Chapter 9

Currency Options (2): Hedging and Valuation
Overview

The Binomial Logic: One-period pricing
- The Replication Approach
- The Hedging Approach
- The Risk-adjusted Probabilities

Multiperiod Pricing: Assumptions
- Notation
- Assumptions
- Discussion

Stepwise Multiperiod Binomial Option Pricing
- Backward Pricing, Dynamic Hedging
- What can go wrong?
- American-style Options

Towards Black-Merton-Scholes
- STP-ing of European Options
- Towards the Black-Merton-Scholes Equation
- The Delta of an Option
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Binomial Models—What & Why?

◊ **Binomial Model**

- given $S_t$, there only two possible values for $S_{t+1}$, called “up” and “down”.

◊ **Restrictive?—Yes, but ...**

- the distribution of the total return, after many of these binomial price changes, becomes bell-shaped
- the binomial option price converges to the BMS price
- the binomial math is much more accessible than the BMS math
- BinMod can be used to value more complex derivatives that have no closed-form Black-Scholes type solution.

◊ **Ways to explain the model—all very similar:**

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Our Example

◇ Data

▷ $S_0 = \text{INR/MTL 100}$, $r = 5\%p.p.; r^* = 3.9604\%$. Hence:

$$F_{0,1} = S_0 \frac{1 + r_{0,1}}{1 + r^*_{0,1}} = 100 \frac{1.05}{1.039604} = 101.$$

▷ $S_1$ is either $S_{1,u} = 110$ ("up") or $S_{1,d} = 95$ ("down").

▷ 1-period European-style call with $X=\text{INR/MTL 105}$

slope of exposure line (exposure):

$$\text{exposure} = \frac{C_{1,u} - C_{1,d}}{S_{1,u} - S_{1,d}} = \frac{5 - 0}{110 - 95} = \frac{5}{15} = \frac{1}{3}$$
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The Replication Approach

**Step 1** Replicate the payoff from the call—[5 if $u$] and [0 if $d$]:

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<th>(a) = forward contract (buy MTL 1/3 at 101)</th>
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**Step 2** Time-0 cost of the replicating portfolio:

- forward contract is free
- deposit will cost INR $2/1.05 = INR 1.905$

**Step 3** Law of One Price: option price = value portfolio

$$C_0 = INR 1.905$$
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The Hedging Approach

Replcation: call = forward position + riskfree deposit
Hedging: call – forward position = riskfree deposit

◊ Step 1  Hedge the call

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◊ Step 2  time-0 value of the riskfree portfolio

\[
\text{value} = \text{INR} \frac{2}{1.05} = \text{INR} 1.905
\]

◊ Step 3  Law of one price: option price = value portfolio

\[C_0 + \text{[time-0 value of hedge]} = \text{INR} 1.905 \Rightarrow C_0 = \text{INR} 1.905\]

... otherwise there are arbitrage possibilities.
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| Hedging:     | call – forward position = riskfree deposit |

- **Step 1** Hedge the call

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  (sell MTL 1/3 at 101)
  (b) = call
  (a)+(b)

| $S_1 = 95$ | $1/3 \times (101 - 95) = 2$ | 0 | 2 |
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- **Step 2** time-0 value of the riskfree portfolio

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The Risk-adjusted Probabilities

◊ **Overview:** Implicitly, the replication/hedging story ...

▷ extracts a risk-adjusted probability “up” from the forward market,

▷ uses this probability to compute the call’s risk-adjusted expected payoff, $\text{CEQ}_0(\tilde{C}_1)$; and

▷ discounts this risk-adjusted expectation at the riskfree rate.
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§ Step 1  Extract risk-adjusted probability from $F$:

- **Ordinary expectation:** $E_0(\tilde{S}_1) = p \times 110 + (1 - p) \times 95$
- **Risk-adjusted expectation:** $CEQ_0(\tilde{S}_1) = q \times 110 + (1 - q) \times 95$
- **We do not know how/why the market selects** $q$, but $q$ is revealed by $F_{0,1} (= 101)$:

$$101 = 95 + q \times (110 - 95) \Rightarrow q = \frac{101 - 95}{110 - 95} = \frac{6}{15} = 0.4$$

