On the Robust Stability of the Hill Equation with a Delay Term: A Frequency-Domain Approach

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Abstract—The paper considers the problem of robust stability of the Hill equation with a damping and a time-delay term. Analytical results are presented in terms of the coefficients of the equation. The derivation is based on the frequency-domain approach, specifically, the geometric interpretation of the Yakubovich stability criterion for linear time-periodic systems. There are two approaches to this interpretation: One involves analysis of the so-called Lipatov plots and the other is based directly on the Nyquist hodograph.

I. INTRODUCTION

There is vast amount of literature on both linear periodic systems, especially the Mathieu and the Hill equations, and systems with retarded argument. Two very incomplete lists of works covering these subjects are, respectively, the monographs [1]-[3] with references therein and [4]-[13]. In addition, there are chapters and sections on the Hill equation in several books, including [14]-[19].

By contrast, linear systems that include both time-periodic and time-delay terms have not been studied nearly as extensively. A well-known result (see, for example, [5]) is the applicability of the Floquet multiplier theory, except that the number of such multipliers becomes infinite when the delay term is introduced [18]. In addition, the paper by Lampe and Rosenwasser [20] and references therein point out a number of the difficulties that arise in extending some of the known methods of analysis to systems of this type.

One method that appears to be directly extendable is the one that uses frequency-domain criteria for absolute stability. Specifically, the periodic term is treated as nonlinearity and all the other terms (including delay) are considered to be part of the linear block. This approach has been used in the analysis of linear time-periodic systems without delays [21]-[23]. In this paper the results from [23] will be used to derive robust stability criteria for systems with a time delay term.

Our objective will be to derive global asymptotic stability criteria for the following equation:

\[ \dot{x}(t) + a_1 x(t) + a_0 x(t) + bx(t-r) + p(t)x(t) = 0. \]  

(1)

It will be assumed that the function \( p(t) \) is periodic with the period \( T \) and, in addition, it satisfies the double inequality:

\[ 0 \leq p(t) \leq \kappa. \]  

(2)

Note that there is no loss in generality in making the lower bound of this function equal to zero since it can always be made so by sector rotation.

Furthermore, we are interested in the robust stability of (1) that is we seek stability criteria that are applicable to all functions \( p(t) \) having a period \( T \) and satisfying (2).

The outline of the paper will be as follows. In Section II we will review some background material, including the Yakubovich stability criterion from [24] and its geometric interpretation from [23]. These results will then be applied to (1), including the undamped case of \( a_1 = 0 \) (Section III.A) and the more general damped case (Section III.B). Based on the results from these two sections, a conjecture will be formulated (Section III.C) In Section IV we describe an alternative geometric interpretation of the Yakubovich criterion and apply it to the problem under consideration. Conclusions are summarized in Section V.

II. BACKGROUND MATERIAL

For the sake of making the paper self-contained, let us review some background material.

Let us introduce the transfer function of the stationary part of the equation:

\[ W(s) = \frac{1}{s^2 + a_1 s + a_0 + b e^{-s\tau}}. \]  

(3)

Let \( K(t) \) and \( \alpha(t) \) be the inverse Laplace transforms of \( W(s) \) and \( W(s)[x(0)s + x(0) + \dot{x}(0)] \), respectively. Then we can rewrite (1) in the form of the Volterra integral equation:

\[ x(t) - \alpha(t) + \int_0^t K(t-r)p(r)x(r)dr = 0. \]  

(4)

Yakubovich criterion, originally proved in [24] for differential equations without delays and extended to equations in the form (4) in [23], says (as adapted to the problem under consideration) that the zero solution is globally asymptotically stable if there exists an odd periodic function \( \vartheta(\omega) \) with the period \( 2\pi/T \), such that the following frequency condition holds for all real values of \( \omega \):

\[ \text{Re}\left[ \left( \kappa^{-1} + W(i\omega) \right)ight] > 0. \]  

(5)

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Application of this criterion requires finding of the function $\mathcal{B}(\omega)$. This can be accomplished using the geometric interpretation.

One such geometric interpretation was discussed in [23]. Define the new function:

$$\Phi_\kappa(\omega) = \frac{\kappa^{-1} + P(\omega)}{Q(\omega)},$$

where $P(\omega) = \text{Re} W(i\omega)$ and $Q(\omega) = \text{Im} W(i\omega)$.

The class of periodic functions $p(t)$ satisfying (2) includes constants. It follows from the Nyquist criterion that for all points where $Q(\omega) = 0$, the inequality

$$\kappa^{-1} + P(\omega) > 0 \quad (7)$$

is a necessary and sufficient condition for stability for this included special case. It is, therefore, a necessary condition for robust stability of (1) that (7) must hold for all points where the Nyquist hodograph of $W(s)$ intersects the real axis.

