudes.

The lotteries p' and p'' were purposely constructed so that the average payoff of p' is equal to the sure payoff from p'': $\frac{7}{12} \times 4 + \frac{5}{12} \times 16 = 9$. Hence, *on average,* both lotteries offer the player the same amount of money, but one is a sure thing while the other is uncertain. If the player chooses p' instead of p'', he faces the risk of getting \$5 less, but he also has the chance of getting \$7 more. How then do his choices imply something about his attitude toward risk?

Imagine that the player is indifferent between the two lotteries, implying that $\frac{7}{12}u(4) + \frac{5}{12}u(16) = u(9)$. In this case we say that the player is **risk neutral,** because replacing a sure thing with an uncertain lottery that has the same expected monetary payout has no effect on his well-being. More precisely we say that a player is risk neutral if he is willing to exchange any sure payout with any lottery that promises *the same expected* monetary payout.

Alternatively the player may prefer not to be exposed to risk for the same expected payout, so that $\frac{7}{12}u(4) + \frac{5}{12}u(16) < u(9)$. In this case we say that the player is **risk averse.** More precisely a player is risk averse if he is *not* willing to exchange a sure payout with any (nondegenerate) lottery that promises *the same expected* monetary payout. Finally a player is **risk loving** if the opposite is true: he *strictly prefers* any lottery that promises the same expected monetary payout.

Remark Interestingly risk attitudes are related to the fact that the payoff representation of preferences matters above and beyond the rank order of outcomes, as discussed in Section 2.2.3. To see this imagine that $u(x) = x$. This immediately implies that the player is risk neutral: $\frac{7}{12}u(4) + \frac{5}{12}u(16) = 9 = u(9)$. In addition it is obvious from $u(\cdot)$ that the preference ranking is $\$16 \succ \$9 \succ \$4$. Now imagine that we use a different payoff representation for the same preference ranking: $u(x) = \sqrt{x}$. Despite the fact that the ordinal ranking is preserved, we now have $\frac{7}{12}u(4) + \frac{5}{12}u(16) = \frac{17}{6} < 3 = u(9)$. Hence a player with this modified payoff function, which *preserves the ranking* among the outcomes, will exhibit different risk attitudes.

2.2.5 The St. Petersburg Paradox

Some trace the first discussion of risk aversion to the St. Petersburg Paradox, so named in Daniel Bernoulli's original presentation of the problem and his solution, published in 1738 in the *Commentaries of the Imperial Academy of Science of Saint Petersburg.* The decision problem goes as follows.

You pay a fixed fee to participate in a game of chance. A "fair" coin (each side has an equal chance of landing up) will be tossed repeatedly until a "tails" first appears, ending the game. The "pot" starts at \$1 and is doubled every time a "head" appears. You win whatever is in the pot after the game ends. Thus you win \$1 if a tail appears on the first toss, \$2 if it appears on the second, \$4 if it appears on the third, and so on. In short, you win 2^{k-1} dollars if the coin is tossed k times until the first tail appears. (In the original introduction, this game was set in a hypothetical casino in St. Petersburg, hence the name of the paradox.)

The probability that the first "tail" occurs on the kth toss is equal to the probability of the "head" appearing $k-1$ times in a row and the "tail" appearing once. The probability of this event is $\left(\frac{1}{2}\right)^k$, because at any given toss the probability of any side coming up is $\frac{1}{2}$. We now calculate the expected *monetary value* of this lottery, which takes expectations over the possible events as follows: You win \$1 with probability $\frac{1}{2}$; \$2 with probability $\frac{1}{4}$; \$4 with probability $\frac{1}{8}$, and so on. The expected value of this lottery is

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \times 2^{k-1} = \sum_{k=1}^{\infty} \frac{1}{2} = \infty.$$

Thus the expected monetary value of this lottery is infinity! The reason is that even though large sums are very unlikely, when these events happen they are huge. For example, the probability that you will win more than \$1 million is less than one in 500,000!

When Bernoulli presented this example, it was very clear that no reasonable person would pay more than a few dollars to play this lottery. So the question is: where is the paradox? Bernoulli suggested a few answers, one being that of decreasing marginal payoff for money, or a concave payoff function over money, which is basically risk aversion. He correctly anticipated that the value of this lottery should not be measured in its expected monetary value, but instead in the monetary value of its *expected payoff.*

Throughout the rest of this book we will make no more references to risk preferences but instead assume that every player's preferences can be represented using expected payoffs. For a more in-depth exposition of attitudes toward risk, see Chapter 3 in Kreps (1990a) and Chapter 6 in Mas-Colell et al. (1995).

2.3 Rational Decision Making with Uncertainty

2.3.1 Rationality Revisited

We defined a rational player as one who chooses an action that maximizes his payoff among the set of all possible actions. Recall that the four rational choice assumptions in Section 1.2 included a requirement that the player know "exactly how each action affects which outcome will materialize."

For this knowledge to be meaningful and to guarantee that the player is correctly perceiving the decision problem when outcomes can be stochastic, it must be the case that he fully understands how each action translates into a lottery over the set of possible outcomes. In other words, the player knows that by choosing actions he is choosing lotteries, and he knows exactly what the probability of each outcome is, conditional on his choice of an action.

Understanding the requirements for rational decision making under uncertainty, together with the adoption of expected payoff as a means of evaluating random outcomes, offers a natural way to define rationality for decision problems with random outcomes:

Definition 2.5 A player facing a decision problem with a payoff function $u(\cdot)$ over outcomes is rational if he chooses an action $a \in A$ that maximizes his expected payoff. That is, $a^* \in A$ is chosen if and only if $v(a^*) = E[u(x)|a^*] \geq E[u(x)|a] = v(a)$ for all $a \in A$.

That is, the player, who understands the stochastic consequences of each of his actions, will choose an action that offers him the highest expected payoff. In the R&D problems described in Figures 2.1 and 2.3 the choice that maximizes expected payoff was to go ahead with the project and choose g.