§ Step 2  CEQ of the call’s payoff:

$$CEQ_0(\tilde{C}_1) = (0.4 \times 5) + (1 - 0.4) \times 0 = 2$$

§ Step 3  Discount at $r$:

$$C_0 = \frac{CEQ_0(\tilde{C}_1)}{1 + r_{0,1}} = \frac{2}{1.05} = 1.905$$
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Multiperiod Pricing: Notation

- **Subscripts**: \( n, j \) where
  - \( n \) says how many jumps have been made since time 0
  - \( j \) says how many of these jumps were *up*

- **General pricing equation**:

\[
C_{t,j} = \frac{C_{t+1,u} \times q_t + C_{t+1,d} \times (1 - q_t)}{1 + r_{t,\text{1 period}}},
\]

where

\[
q_t = \frac{F_{t,t+1} - S_{t+1,d}}{S_{t+1,u} - S_{t+1,d}},
\]

\[
S_t \frac{1 + r_{t,t+1}}{1 + r^{*}_{t,t+1}} - S_t d_t
= \frac{S_t u_t - S_t d_t}{S_t u_t - S_t d_t},
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\[
d_t = \frac{S_{t+1,d}}{S_t}, \quad u_t = \frac{S_{t+1,u}}{S_t}.
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Assumptions

- **A1 \((r \text{ and } r^*)\)**: The risk-free one-period rates of return on both currencies are constant
  - denoted by unsubscripted \(r\) and \(r^*\)
  - Also assumed in Black-Scholes.

- **A2 \((u \text{ and } d)\)**: The multiplicative change factors, \(u\) and \(d\), are constant.
  - Also assumed in Black-Scholes:
    - no jumps (sudden de/revaluations) in the exchange rate process, and
    - a constant variance of the period-by-period percentage changes in \(S\).

- **Implication of A1-A2**: \(q_t\) is a constant.

- **A2.01 (no free lunch in \(F\))**:
  \[
d < \frac{1 + r}{1 + r^*} < u \iff S_{t+1,d} < F_t < S_{t+1,u} \iff 0 < q < 1
  \]

Q: what would you do if \(S_1 = [95 \text{ or } 110]\) while \(F=90\)? 115?
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How such a tree works

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Discussion

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Currency Options (2): Hedging and Valuation

P. Sercu, *International Finance: Theory into Practice*

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The emerging bell-shape

let \( p = \frac{1}{2} \)

\[
\begin{align*}
1 & \quad 1/2 \\
1/2 & \quad 1/2 \\
1/4 & \quad 1/4 \\
1/8 & \quad 1/8 \\
1/16 & \quad 1/16 \\
4/16 & \quad 4/16 \\
6/16 & \quad 6/16 \\
4/16 & \quad 4/16 \\
1/16 & \quad 1/16 \\
C = 4!/4!0! = 1 & \quad C = 4!/3!1! = 24/6 = 4 \\
C = 4!/2!2! = 24/6 = 6 & \quad C = 4!/1!3! = 24/6 = 4 \\
C = 4!/4!0! = 1 &
\end{align*}
\]
What Emerging Bellshape?

◇ Chosing between two oversimplifications:

- additive

  100 ← 110 ← 120 ← 130
  90 ← 100 ← 90 ← 70

- multiplicative

  100 ← 110 ← 121 ← 133.1
  90 ← 99 ← 81 ← 72.9

- cents v percent: we prefer a constant distribution of percentage price changes over a constant distribution of dollar price changes.

- non-negative prices: with a multiplicative, the exchange rate can never quite reach zero even if it happens to go down all the time.

- invertible: we get a similar multiplicative process for the exchange rate as viewed abroad, 
  \[ S^* = 1/S \]
  (with \( d^* = 1/u, \ a^* = 1/d \)).