For the derivative of the function $\Phi_\kappa(\omega)$ we have:

$$\Phi'_\kappa(\omega) = \frac{Q(\omega)P'(\omega) - [\kappa^{-1} + P(\omega)]Q'(\omega)}{Q^2(\omega)}. \quad (8)$$

It is easy to see that, because of (7), the signs of the numerator in (8) alternate at the points where $Q(\omega) = 0$.

The above considerations imply the following about the function $\Phi_\kappa(\omega)$. First, it changes sign at its asymptotes defined by $\text{Im} W(i\omega) = 0$. These asymptotes partition the graph of this function into branches. Furthermore, both endpoints of each branch are directed, alternatingly, upward (such branches are called stalactites) or downward (such branches are called stalagmites). The frequency condition (5) holds if the graph of the function $\mathcal{B}(\omega)$ separates stalactites from stalagmites. The special case of $\mathcal{B}(\omega) = 0$ (the horizontal line) corresponds to the circle criterion. We call the joint graph of the functions $\Phi_\kappa(\omega)$ and $\mathcal{B}(\omega)$ the Lipatov plot after the paper [25] where this approach was first introduced (for stationary nonlinearities).

As discussed in [23], when there is no delay, the function $\Phi_\kappa(\omega)$ takes the form:

$$\Phi_\kappa(\omega) = \frac{\kappa^{-1} - (1 + 2a_0 - \kappa^{-1}a_0^2)\omega^2 + a_0 + \kappa^{-1}a_0^2}{a_1\omega} \quad (9)$$

Clearly, this curve has only one asymptote at $\omega = 0$. If the right half-plane branch of this curve does not intersect the abscissa axis, the system is stable by the circle criterion. If there are two intersection points, say $\omega_1$ and $\omega_2$ with $\omega_1 < \omega_2$, one can always find the required function $\mathcal{B}(\omega)$ with the period $2\pi/T \geq 2\omega_2$, and, therefore, the zero solution is stable if $T < \pi/\omega_2$. It turns out that this argument can be extended to systems with a delay.

The other geometric interpretation of (5) involves the use of a certain modified form of the Nyquist hodograph and will be introduced in Section IV.

### III. Lipatov Plot Analysis

#### A. The Undamped Case

Let us first consider the undamped case, that is $a_1 = 0$.

The expression for the function $\Phi_\kappa(\omega)$ takes the form:

$$\Phi_\kappa(\omega) = a_2 + \kappa^{-1}(b^2 + a_2^2) - (1 + 2\kappa^{-1}a_2)\omega^2 + \kappa^{-1}\omega^4 \quad (10)$$

$$+ \frac{b[1 + 2\kappa^{-1}(a_2 - \omega^2)]\cos\omega T}{b\sin\omega T}$$

While it is clear that this curve has infinitely many branches, its intersections with the abscissa axis are defined by the equation:

$$a_2 + \kappa^{-1}(b^2 + a_2^2) - (1 + 2\kappa^{-1}a_2)\omega^2 + \kappa^{-1}\omega^4$$

$$+ b[1 + 2\kappa^{-1}(a_2 - \omega^2)]\cos\omega T = 0. \quad (11)$$

Note that

$$a_2 + \kappa^{-1}(b^2 + a_2^2) - (1 + 2\kappa^{-1}a_2)\omega^2 + \kappa^{-1}\omega^4$$

$$+ b[1 + 2\kappa^{-1}(a_2 - \omega^2)]\cos\omega T$$

$$\geq \kappa^{-1}(\omega^2 - b^2)^2 - (1 + 2\kappa^{-1}a_2)\omega^2$$

$$+ \kappa^{-1}a_2^2 + (1 - 2\kappa^{-1})a_2 - b. \quad (12)$$

Because of the periodicity of the denominator in (10), it is sufficient to investigate only the behavior of the numerator, which is bounded below by the function

$$\Gamma(\omega) = \kappa^{-1}(\omega^2 - b^2)^2 - (1 + 2\kappa^{-1}a_2)\omega^2$$

$$+ \kappa^{-1}a_2^2 + (1 - 2\kappa^{-1})a_2 - b. \quad (13)$$

The graph of this function intersects the positive half of the abscissa axis at the points $\omega_1$ and $\omega_2$ with $\omega_1 < \omega_2$, which can be easily found by solving (analytically) the equation $\Gamma(\omega) = 0$. The expressions for these values are lengthy and are, therefore, omitted. In the special case of $a_2 = 0$ we find:
\[ \omega_2 = \sqrt{\frac{b^2 + \omega^2}{2}}. \] (14)

Therefore, we can conclude, just as in case of no delay, that the required curve with the period \( 2\pi/T \geq 2\omega_b \) can be drawn, and this inequality is a sufficient condition for stability. Let us state this result as a theorem.