2.3.2 Maximizing Expected Payoffs

As another illustration of maximizing expected payoff with a finite set of actions and outcomes, consider the following example. Imagine that you have been working after college and now face the decision of whether or not to get an MBA at a prestigious institution. The cost of getting the MBA is 10. (Again, you can decide on the denomination, but rest assured that this sum includes the income lost over the course of the two years you will be studying!) Your future value is your stream of income, which depends on the strength of the labor market for the next decade. If the labor market is strong then your income value from having an MBA is 32, while your income value from your current status is 12. If the labor market is average then your income value from having an MBA is 16, while your income value from your current status is 8. If the labor market is weak then your income value from having an MBA is 12, while your income value from your current status is 4. After spending some time researching the topic, you learn that the labor market will be strong with probability 0.25, average with probability 0.5, and weak with probability 0.25. Should you get the MBA?

This decision problem is depicted in Figure 2.4. Notice that following the decision of whether or not to get an MBA, we subtract the cost of the degree from the income benefit in each of the three states of nature. To solve this decision problem we first evaluate the expected payoff from each action. We have

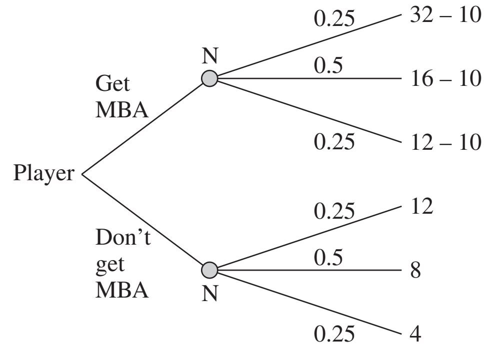

FIGURE 2.4 The MBA degree decision problem.

$$v(\text{Get MBA}) = 0.25 \times 22 + 0.5 \times 6 + 0.25 \times 2 = 9$$
$$v(\text{Don't get MBA}) = 0.25 \times 12 + 0.5 \times 8 + 0.25 \times 4 = 8.$$

Thus we conclude that, given the parameters of the problem, it is worth getting the MBA.

To illustrate the maximization of expected payoffs when there is a continuous set of actions and outcomes, consider the following example, which builds on the tomato growing problem from Section 2.2.2, with $A = [0, 50]$, $X = [0, 100]$ and the distribution of x conditional on a is uniform given by $x|a \sim U[0, 2a]$. We showed in the example in Section 2.2.2 that if the player's payoff from quantity x is given by $u(x)$ then his expected payoff from any choice $a \in A$ is given by

$$v(a) = E[u(x)|a] = \frac{1}{2a}\int_0^{2a} u(x)dx.$$

To account for the cost of water, assume that choosing $a \in A$ imposes a payoff cost of $2a$ (you can think of 2 being the cost of a gallon of water). Also assume that the payoff value from quantity x is given by the function $u(x) = 18\sqrt{x}$. (The square root function implies that the added value of every extra unit is less than the added value of the previous unit because this function is concave.) Then the player wishes to maximize his *expected net payoff*. This will be obtained by choosing the amount of water a that maximizes the difference between the expected benefit from choosing some $a \in A$ (given by $E[18\sqrt{x}|a]$) and the actual cost of $2a$. Thus the player's mathematical representation of the decision problem is given by

$$\max_{a\in[0,50]} \frac{1}{2a}\int_0^{2a} 18\sqrt{x}dx - 2a.$$

Solving for the integral, this is equivalent to maximizing the objective function $12\sqrt{2a} - 2a$. Differentiating this gives us the first-order condition for finding an optimum, which is

$$\frac{12}{\sqrt{2a}} - 2 = 0,$$

resulting in the solution $a = 18$, the quantity of water that maximizes the player's net expected payoff.[5] Plugging this back into the objective function yields the expected net payoff of $12\sqrt{2 \times 18} - 2 \times 18 = 36$.

2.4 Decisions over Time

The setup we have used up to now fits practically any decision problem in which you need to choose an action that has consequences for your well-being. Notice, however, that the approach we have used has a timeless, static structure. Our decision problems have shared the feature that you have to make a choice, after which some outcome will materialize and the payoff will be obtained. Many decision problems, however, are more involved; they require some decisions to follow others, and the sequencing of the decision making is imposed by the problem at hand.

2.4.1 Backward Induction

Once again, let's consider a modification of the R&D problem described in Figure 2.2, with a slight twist. Imagine that after the stage at which Nature first randomizes over whether or not the R&D project succeeds, the player faces another decision of whether to engage in a marketing campaign (m) or do nothing (d). The campaign can be launched only if the R&D project was executed. The marketing campaign will double the profits if the new product line is a success but not if it is a failure, and the campaign costs 6. The resulting modified decision problem is presented in Figure 2.5. For example, if the R&D effort succeeds and the marketing campaign is launched then profits from the product line will be 20 (double the original 10 in Figure 2.2), but the cost of the campaign, equal to 6, must be accounted for. This explains the payoff of $20 - 6 = 14$.

Now the question of whether the player should choose g or s is not as simple as before, because it may depend on what he will do later, after the fate of the R&D project is determined by Nature. Our assumption that the player is rational has some strong implications: he will be rational *at every stage* at which he faces a decision. At the beginning of the problem the player knows that he will act optimally to maximize his expected payoff at later stages, hence he can predict what he will do there.

This logic is the simple idea behind the optimization procedure known as **dynamic programming** or **backward induction.** To explain this procedure it is useful to separate the player's decision nodes into separate groups as follows: Group 1 will include all the nodes after which no more decision nodes exist, so that only Nature or final payoffs follow such nodes. For example, in Figure 2.5 Group 1 would include both decision nodes at which the player must decide between m and d. Then define Group 2 nodes as follows: a node k will belong to Group 2 if and only if the only decision nodes that follow any action at node k are decision nodes of Group 1. Define higher-order groups similarly.