◇ Corresponding Limiting Distributions:

- additive: \( \tilde{S}_n = S_0 + \sum_{t=1}^{n} \tilde{\Delta}_t \) where \( \tilde{\Delta} = \{+10, -10\} \)
  \( \iff \) \( \tilde{S}_n \) is normal if \( n \) is large (CLT)

- multiplicative: \( \tilde{S}_n = S_0 \times \prod_{t=1}^{n} (1 + \tilde{r}_t) \) where \( \tilde{r} = \{+10\%, -10\%\} \)
  \( \iff \) \( \ln \tilde{S}_n = \ln S_0 + \sum_{t=1}^{n} \tilde{\rho}_t \) where \( \tilde{\rho} = \ln(1 + \tilde{r}) = \{-0.095, -0.095\} \)
  \( \iff \) \( \ln \tilde{S}_n \) is normal if \( n \) is large \( \iff \) \( \tilde{S}_n \) is lognormal.
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◊ Chosing between two oversimplifications:

- **cents v percent**: we prefer a constant distribution of percentage price changes over a constant distribution of dollar price changes.

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  \[ \tilde{S}_n = S_0 + \sum_{t=1}^{n} \Delta_t \]
  where \( \Delta = \{+10, -10\} \)
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- **multiplicative**: 
  \[ \tilde{S}_n = S_0 \times \prod_{t=1}^{n} (1 + \tilde{r}_t) \]
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\[ S^* = \frac{1}{S} \text{ (with } d^* = \frac{1}{u}, \ u^* = \frac{1}{d}) \]

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\[ \tilde{S}_n = S_0 + \sum_{t=1}^{n} \Delta_t \text{ where } \Delta = \{+10, -10\} \]

\[ \iff \tilde{S}_n \text{ is normal if } n \text{ is large (CLT)} \]

— **multiplicative**: 

\[ \tilde{S}_n = S_0 \times \prod_{t=1}^{n} (1 + \tilde{r}_t) \text{ where } \tilde{r} = \{+10\%, -10\%\} \]

\[ \iff \ln \tilde{S}_n = \ln S_0 + \sum_{t=1}^{n} \tilde{r}_t \text{ where } \tilde{r} = \ln(1 + \tilde{r}) = \{+0.095, -0.095\} \]

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**additive**

\[ \begin{array}{ccc}
100 & 110 & 120 \\
90 & 100 & 110 \\
80 & 90 & 70 \\
\end{array} \]

**multiplicative**

\[ \begin{array}{ccc}
100 & 110 & 121 \\
90 & 99 & 81 \\
80 & 72.9 & 72.9 \\
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**An N-period European Call: The Problem**

\[
\begin{array}{c}
S_0 = 100 \\
S_{1,0} = 90 \\
S_{1,1} = 110 \\
S_{2,0} = 81 \\
S_{2,1} = 99 \\
S_{2,2} = 121 \\
\end{array}
\]

- payoff
- \( u = 1.1 \);
- \( d = 0.9 \);
- \( 1 + r = 1.05 \);
- \( 1 + r^* = 1.0294118 \);
- forward factor \( \frac{1 + r}{1 + r^*} = 1.02 \);
- \( q = \frac{1.02 - 0.9}{1.1 - 0.9} = 0.60 \)

**A4.** At any discrete moment in the model, investors can trade and adjust their portfolios of HC-FC loans.

**Black-Scholes:** trading is continuous
Backward Pricing, Dynamic Hedging

Bow the binomial pricing of European options.

\[ S_0 = 100 \]

\[ S_{1,0} = 90 \]

\[ S_{1,1} = 110 \]

\[ S_{2,0} = 81 \]

\[ S_{2,1} = 99 \]

\[ S_{2,2} = 121 \]

\[ \text{payoff} \]

\[ 26 \]

\[ 4 \]

\[ 0 \]

\[ u = 1.1; d = 0.9; 1+r = 1.05; 1+r^* = 1.0294118; \]

\[ \text{forward factor} \frac{1+r}{1+r^*} = 1.02 \]

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◊ if we land in node \((1,1)\):

\[ b_{1,1} = \frac{26 - 4}{121 - 99} = 1 \]

\[ C_{1,1} = \frac{(26 \times 0.6) + (4 \times 0.4)}{1.05} = 16.38 \]