**Theorem 1.** Assume that \( T < \pi/\omega_b \) where \( \omega_b \) is the largest root \( \omega \) of the equation

\[ \kappa^{-1}(\omega^2 - b^2)^2 - (1 + 2\kappa^{-1}a_2)\omega^2 + \kappa^{-1}a_2^2 + (1 - 2\kappa^{-1})a_2 - b = 0. \] (15)

Then the zero solution of the equation

\[ \ddot{x}(t) + ax(t) + bx(t - \tau) + p(t)x(t) = 0 \] (16)

is globally asymptotically stable for all functions \( p(t) \) having a period \( T \) and satisfying the inequality (2).

**B. The Damped Case**

We now turn our attention to the case when \( \omega \neq 0 \). The expression for the function \( \Phi_\kappa(\omega) \) takes the form:

\[ \Phi_\kappa(\omega) = \frac{a_2 + \kappa^{-1}(b^2 + a_2^2) - [1 + \kappa^{-1}(2a_2 - a_1^2)]\omega^2}{b\sin \omega \tau - a_1 \omega} + \frac{\kappa^{-1}\omega^4 + b[1 + 2\kappa^{-1}(a_2 - \omega^2)]\cos \omega \tau}{b\sin \omega \tau - a_1 \omega} - \frac{2ba_1\kappa^{-1}\sin \omega \tau - a_2 \omega}{b\sin \omega \tau - a_1 \omega}. \] (17)

The number of asymptotes this function has can be determined by the following elementary observations. First, note that if \( a_1 > b\tau \) the only asymptote is \( \omega = 0 \) and this curve has only one branch (the stalactite) in the right half-plane.

The equation \( b\sin \omega \tau - a_1 \omega = 0 \) acquires an additional root when the straight line \( a_1 \omega \) is tangent to the sinusoid \( b\sin \omega \tau \), that is the equation \( a_1 = b\tau \cos \omega \tau \) holds. These two equations are equivalent to the pair of equations:

\[ \tan \omega \tau = \omega \tau; \]
\[ \omega^2 + \frac{1}{\tau^2} = \frac{b^2}{a_1}. \] (18)

These two equations imply that the straight line \( a_1 \omega \) is tangent to the sinusoid \( b\sin \omega \tau \) if \( a_1^2(1 + \chi^2) = b^2\tau^2 \) where \( \chi \) is the smallest positive root of the equation \( \chi = \tan \chi \).

From these considerations we can conclude that the graph of the function \( \Phi_\kappa(\omega) \) has at most two branches (one stalactite and one stalagmite) in the right half-plane if

\[ a_1 > \frac{b\tau}{\sqrt{1 + \chi^2}}. \] (19)

Therefore, if (19) holds, we can apply the same logic as in the case of no delay terms. In order to do so, we must locate the points where the curve \( \Phi_\kappa(\omega) \) intersects the abscissa axis (If there are no such intersections, stability follows from the circle criterion), i.e., to solve the equation:

\[ a_2 + \kappa^{-1}(b^2 + a_2^2) - [1 + \kappa^{-1}(2a_2 - a_1^2)]\omega^2 + \kappa^{-1}\omega^4 + b[1 + 2\kappa^{-1}(a_2 - \omega^2)]\cos \omega \tau - 2ba_1\kappa^{-1}\omega \sin \omega \tau = 0. \] (20)

Note that

\[ a_2 + \kappa^{-1}(b^2 + a_2^2) - [1 + \kappa^{-1}(2a_2 - a_1^2)]\omega^2 + \kappa^{-1}\omega^4 + b[1 + 2\kappa^{-1}(a_2 - \omega^2)]\cos \omega \tau - 2ba_1\kappa^{-1}\omega \sin \omega \tau \geq a_2 + \kappa^{-1}(b^2 + a_2^2) - [1 + \kappa^{-1}(2a_2 - a_1^2)]\omega^2 + \kappa^{-1}\omega^4 - b[1 + 2\kappa^{-1}(a_2 - \omega^2)] \] (21)

\[ -2ba_1\kappa^{-1}\omega \sin \omega \tau = 0. \]

Let \( \omega \) be the largest root \( \omega \) of the equation (It can be easily found analytically, but the expression is lengthy and is, therefore, omitted):

\[ a_2 + \kappa^{-1}(b^2 + a_2^2) - [1 + \kappa^{-1}(2a_2 - a_1^2)]\omega^2 + \kappa^{-1}\omega^4 - b[1 + 2\kappa^{-1}(a_2 + \omega^2)] \] (22)

Then, similarly to the undamped case and the case of no delay, the required curve with the period \( 2\pi/T \geq 2\omega_b \) can be drawn and we can conclude that this inequality, together with (19), is a sufficient condition for stability. Let us state the result as a theorem.