Now consider all the nodes in Group 1 and figure out the optimal action of the player at each of these nodes. Once this is done we can compute the expected payoff from optimizing at that node, and that will be the value of the decision node.

5. We also need to check the second-order condition, that the second derivative of the objective function is negative at the proposed candidate. This is indeed the case, since the second derivative of the objective function is $-12(2a)^{\frac{3}{2}}$, which is always negative (the objective function is concave).

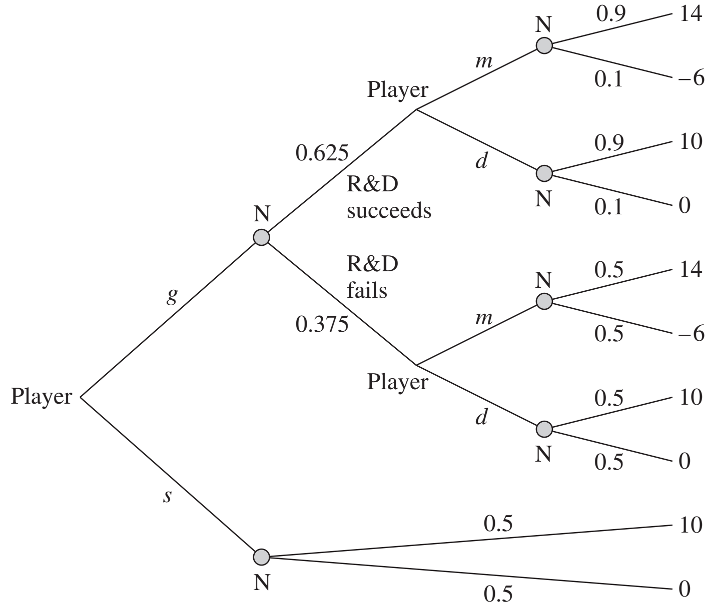

FIGURE 2.5 The R&D problem with a marketing phase.

Effectively we can throw away all the branches that follow the decision nodes of Group 1 and, assuming rationality, associate these nodes with the expected payoff from acting optimally. We can continue this process "backward" through the tree until we cover all the decision nodes of the player.

We can use the decision problem in Figure 2.5 to illustrate this procedure. First, we can compute the expected payoff of the player from choices m and d at the node after it has been determined by Nature that the R&D project was a success. We have

$$E[u(x)|\text{R\&D succeeds and } m] = 0.9 \times (20 - 6) + 0.1 \times (-6) = 12$$
$$E[u(x)|\text{R\&D succeeds and } d] = 0.9 \times 10 + 0.1 \times 0 = 9,$$

which implies that at this node the player will choose m in anticipation of an expected payoff of 12. Now consider his same choice problem at the node after it has been determined by Nature that the R&D project was a failure. We have

$$E[u(x)|\text{R\&D fails and } m] = 0.5 \times (20 - 6) + 0.5 \times (-6) = 4$$
$$E[u(x)|\text{R\&D fails and } d] = 0.5 \times 10 + 0.5 \times 0 = 5,$$

which implies that at this node the player will choose D in anticipation of an expected payoff of 5.

Imposing these rational decisions in the Group 1 nodes allows us to rewrite the decision tree as a simpler tree that already folds in the optimal decisions of the player at the Group 1 nodes. This "reduced" decision tree is depicted in Figure 2.6. The player's choice at the beginning of the tree is now easy to analyze. Taking into account his optimal actions after the R&D project's fate is determined, his expected payoff from choosing g is

$$v(g) = 0.625 \times 12 + 0.375 \times 5 = 9.375.$$

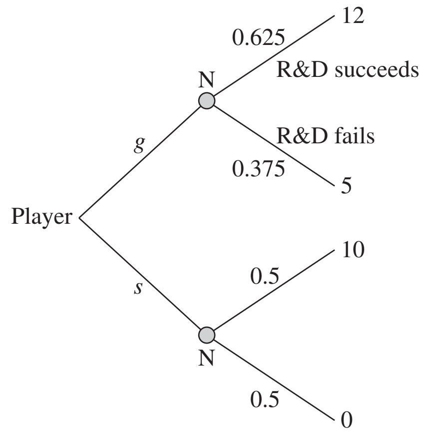

FIGURE 2.6 The marketing staged reduction of the R&D problem.

Because his expected payoff from choosing s is 5, it is clear that, anticipating his future decisions, his first decision should be to choose g.

2.4.2 Discounting Future Payoffs

In the R&D example as we have analyzed it so far, a player treated his costs and benefits equally. That is, even though the costs of the R&D project were incurred at the beginning, and the benefits came some (unspecified) time later, a dollar "today" was worth a dollar "tomorrow." However, this is often not the way current and future payments are evaluated. The convention in decision analysis is to *discount future payoffs* so that a dollar tomorrow is worth less than a dollar today.

For those who have had some experience with finance, the motivation for discounting future financial payoffs is simple. Imagine that you can invest money today in an interest-bearing savings account that yields 2% interest a year. If you invest \$100 today then you can receive $\$100 \times 1.02 = \$102$ in a year, $\$100 \times (1.02)^2 = \$104.04$ in two years, and similarly $\$100 \times (1.02)^t$ in t years. As we can see, any amount today will be worth more and more in *nominal* terms as we move further into the future. As a consequence, the opposite should be true: any amount $\$x$ that is expected in t years will be worth $\$v = \frac{x}{(1.02)^t}$ today precisely because we need only to invest $\$v$ today in order to get $\$x$ in t years. More generally, if the interest rate is r% per period, then any amount $\$x$ that is received in t periods is discounted and is worth only $\frac{x}{(1+r)^t}$ today.