◊ if we land in node \((1,0)\):

\[ b_{1,0} = \frac{4 - 0}{99 - 81} = 0.222 \]

\[ C_{1,0} = \frac{(4 \times 0.6) + (0 \times 0.4)}{1.05} = 2.29 \]
Backward Pricing, Dynamic Hedging

\[ b_{1,1} = \frac{26 - 4}{121 - 99} = 1 \]
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Backward Pricing, Dynamic Hedging

\[ C_1 = \begin{cases} 16.38 & \text{if } S_1 = 110 \\ 2.29 & \text{if } S_1 = 90 \end{cases} \]

\[ b_0 = \frac{16.38 - 2.29}{110 - 90} = 0.704 \]

\[ C_{1,1} = \frac{(16.38 \times 0.6) + (2.29 \times 0.4)}{1.05} = 10.23 \]

**Summary:**

- We hedge dynamically:
  - Start the hedge at time 0 with 0.704 units sold forward.
  - The time-1 hedge will be to sell forward 1 or 0.222 units of foreign currency, depending on whether the rate moves up or down.

- We price backward, step by step
Backward Pricing, Dynamic Hedging

\[
C_1 = \begin{cases} 
16.38 & \text{if } S_1 = 110 \\
2.29 & \text{if } S_1 = 90 
\end{cases}
\]

◊ at time 0 we do have a two-point problem:

\[
b_0 = \frac{16.38 - 2.29}{110 - 90} = 0.704
\]

\[
C_{1,1} = \frac{(16.38 \times 0.6) + (2.29 \times 0.4)}{1.05} = 10.23
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Summary:

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### Hedging Verified

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<tr>
<th>Step</th>
<th>Value if up</th>
<th>Value if down</th>
</tr>
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<tbody>
<tr>
<td><strong>at time 0:</strong></td>
<td>invest 10.23 129 at 5%, buy fwd MTL 0.704 762 at 100 × 1.02 = 102</td>
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</tr>
<tr>
<td>— value if up</td>
<td>10.23 129 × 1.05 + 0.704 762 × (110 − 102) = 16.380 95</td>
<td>10.23 129 × 1.05 + 0.704 762 × (90 − 102) = 2.295 71</td>
</tr>
<tr>
<td>— value if down</td>
<td>10.23 129 × 1.05 + 0.704 762 × (90 − 102) = 2.295 71</td>
<td>10.23 129 × 1.05 + 0.704 762 × (81 − 91.8) = 0.000 00</td>
</tr>
<tr>
<td><strong>if in node (1,1):</strong></td>
<td>invest 16.380 95 at 5%, buy fwd MTL 1 at 100 × 1.02 = 112.2</td>
<td>invest 16.380 95 at 5%, buy fwd MTL 1 at 100 × 1.02 = 112.2</td>
</tr>
<tr>
<td>— value if up</td>
<td>16.380 95 × 1.05 + 1.000 000 × (121 − 112.2) = 26.000 00</td>
<td>16.380 95 × 1.05 + 1.000 000 × (99 − 112.2) = 4.000 00</td>
</tr>
<tr>
<td>— value if down</td>
<td>16.380 95 × 1.05 + 1.000 000 × (99 − 112.2) = 4.000 00</td>
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</tr>
<tr>
<td><strong>if in node (1,0):</strong></td>
<td>invest 2.295 71 at 5%, buy fwd MTL 0.222 222 at 90 × 1.02 = 91.8</td>
<td>invest 2.295 71 at 5%, buy fwd MTL 0.222 222 at 90 × 1.02 = 91.8</td>
</tr>
<tr>
<td>— value if up</td>
<td>2.295 71 × 1.05 + 0.222 222 × (99 − 91.8) = 4.000 00</td>
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What can go wrong?