**Theorem 2.** Define \( \chi \) to be the smallest positive root of the equation \( \chi = \tan \chi \) and let \( \omega_b \) be the largest root \( \omega \) of the equation (22). Assume that the following two inequalities hold: \( T < \pi/\omega_b \) and \( b\tau < a_1\sqrt{1 + \chi^2} \). Then the zero solution of the equation (1) is globally asymptotically stable for all functions \( p(t) \) having a period \( T \) and satisfying the inequality (2).
C. A Conjecture

Theorems 1 and 2 from the previous sections suggest the following conjecture:

The zero solution of the equation (1) is globally asymptotically stable for all functions $p(t)$ having a period $T$ and satisfying the inequality (2) if $T < \pi/\omega_0$, where $\omega_0$ is the largest root of the equation $\Phi(\omega) = 0$.

Note that both Theorems 1 and 2 represent special cases of this conjecture except that in their statements $\omega_0$ has been estimated analytically. It, therefore, remains to prove or refute it for the case of $0 < a_1 < \sqrt{1 + \chi^2}/b \tau$.

The conjecture presupposes that the number of roots of the equation $\Phi(\omega) = 0$ is finite. This has been established in the course of the proofs of Theorems 1 and 2.

It is also worth noting that this conjecture is generally false for equations of higher order as follows from some of the results in [23]. Finding out for which equations (if any) it is true is a question of independent research significance.

IV. Hodograph Analysis

Let us chose $\mathcal{H}(\omega) = \theta \sin \omega T$, where $\theta$ is a real constant (to be determined later). In the above notations for $P(\omega)$ and $Q(\omega)$, the inequality (5) takes the form:

$$
\kappa^{-1} + P(\omega) - \theta Q(\omega) \sin \omega T > 0. \quad (23)
$$

This inequality means that the curve in the complex plane, defined parametrically $(P(\omega), Q(\omega) \sin \omega T)$ must lie entirely to the right of the straight line with the slope $\theta$ and the x-intercept of $-\kappa^{-1}$, essentially the Popov line.

Consider a numerical example with $a_1 = 0.5$, $a_2 = 2$, $b = 0.7$, $T = 2.1$, and $\tau = 3$. The parametric plot with $\theta = -2.5$ and $\kappa^{-1} = 0.35$ is shown in Figure 1. The curve is drawn as a solid line and the Popov line is broken.

This plot shows that the system is stable for these values of parameters. Furthermore, since the Popov line is tangent to the curve, this plot gives the upper limit for the sector boundary $\kappa$ that assures system stability for the given values of the transfer function parameters.

This geometric interpretation can also be formulated algebraically, similarly to the way described by Rasvan [7] for the Popov criterion. The system is stable for the values of the parameter $\kappa$ found by solving the minimax problem:

$$
\kappa^{-1} > \min_{\theta} \max_{\omega} \left[ -P(\omega) + \theta Q(\omega) \sin \omega T \right]. \quad (24)
$$

In essence, the geometric approach solves this minimax problem graphically. Algebraic formulation suggests that it can also be solved numerically.

In illustrating this approach we have used one specific form of the function $\mathcal{H}(\omega)$. It is certainly possible, at a price of increased computational complexity, to use other forms.

V. Conclusions

We have considered two possible geometric approaches to investigating stability of the Hill equation with a time delay term. Both approaches illustrate some of the difficulties that arise when transfer function contains a transcendental term. Introduction of this term increases the complexity of both algebraic and geometric analysis of the frequency-domain inequality.

Nevertheless, it turned out to be possible to obtain some stability results using two alternative geometric approaches. The first approach, based on the Lipatov plot, is more suited to determining the bounds of the period $T$ and/or of the delay for a given transfer function and the value of the parameter $\kappa$. By contrast, the use of the modified form of the Nyquist hodograph is better for solving the "inverse" of this problem, that is to find the boundary of the parameter $\kappa$ for a given transfer function and the period of oscillations.

Further research into this problem may proceed along the lines of investigating the conjecture stated in Section III.C, extending these results to systems of higher order, including delays involving derivatives. However, such systems will certainly be more difficult to treat, both analytically and computationally.

REFERENCES