Another motivation for discounting future payoffs is uncertainty over the future coupled with expected future values. Most people are quite certain that they will be alive and well a year from today. That said, there is always that small chance that one's future may be cut short due to illness or accident. (This is the reason that life insurance companies use actuarial tables.) Imagine that a player assesses that with probability $\delta \in (0, 1)$ he will be alive and well in one period (a year, a month, and so forth), while with probability $1 - \delta$ he will not. This implies that if he is offered a payoff of x in one period then his expected utility is $v = \delta x + (1 - \delta)0 = \delta x < x$. Similarly, if he is promised a payoff of x in t periods, then he would be willing to trade that promise for a payoff of $v = \delta^t x$ today.

More generally, if a player expects to receive a stream of payments $x_1, x_2, \ldots, x_T$ over the periods $t = 1, 2, \ldots, T$, and he evaluates payments with the utility function

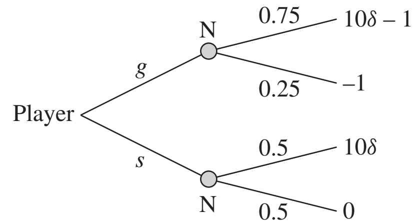

FIGURE 2.7 The R&D decision problem with discounting.

$u(x)$ in every period, then in the first period $t = 1$ his *discounted sum of future payoffs* is defined by

$$v(x_1, x_2, \ldots, x_T) = u(x_1) + \delta u(x_2) + \cdots + \delta^{T-1}u(x_T) = \sum_{t=1}^{T} \delta^{t-1}u(x_t).$$

The reason that discounting future payoffs is important is that changes in the discount rate will change the way decisions today are traded off against future payoffs. Consider the simple R&D problem. A successful product line, worth 10 in one period, will occur with probability 0.75 if the player chooses g, while only with probability 0.5 if he chooses s. The cost today of the R&D project is 1. If future payoffs are discounted at a rate of δ, then the problem is described by the decision tree depicted in Figure 2.7.

The expected payoffs for the player from choosing g or s are given by

$$v(g) = E[u(x)|g] = 0.75 \times (10\delta - 1) + 0.25 \times (-1) = 7.5\delta - 1$$

and

$$v(s) = E[u(x)|s] = 0.5 \times 10\delta + 0.5 \times 0 = 5\delta.$$

It is easy to see that the optimal decision will depend on the discount factor δ. In particular the player will choose to go ahead with the R&D development if and only if $7.5\delta - 1 > 5\delta$, or $\delta > 0.4$. The intuition is quite straightforward: the cost of the investment equal to 1 is borne today, and hence evaluated at a value of 1 regardless of the discount factor δ. Future payoffs, however, are discounted at the rate of δ. Hence as δ gets smaller the value of the future benefits from R&D decreases, while the value of today's costs remains the same, and once δ drops below the critical value of 0.4 the costs are no longer covered by the added benefits.

2.5 Applications

2.5.1 The Value of Information

When decisions lead to stochastic outcomes, our rational player chooses his action to maximize his expected payoff so that *on average* he is making the right choice. If the player actually knew the choice of Nature ahead of time then he might have chosen something else. Recall the example in Section 2.3.2, the decision of whether or not to get an MBA, which is depicted in Figure 2.4. As our previous analysis indicated, the expected payoff from getting an MBA is 9, while the expected payoff from not getting an MBA is only 8. Hence our rational decision maker will leave his job and go back to school, making what is, *on average,* the right choice.

Now imagine that an all-knowing oracle appears before the player the moment before he is about to resign from his job and says, "I know what the state of the labor market will be and I can tell you for a price." If you are the player, the questions that you need to answer are, first, is this information valuable, and second, how much is it worth?

It is quite easy to answer the first question: the information is valuable if it may cause you to change the decision you would have made without it. Looking at the decision problem, the answer is clear: If you learn that the labor market is strong then you will not change your decision. In contrast, if you learn that the labor market is weak then you would rather forgo the MBA program because it gives you a payoff of 2 while staying at your current job gives you a payoff of 4. The same decision would be made if you learn that the labor market is average in its strength. Hence it is quite clear that the oracle has what may be considered valuable information.

Now to our second question: how much is this information worth? This question is also not too hard to answer by considering the decision problem the player *would face* after the oracle's announcement. In particular we can calculate the expected payoff that the player anticipated before making a decision *without the advice of the oracle* and compare it to the expected payoff the player anticipated before making a decision *with the oracle's advice.* The difference is clearly due to the oracle's information, and this will be our measure of the value of the information.

We concluded that the choice the player would have made without the oracle's advice was to get an MBA, and that the expected payoff of this choice was 9, which is the expected value of the decision problem depicted in Figure 2.4. With the oracle's advice, though, the player can make a decision *after* learning the state of the labor market. How does this affect his expected payoff? The correct way to calculate this is to maintain the probability distribution over the three states of Nature, but to take into account that the player will make different choices that *depend on the state of Nature,* unlike the case in which he has to make a choice that applies to *all states of Nature.*

This new decision problem is shown in Figure 2.8, and as we argued, the player will condition his behavior on the oracle's advice. In particular the labor market will be strong with probability 0.25, and in this case the player will get an MBA and will have a payoff of 22. The labor market will be average with probability 0.5, and in this case the player will not get an MBA and will have a payoff of 8. Finally the labor market will be weak with probability 0.25, and in this case the player will not get an MBA and will have a payoff of 4. This implies that, *before* hearing the oracle's advice, with the anticipation of using the oracle's advice, the player will have an expected payoff of

$$E[u] = 0.25 \times 22 + 0.5 \times 8 + 0.25 \times 4 = 10.5.$$

Thus we can conclude that with the oracle's advice the expected payoff to the player is 10.5, which is 1.5 more than his expected payoff without the oracle's advice. This answers our second question: the oracle's information is worth 1.5 to the player.