Everything can and will go wrong:

Change of risk: ±20% if up, ±5% if down, instead of the current ±10%:

\[
C_{1,1} = \frac{37 \times 0.55 + 0}{1.05} = 19.36, \text{ not } 16.38,
\]

\[
C_{1,0} = \frac{0 + 0}{1.05} = 0.00, \text{ not } 2.29,
\]

You would have mishedged:

- You would lose, as a writer, in the upstate (risk up)
- You would gain, as a writer, in the downstate (risk down)
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\]

You would have mishedged:

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– You would gain, as a writer, in the downstate (risk down)
American-style Options

\[ u = 1.1, \ d = 0.9, \ r = 5\%, \ \frac{(1+r)}{(1+r^*)} = 1.02, \ q = 0.60 \]

- **Node (1,1)** In this node the choices are
  - PV of later exercise (0 or 1): 0.381
  - Value of immediate exercise: 0 — so we wait; \( V_{1,1} = 0.381 \)

- **Node (1,0)** Now the choices are
  - PV of later exercise (0 or 19): 7.81
  - Value of immediate exercise: 10 — so we exercise; \( V_{1,0} = 10 \) not 7.81

- **Node (0)** We now choose between
  - PV of later exercise (0 or 1 at time 2, or 10 at time 1):
    \[
    P_{0}^{alive} = \frac{0.381 \times 0.60 + 10 \times 0.40}{1.05} = 4.03
    \]
  - Value of immediate exercise: 0 — so we wait; \( V_{0} = 4.03 \)
American-style Options

\[ u = 1.1, \quad d = 0.9, \quad r = 5\%, \quad \frac{1 + r}{1 + r^*} = 1.02, \quad q = 0.60 \]

\[ \text{Exchange rate} \]

\[
\begin{array}{c|c|c}
0 & 1 & 0 \\
100 & 110 & 121 \\
90 & 99 & 81 \\
\end{array}
\]

\[ \text{European Put with} \]
\[ X = 100 \]

\[
\begin{array}{c|c|c}
0 & 1 & 0 \\
3.193 & .381 (0) & 0 \\
7.81 & 7.81 (10) & 19 \\
\end{array}
\]

\[ \text{American Put with} \]
\[ X = 100 \]

\[
\begin{array}{c|c|c}
0 & 1 & 0 \\
4.03 & .381 (0) & 1 \\
7.81 & 7.81 (10) & 19 \\
\end{array}
\]

- **Node (1,1)** In this node the choices are
  - PV of later exercise (0 or 1): 0.381
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    \[
    p_{alive}^0 = \frac{0.381 \times 0.60 + 10 \times 0.40}{1.05} = 4.03
    \]
  - Value of immediate exercise: 0 — so we wait; \( V_0 = 4.03 \)
## American-style Options

### Node (1,1)
In this node the choices are
- **PV of later exercise (0 or 1):** 0.381
- **Value of immediate exercise:** 0 — so we wait; \( V_{1,1} = 0.381 \)

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Now the choices are
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Straight-Through-Pricing a 3-period Put

The long way:

\[
\begin{align*}
C_{2,2} &= \frac{0.00 \times 0.6 + 0.00 \times 0.4}{1.05} = 0.00, \\
C_{2,1} &= \frac{0.00 \times 0.6 + 10.9 \times 0.4}{1.05} = 4.152, \\
C_{2,0} &= \frac{10.0 \times 0.6 + 27.1 \times 0.4}{1.05} = 16.55, \\
C_{1,1} &= \frac{0.000 \times 0.6 + 4.152 \times 0.4}{1.05} = 1.582, \\
C_{1,0} &= \frac{4.152 \times 0.6 + 16.55 \times 0.4}{1.05} = 8.678, \\
C_0 &= \frac{1.582 \times 0.6 + 8.678 \times 0.4}{1.05} = 4.210.
\end{align*}
\]
Straight-Through-Pricing a 3-period Put

The long way:

\[
\begin{align*}
C_{2,2} &= \frac{0.00 \times 0.6 + 0.00 \times 0.4}{1.05} = 0.00, \\
C_{2,1} &= \frac{0.00 \times 0.6 + 10.9 \times 0.4}{1.05} = 4.152, \\
C_{2,0} &= \frac{10.0 \times 0.6 + 27.1 \times 0.4}{1.05} = 16.55, \\
C_{1,1} &= \frac{0.000 \times 0.6 + 4.152 \times 0.4}{1.05} = 1.582, \\
C_{1,0} &= \frac{4.152 \times 0.6 + 16.55 \times 0.4}{1.05} = 8.678, \\
C_0 &= \frac{1.582 \times 0.6 + 8.678 \times 0.4}{1.05} = 4.210.
\end{align*}
\]
Straight-Through-Pricing a 3-period Put

100<br>90<br>121<br>99<br>133.1<br>108.9<br>89.1<br>72.9\[\Rightarrow\]
4.21
1.58
6.68
4.15
10.9
27.1

The fast way:

▷ \( pr_3 = \ldots \)
▷ \( pr_2 = \ldots \)
▷ \( pr_1 = \ldots \)
▷ \( pr_0 = \ldots \)
▷ The (risk-adjusted) chance of ending in the money is ...
▷ \( C_0 = \ldots \times \ldots \times \ldots \times \ldots \times \ldots = 4.21 \)
Straight-Through-Pricing: 2-period Math

\[ C_{1,1} = \frac{q C_{2,2} + (1 - q)C_{2,1}}{1 + r}, \]
\[ C_{1,0} = \frac{q C_{2,1} + (1 - q)C_{2,0}}{1 + r}, \]
\[ C_0 = \frac{q C_{1,1} + (1 - q)C_{1,0}}{1 + r}, \]
\[ =q \left[ \frac{q C_{2,2} + (1 - q)C_{2,1}}{1 + r} \right] + (1 - q) \left[ \frac{q C_{2,1} + (1 - q)C_{2,0}}{1 + r} \right] \]
\[ = \frac{q^2 C_{2,2} + 2q (1 - q)C_{2,1} + (1 - q)^2 C_{2,0}}{(1 + r)^2} \]
Straight-Through-Pricing: 3-period Math

\[ C_{1,1} = \frac{q^2 C_{3,3} + 2q(1-q)C_{3,2} + (1-q)^2 C_{3,1}}{(1+r)^2} \]

\[ C_{1,0} = \frac{q^2 C_{3,2} + 2q(1-q)C_{3,1} + (1-q)^2 C_{3,0}}{(1+r)^2} \]

\[ C_0 = \frac{q C_{1,1} + (1-q)C_{1,0}}{1+r} \]

\[ = q \left[ q^2 C_{3,3} + 2q(1-q)C_{3,2} + (1-q)^2 C_{3,1} \right] \]
\[ + (1-q) \left[ q^2 C_{3,2} + 2q(1-q)C_{3,1} + (1-q)^2 C_{3,0} \right] \]
\[ = \frac{q^3 C_{3,3} + 3q^2(1-q)C_{3,2} + 3q(1-q)^2 C_{3,1} + (1-q)^3 C_{3,0}}{(1+r)^3} \]
Toward BMS 1: two terms

Let \( pr_{n,j}^{(Q)} \) = risk-adjusted chance of having \( j \) ups in \( n \) jumps

\[
= \frac{n!}{j! (n-j)!} \times q^j (1-q)^{N-j} = \left( \begin{array}{c} N \\ j \end{array} \right) q^j (1-q)^{N-j}
\]

\# of paths with \( j \) ups
prob of such a path

and let \( a \) : \( \{ j \geq a \} \iff \{ S_{n,j} \geq X \} \);

then \( C_0 \)

\[
= \frac{\sum_{j=0}^{N} pr_{n,j}^{(Q)} C_{n,j}}{(1+r)^N} = \frac{\text{CEQ}_0(\tilde{C}_N)}{\text{discounted}},
\]

\[
= \frac{\sum_{j=0}^{N} pr_{n,j}^{(Q)} (S_{n,j} - X)_+}{(1+r)^N},
\]

\[
= \frac{\sum_{j=a}^{N} pr_{n,j}^{(Q)} (S_{n,j} - X)}{(1+r)^N},
\]

\[
= \frac{\sum_{j=a}^{N} pr_{n,j}^{(Q)} S_{n,j}}{(1+r)^N} - \frac{X}{(1+r)^N} \sum_{j=a}^{N} pr_{n,j}^{(Q)}. \quad (2)
\]
Toward BMS 2: two probabilities

Recall: \( C_0 = \frac{\sum_{j=a}^{N} p_r(Q) s_{n,j}}{(1 + r)^N} - \frac{X}{(1 + r)^N} \sum_{j=a}^{N} p_r(Q). \)