In general when a decision maker is faced with the option of whether or not to acquire information and how much to pay for it, then by comparing the decision problem with the added information to the decision problem without the additional information, the player will be able to calculate the value of the information. Of

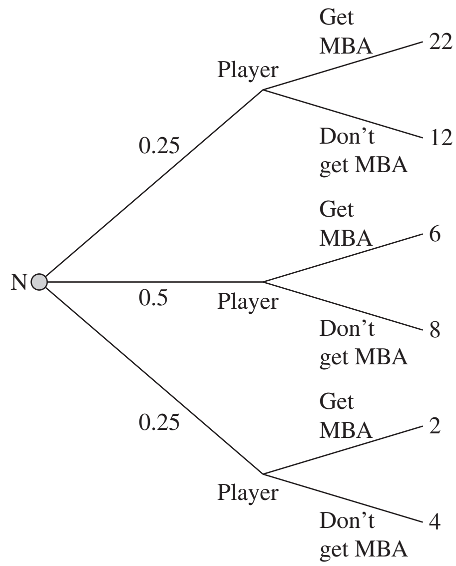

FIGURE 2.8 The MBA decision problem with the oracle's information.

course, this approach assumes that the player knows exactly what kind of information he can receive, how each piece of information will affect his payoffs, and what are the probabilities that each event will happen. Despite relying on a very demanding set of assumptions, this approach is valuable in offering a framework for decision making and valuing information.

2.5.2 Discounted Future Consumption

Usually we receive income or monetary gifts every so often, but we need to consume over several periods of time in between these income events. For example, you may receive a paycheck every month, but after paying your monthly costs, like rent and utilities, you need to buy groceries every week. If you spend too much during the first week, you may go hungry toward the end of the month. This kind of problem is known as *choosing consumption over time.*

Imagine a player who has $\$K$ today that need to be consumed over the next two periods, $t = 1, 2$. The utility over consuming $\$x$ in any period is given by the concave utility function $u(x)$, with $u'(x) > 0$ and $u''(x) < 0$. At period $t = 1$, the player values his utility from consuming x_2 in period $t = 2$ at the discounted value of $\delta u(x_2)$, so that at period $t = 1$ the player maximizes his present value of utility given by

$$\max_{x_1} \; u(x_1) + \delta u(K - x_1),$$

and the player's first-order condition is therefore[6]

$$u'(x_1) = \delta u'(K - x_1). \tag{2.1}$$

6. Because $u(\cdot)$ is concave, the second-order condition (that the second derivative be zero) is satisfied, implying that the solution to the first-order condition is the maximum.

For example, if $u(x) = \ln(x)$, so that $u'(x) = \frac{1}{x}$, the solution to the player's problem will be

$$\frac{1}{x_1} = \frac{\delta}{K - x_1}$$

or

$$x_1 = \frac{K}{1+\delta}. \tag{2.2}$$

Turning back to the player's first-order condition in equation (2.1), the solution has a nice intuitive flavor that will be familiar to students of economics. Because the player needs to allocate a scarce resource, K, across the two periods, the optimal solution equates the *marginal values* across the two periods, taking into account that the second period is discounted by the factor δ. Turning to the specific form using the natural log function in (2.2), we can see that when period $t = 2$ is not discounted ($\delta = 1$) then the player splits K equally between the two periods. This is the case in which the player perfectly *smooths* his first-period resource across the two periods. As discounting becomes stronger, more of the scarce resource is consumed in the first period, and when the second period is completely discounted (δ goes to zero) then the player consumes all of the resource in the first period.

2.6 Theory versus Practice

We required the player to be rational in that he fully understands the decision problem he is facing. Hence if we present the same decision problem in different ways to the player, then as long as our presentation is loyal to the facts and includes all the relevant information, he should be able to decipher the true problem regardless of the way we present it to him. Yet Amos Tversky and Daniel Kahneman (1981) have shown that "framing," or the way in which a problem is presented, can affect the choices made by a decision maker. This would imply that our fundamental assumptions of rational choice do not hold.

Tversky and Kahneman demonstrated systematic "preference reversals" when the same problem was presented in different ways, in particular for the "Asian disease" decision problem. Physician participants were asked to "imagine that the U.S. is preparing for the outbreak of an unusual Asian disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed. Assume the exact scientific estimates of the consequences of the programs are as follows." The first group of participants was presented with a decision problem between two programs:

Program A 200 people will be saved.
Program B There is a $\frac{1}{3}$ probability that 600 people will be saved and a $\frac{2}{3}$ probability that no people will be saved.

When presented with these alternatives, 72% of participants preferred program A and 28% preferred program B. The second group of participants was presented with the following choice:

Program C 400 people will die.
Program D There is a $\frac{1}{3}$ probability that nobody will die, and a $\frac{2}{3}$ probability that 600 people will die.

For this decision problem, 78% preferred program D, while the remaining 22% preferred program C. Notice, however, that programs A and C *are identical,* as are programs B and D. A change in the framing of the decision problem between the two groups of participants produced a "preference reversal," with the first group preferring program A/C and the second group preferring program B/D.

Many argue today that framing biases affect a host of decisions made by people in their daily lives. Preference reversals and other associated phenomena have been key subjects for research in the flourishing field of behavioral economics, which is primarily involved in the study of behavior that contradicts the predictions of rational choice theory.

Nevertheless the rather naive and simple framework of rational choice theory goes a long way toward helping us understand the decisions of many individuals. Furthermore many argue that the "rational" decision makers will drive out the irrational ones when the stakes are high enough, and hence when we look at behavior that persists in situations for which some people have the opportunity to learn the environment, that behavior will often be consistent with rational choice theory. I will use the rational choice framework throughout this book, and I will leave its defense to others.