We can factor out \( S_0 \), in the first term, by using

\[ S_{n,j} = S_0 u^j d^{N-j}. \]

We also use

\[ \frac{1}{(1 + r)^N} = \frac{1}{(1 + r^*)^N} \left( \frac{1 + r^*}{1 + r} \right)^j \left( \frac{1 + r^*}{1 + r} \right)^{N-j} \]

\[
\sum_{j=a}^{N} p_r(Q) s_{n,j} = \frac{S_0}{(1 + r^*)^N} \sum_{j=a}^{N} \binom{N}{j} \left( q \frac{1 + r^*}{1 + r} \right)^j \left( 1 - q \frac{1 + r^*}{1 + r} \right)^{N-j}
\]

\[
= \frac{S_0}{(1 + r^*)^N} \sum_{j=a}^{N} \binom{N}{j} \pi^j (1 - \pi)^{N-j}
\]

where \( \pi := q \frac{1 + r^*}{1 + r} \Rightarrow 1 - \pi = (1 - q) \frac{1 + r^*}{1 + r}. \)
Towards BMS 3: the limit

\[ C_0 = \frac{S_0}{(1 + r^*)^N} \left\{ \sum_{j=a}^{N} \binom{N}{j} \pi^j (1 - \pi)^{N-j} \right\} \]

\[ \left\{ \frac{X}{(1 + r)^N} \right\} \sum_{j=a}^{N} pr(Q)_{n,j} \]  

\( C_0 \) = price of the underlying FC PN

\( a \geq a \) probability-like expression

\( j \geq a \) discounted strike

\( Q \) of

\( j \geq a \)

\( pr(Q) \)

\( N \rightarrow \infty \) (and suitably adjusting \( u, d, r, r^* \))

\( j/N \) becomes Gaussian, so we get Gaussian probabilities

\( \text{first prob typically denoted } N(d_1), d_1 = \frac{\ln(F_t,T/X) + (1/2)\sigma_t^2}{\sigma_t \sqrt{T}} \), with \( \sigma_t \)

\( \text{the effective stdev of } \ln(S_T) \text{ as seen at time } t \)

\( \text{second prob typically denoted } N(d_2), d_2 = \frac{\ln(F_t,T/X) - (1/2)\sigma_t^2}{\sigma_t \sqrt{T}} \)

\( \text{Special case } a = 0: \)

\( a = 0 \) means that ...

so both probabilities become ...

and we recognize the value of ...

\( \text{Towards BMS} \)
Towards BMS 3: the limit

\[
C_0 = \frac{S_0}{(1 + r^*)^N} \sum_{j=a}^N \binom{N}{j} \pi^j (1 - \pi)^{N-j} \left( \frac{X}{(1 + r)^N} \right) \sum_{j=a}^N pr_{n,j}^{(Q)} \tag{3}
\]

- a “\( j \geq a \)” probability-like expression
- price of the underlying FC PN
- discounted strike
- prob\((Q)\) of \( j \geq a \)

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\[ C_0 = \frac{S_0}{(1 + r^*)^N} \sum_{j=a}^{N} \left( \binom{N}{j} \pi^j (1 - \pi)^{N-j} \right) X \left( \frac{1}{1 + r} \right)^N - \sum_{j=a}^{N} pr_{n,j}^{(Q)}. \]  

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\[ j \geq a \]

\[ \text{price of the underlying FC PN} \]

\[ \text{discounted strike} \]

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\[ \text{STP-ing of European Options} \]

\[ \text{Towards BlackMertonScholes} \]

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The Delta of an Option

◊ **Replication:** in BMS the option formula is still based on a portfolio that replicates the option (over the short time period $dt$):

- a fraction $\sum_{j=a}^n \pi_j$ or $N(d_1)$ of a FC PN with face value unity, and
- a fraction $\sum_{j=a}^n pr_j$ or $N(d_2)$ of a HC PN with face value $X$.

◊ **Hedge:** since hedging is just replication reversed, you can use the formula to hedge:

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◇ **Why binomial?**
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  - but that’s because we implicitly use $q$ instead:
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  - for American-style options, we also compare to the value dead

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