2.7 Summary

- When prospects are uncertain, a rational decision maker needs to put structure on the problem in the form of probability distributions over outcomes that we call *lotteries.*
- Whether the acts of Nature evolve over time or whether they are chosen once and for all, a rational player cares only about the distribution over final outcomes. Hence any series of *compound lotteries* can be compressed to its equivalent *simple lottery.*
- When evaluating lotteries, we use the expected payoff criterion. Hence every lottery is evaluated by the expected payoff it offers the player.
- Unlike certain outcomes in which only the order of preferences matters, when random outcomes are evaluated with expected payoffs, the magnitude of payoffs matters as well. The difference in payoff values between outcomes will also be related to a player's risk preferences.
- Rational players will always choose the action that offers them the highest expected payoff.
- When decisions need to be made over time, a rational player will solve his problem "backwards" so that early decisions take into account later decisions.
- Payoffs that are received in future periods will often be discounted in earlier periods to reflect impatience, costs of capital, or uncertainty over whether future periods will be relevant.

2.8 Exercises

2.1 **Getting an MBA:** Recall the decision problem in Section 2.3.2, and now assume that the probability of a strong labor market is p, that of an average labor market is 0.5, and that of a weak labor market is $0.5 - p$. All the other values are the same.

a. For which values of p will you decide not to get an MBA?
b. If $p = 0.4$, what is the highest price the university can charge for you to be willing to go ahead and get an MBA?

2.2 **Recreation Choices:** A player has three possible activities from which to choose: going to a football game, going to a boxing match, or going for a hike. The payoff from each of these alternatives will depend on the weather. The following table gives the agent's payoff in each of the two relevant weather events:

Alternative	Payoff if rain	Payoff if shine
Football game	1	2
Boxing match	3	0
Hike	0	1

Let p denote the probability of rain.

a. Is there an alternative that a rational player will never take regardless of p?
b. What is the optimal decision as a function of p?

2.3 **At the Dog Races:** You're in Las Vegas, and you must decide what to do at the dog-racing betting window. You may choose not to participate or you may bet on one of two dogs as follows. Betting on Snoopy costs \$1, and you will be paid \$2 if he wins. Betting on Lassie costs \$1, and you will be paid \$11 if she wins. You believe that Snoopy has probability 0.7 of winning and that Lassie has probability 0.1 of winning (there are other dogs on which you are not considering betting). Your goal is to maximize the expected monetary return of your action.

a. Draw the decision tree for this problem.
b. What is your best course of action, and what is your expected value?
c. Someone offers you gambler's "anti-insurance," which you may accept or decline. If you accept it, you get paid \$2 up front and you agree to pay back 50% of any winnings you receive. Draw the new decision tree and find the optimal action.

2.4 **Drilling for Oil:** An oil drilling company must decide whether or not to engage in a new drilling venture before regulators pass a law that bans drilling on that site. The cost of drilling is \$1 million. The company will learn whether or not there is oil on the site only after drilling has been completed and all drilling costs have been incurred. If there is oil, operating profits are estimated at \$4 million. If there is no oil, there will be no future profits.

a. Using p to denote the likelihood that drilling results in oil, draw the decision tree for this problem.
b. The company estimates that $p = 0.6$. What is the expected value of drilling? Should the company go ahead and drill?
c. To be on the safe side, the company hires a specialist to come up with a more accurate estimate of p. What is the minimum value of p for which it would be the company's best response to go ahead and drill?

2.5 **Discount Prices:** A local department store sells products at a given initial price, and every week a product goes unsold, its price is discounted by 25% of the original price. If it is not sold after four weeks, it is sent back to the regional warehouse. A set of kitchen knives was just put out for \$200. Your willingness to pay for the knives (your dollar value) is \$180, so if you buy them at a price P, your payoff is $u = 180 - P$. If you don't buy the knives, the chances that they will be sold to someone else, conditional on not having been sold the week before, are as follows:

Week 1	0.2
Week 2	0.4
Week 3	0.6
Week 4	0.8

For example, if you do not buy the knives during the first two weeks, the likelihood that they will be available at the beginning of the third week is the likelihood that they do not sell in either week 1 or week 2, which is $0.8 \times 0.6 = 0.48$.

a. Draw your decision tree for the four weeks after the knives are put out for sale.
b. At the beginning of which week, if any, should you run to buy the knives?
c. Find a willingness to pay for the knives that would make it optimal to buy at the beginning of the first week.
d. Find a willingness to pay that would make it optimal to buy at the beginning of the fourth week.

2.6 **Real Estate Development:** A real estate developer wishes to build a new development. Regulations require an environmental impact study that will yield an "impact score," which is an index number based on the impact the development will likely have on such factors as traffic, air quality, and sewer and water usage. The developer, who has lots of experience, knows that the score will be no less than 40 and no more than 70. Furthermore he knows that any score between 40 and 70 is as likely as any other score between 40 and 70 (use continuous values). The local government's past behavior implies that there is a 35% chance that it will approve the development if the impact score is less than 50 and a 5% chance that it will approve the development if the score is between 50 and 55; if the score is greater than 55 then the project will surely be halted. The value of the development to the developer is \$20 million. Assuming that the developer is risk neutral, what is the maximum cost of the impact study such that it is still worthwhile for the developer to have it conducted?

2.7 **Toys:** WakTek is a manufacturer of electronic toys, with a specialty in remote-controlled miniature vehicles. WakTek is considering the introduction of a new product, a remote-controlled hovercraft called WakAtak. Preliminary designs have already been produced at a cost of \$2 million. To introduce a marketable product requires the building of a dedicated production line at a cost of \$12 million. In addition, before the product can be launched a prototype must be built and tested for safety. The prototype can be crafted in the absence of a production line, at a cost of \$0.5 million, but if the prototype is created after the production line is built then its cost is negligible (you can treat it as zero). There is uncertainty over what safety rating WakAtak will get. This could have a significant impact on demand, as a lower safety rating will increase the minimum age required of users. Safety testing costs \$1 million.

The outcome of the safety test is estimated to have a 65% chance of resulting in a minimum user age of 8 years, a 30% chance of a minimum age of 15 years, and a 5% chance that the product will be declared unsafe—in which case it could not be sold at all. (The cost of improving the safety rating of a finished design is deemed prohibitive.) After successful safety testing the product could be launched at a cost of $1.5 million.

There is also uncertainty over demand, which will have a crucial impact on the eventual profits. Currently the best estimate is that the finished product, if available to the 8- to 14-year demographic, has a 50-50 chance of resulting in profits of either $10 million or $5 million from that demographic. Similarly there is a 50-50 chance of either $14 million or $6 million in profits from the 15-year-or-above demographic. These demand outcomes are independent across the demographics. The profits do not take into account the costs previously defined; they are measured in expected present-value terms, so they are directly comparable with the costs.

a. What is the optimal plan of action for WakTek? What is the current expected economic value of the WakAtak project?
b. Suddenly it turns out that the original estimate of the cost of safety testing was incorrect. Analyze the sensitivity of WakTek's optimal plan of action to the cost of safety testing.
c. Suppose WakTek also has the possibility of conducting a market survey, which would tell it exactly which demand scenario is true. This market research costs $1.5 million if done simultaneously for both demographics and $1 million if done for only one demographic. How, if at all, is your answer to part (a) affected?
d. Suppose that demand is not independent across demographics after all, but instead is perfectly correlated (i.e., if demand is high in one demographic then it is for sure high in the other one as well). How, if at all, would that change your answer to part (c)?

2.8 **Juice:** Bozoni is a Swiss maker of fruit and vegetable juice, whose products are sold at specialty stores throughout Western Europe. Bozoni is considering whether to add cherimoya juice to its line of products. "It would be one of our more difficult varieties to produce and distribute," observes Johann Ziffenboeffel, Bozoni's CEO. "The cherimoya would be flown in from New Zealand in firm, unripe form, and it would need its own dedicated ripening facility here in Europe." Three successful steps are absolutely necessary for the new cherimoya variety to be worth producing. The industrial ripening process must be shown to allow the delicate flavors of the cherimoya to be preserved; the testing of the ripening process requires the building of a small-scale ripening facility. Market research in selected limited regions around Europe must show that there is sufficient demand among consumers for cherimoya juice. And cherimoya juice must be shown able to withstand the existing tiny gaps in the cold chain (the temperature-controlled supply chain) between the Bozoni plant and the end consumers (these gaps would be prohibitively expensive to fix). Once these three steps have been completed, there would be about €2,500,000 worth of expenses in launching the new variety of juice. A successful new variety will then yield profits, in expected present-value terms, of €42.5 million.

The three necessary steps can be done in parallel or sequentially, and in any order. Data about these three steps are given in the following table:

Step	Probability of success	Cost
Ripening process	0.7	€1,000,000
Market research	0.3	5,000,000
Cold chain	0.6	500,000

"Probability of success" refers to how likely it is that the step will be successful. If it is not successful, then that means that cherimoya juice cannot be sold at a profit. All probabilities are independent of each other (i.e., whether a given step is successful or not does not affect the probabilities that the other steps will be successful). "Cost" refers to the cost of undertaking the step (regardless of whether it is successful or not).

a. Suppose Mr. Ziffenboeffel calls you and asks your advice about the project. In particular he wants to know (i) should he undertake the three necessary steps in parallel (i.e., all at once) or should he undertake them sequentially, and (ii) if sequentially, what's the correct order in which the steps should be done? What answers do you give him?

b. Mr. Ziffenboeffel calls you back. Since the table was produced, Bozoni has found a small research firm that can perform the necessary tests for the ripening process at a lower cost than Bozoni's in-house research department. At the same time, the European Union (EU) has tightened the criteria for getting approval for new food-producing facilities, which raises the costs of these tests. Mr. Ziffenboeffel would like to know how your answer to (a) changes as a function of the cost of the ripening test. What do you tell him?

c. Mr. Ziffenboeffel calls you back yet again. The good news is that the cost of adhering to the EU regulations and the savings from outsourcing the ripening tests balance each other out, so the cost of the test remains €1,000,000. Now the problem is that his marketing department is suggesting that the probability that the market research will result in good news about demand for the juice could be different in light of recent data on the sales of other subtropical fruit products. He would like to know how your answer to (a) changes as a function of the probability of a positive result from the market research. What do you tell him?

2.9 **Steel:** AK Steel Holding Corporation is a producer of flat-rolled carbon, stainless, and electrical steels and tubular products through its wholly owned subsidiary, AK Steel Corporation. The recent surge in demand for steel significantly increased AK's profits,[7] and it is now engaged in a research project to improve its production of rolled steel. The research involves three distinct

7. See "Demand Sends AK Steel Profit Up 32%," *New York Times,* July 23, 2008, http://www.nytimes.com/2008/07/23/business/23steel.html?partner=rssnyt&emc=rss.

steps, each of which must be successfully completed before the firm can implement the cost-saving new production process. If the research is completed successfully, it will save the firm $4 million. Unfortunately there is a chance that one or more of the research steps might fail, in which case the entire project would be worthless. The three steps are done sequentially, so that the firm knows whether one step was successful before it has to invest in the next step. Each step has a 0.8 probability of success and costs $500,000. The risks of failure in the three steps are uncorrelated with one another. AK Steel is a risk-neutral company. (In case you are worried about such things, the interest rate is zero.)

a. Draw the decision tree for the firm.
b. If the firm proceeds with this project, what is the probability that it will succeed in implementing the new production process?
c. If the research were costless, what would be the firm's expected gain from it before the project began?
d. Should the firm begin the research, given that each step costs $500,000?
e. Once the research has begun, should the firm quit at any point even if it has had no failures? Should it ever continue the research even if it has had a failure?

After the firm has successfully completed the first two steps, it discovers an alternate production process that would cost $150,000 and would lower production costs by $1 million with certainty. This process, however, is a substitute for the three-step cost-saving process; they cannot be used simultaneously. Furthermore, to have this process available, the firm must spend the $150,000 before it knows if it will successfully complete the third step of the three-step research project.

f. Draw the augmented decision tree that includes the possibility of pursuing this alternate production process.
g. If the firm continues the three-step project, what is the chance it would get *any* value from also developing the alternate production process?
h. If developing the alternate production process were costless and if the firm continues the three-step project, what is the expected value that it would get from having the alternate production process available (at the beginning of the third research step)? (This is known as the *option value* of having this process available.)
i. Should the firm
   i. Pursue only the third step of the three-step project?
   ii. Pursue only the alternate production process?
   iii. Pursue both the third step of the three-step project and the alternate production process?
j. If the firm had known of the alternate production process before it began the three-step research project, what should it have done?

2.10 **Surgery:** A patient is very sick and will die in 6 months if he goes untreated. The only available treatment is risky surgery. The patient is expected to live for 12 months if the surgery is successful, but the probability that the surgery will fail and the patient will die immediately is 0.3.

a. Draw a decision tree for this decision problem.

b. Let $v(x)$ be the patient's payoff function, where x is the number of months until death. Assuming that $v(12) = 1$ and $v(0) = 0$, what is the lowest payoff the patient can have for living 3 months so that having surgery is a best response?

For the rest of the problem, assume that $v(3) = 0.8$.

c. A test is available that will provide some information that predicts whether or not surgery will be successful. A positive test implies an increased likelihood that the patient will survive the surgery as follows:

True-positive rate: The probability that the results of this test will be positive if surgery is to be successful is 0.90.

False-positive rate: The probability that the results of this test will be positive if the patient will not survive the operation is 0.10.

What is the probability of a successful surgery if the test is positive?

d. Assuming that the patient has the test done, at no cost, and the result is positive, should surgery be performed?

e. It turns out that the test may have some fatal complications; that is, the patient may die during the test. Draw a decision tree for this revised problem.

f. If the probability of death during the test is 0.005, should the patient opt to have the test prior to deciding on the operation?

2.11 **To Run or Not to Run:** You're a sprinter, and in practice today you fell and hurt your leg. An x-ray suggests that it's broken with probability 0.2. Your problem is deciding whether you should participate in next week's tournament. If you run, you think you'll win with probability 0.1. If your leg is broken and you run, then it will be further damaged and your payoffs are as follows:

$+100$ if you win the race and your leg isn't broken
$+50$ if you win and your leg is broken
0 if you lose and your leg isn't broken
-50 if you lose and your leg is broken
-10 if you don't run and if your leg is broken
0 if you don't run and your leg isn't broken

a. Draw the decision tree for this problem.

b. What is your best choice of action and its expected payoff?

You can gather some more information by having more tests, and you can gather more information about whether you'll win the race by talking to your coach.

c. What is the value of perfect information about the state of your leg?

d. What is the value of perfect information about whether you'll win the tournament?

e. As stated previously, the probability that your leg is broken and the probability that you will win the tournament are independent. Can you use a decision tree in the case that the probability that you will win the race depends on whether your leg is broken?

2.12 **More Oil:** Chevron, the number 2 U.S. oil company, was facing a tough decision. The new oil project dubbed "Tahiti" was scheduled to produce its first commercial oil in mid-2008, yet it was still unclear how productive it would be. “Tahiti is one of Chevron’s five big projects,” said Peter Robertson, vice chairman of the company’s board, to the *Wall Street Journal*.[8] Nevertheless it was unclear whether the project would result in the blockbuster success Chevron was hoping for. As of June 2007 $4 billion had been invested in the high-tech deep sea platform, which sufficed to perform early well tests. Aside from offering information on the type of reservoir, the tests would produce enough oil to just cover the incremental costs of the testing (beyond the $4 billion investment).

Following the test wells, Chevron predicted one of three possible scenarios. The optimistic one was that Tahiti sits on one giant, easily accessible oil reservoir, in which case the company expected to extract 200,000 barrels a day after expending another $5 billion in platform setup costs, with a cost of extraction of about $10 a barrel. This would continue for 10 years, after which the field would have no more economically recoverable oil. Chevron believed this scenario had a 1-in-6 chance of occurring. A less rosy scenario, twice as likely as the optimistic one, was that Chevron would have to drill two more wells at an additional cost of $0.5 billion each (above and beyond the $5 billion setup costs), in which case production would be around 100,000 barrels a day with a cost of extraction of about $30 a barrel, and the field would still be depleted after 10 years. The worst-case scenario involves the oil being tucked away in numerous pockets, requiring expensive water injection techniques, which would involve upfront costs of another $4 billion (above and beyond the $5 billion setup costs) and extraction costs of $50 a barrel; production would be estimated to be at about 60,000 barrels a day for 10 years. Bill Varnado, Tahiti’s project manager, was quoted as giving this least desirable outcome odds of 50-50.

The current price of oil is $70 a barrel. For simplicity assume that the price of oil and all costs will remain constant (adjusted for inflation) and that Chevron faces a 0% cost of capital (also adjusted for inflation).

a. If the test wells would not produce information about which one of three possible scenarios would result, should Chevron invest the setup costs of $5 billion to be prepared to produce under whichever scenario is realized?
b. If the test wells do produce accurate information about which of three possible scenarios is true, what is the added value of performing these tests?

2.13 **Today, Tomorrow, or the Day After:** A player has $100 today that needs to be consumed over the next three periods, $t = 1, 2, 3$. The utility over consuming $\$x_t$ in period t is given by the utility function $u(x) = \ln(x)$, and at period $t = 1$ the player values his net present value from all consumption as $u(x_1) + \delta u(x_2) + \delta^2 u(x_3)$, where $\delta = 0.9$.

a. How will the player plan to spend the $100 over the three periods of consumption?
b. Imagine that the player knows that in period $t = 2$ he will receive an additional gift of $20. How will he choose to allocate his original $100 initially, and how will he spend the extra $20?

8. “Chevron Bets Big on Gulf Output,” *Wall Street Journal,* June 27, 2007, http://online.wsj.com/article/SB118291402301349620.html.