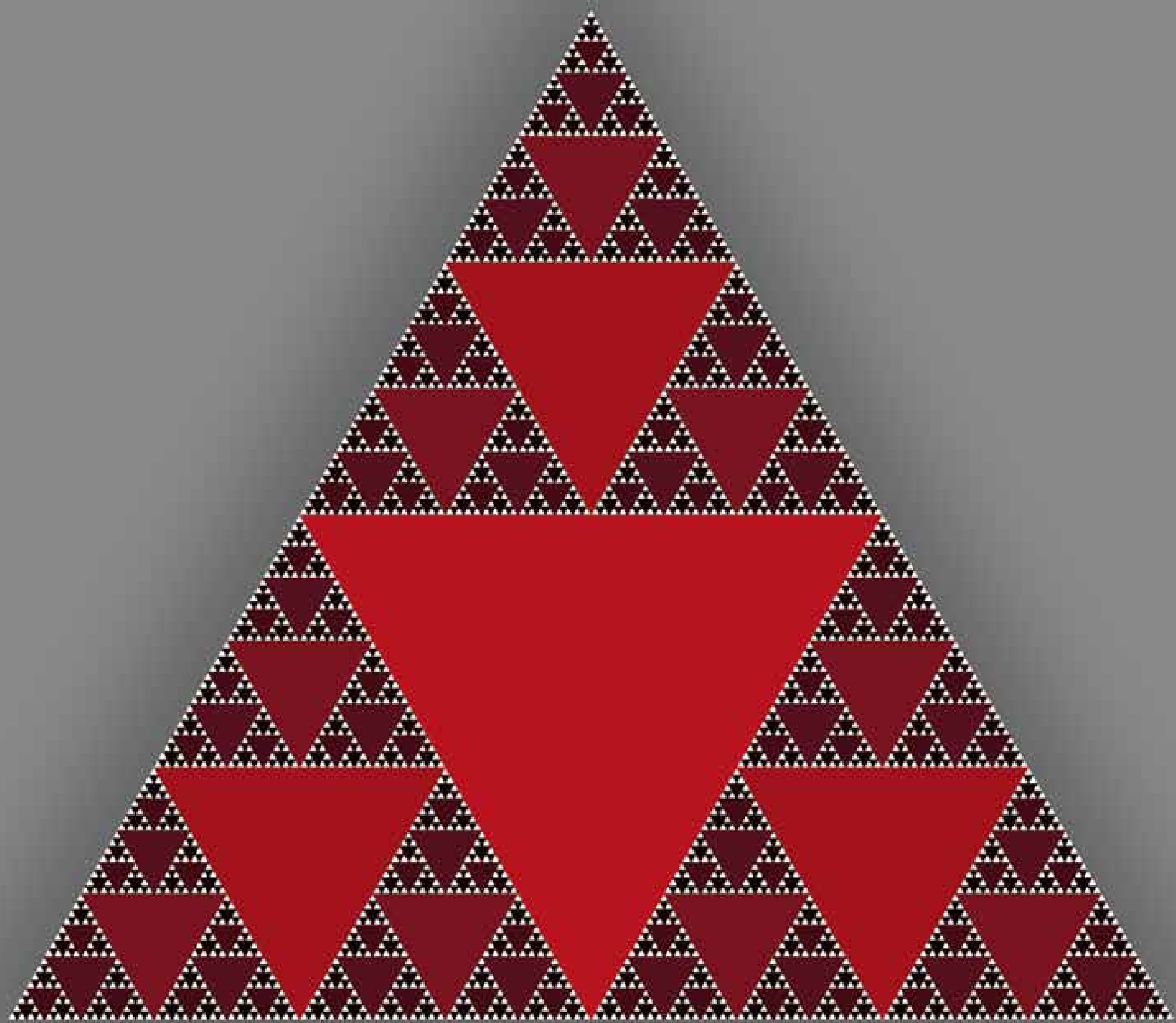

Beautiful Geometry

ELI MAOR AND EUGEN JOST

Frontispiece: *Infinity*

Beautiful Geometry

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Prefaces

ART THROUGH MATHEMATICAL EYES

ELI MAOR

No doubt many people would agree that art and mathematics don't mix. How could they? Art, after all, is supposed to express feelings, emotions, and impressions—a subjective image of the world as the artist sees it. Mathematics is the exact opposite—cold, rational, and emotionless. Yet this perception can be wrong. In the Renaissance, mathematics and art not only were practiced together, they were regarded as complementary aspects of the human mind. Indeed, the great masters of the Renaissance, among them Leonardo da Vinci, Michelangelo, and Albrecht Dürer, considered themselves as architects, engineers, and mathematicians as much as artists.

If I had to name just one trait shared by mathematics and art, I would choose their common search for pattern, for recurrence and order. A mathematician sees the expression $a^2 + b^2$ and immediately thinks of the Pythagorean theorem, with its image of a right triangle surrounded by squares built on the three sides. Yet this expression is not confined to geometry alone; it appears in nearly every branch of mathematics, from number theory and algebra to calculus and analysis; it becomes a pattern, a paradigm. Similarly, when an artist looks at a wallpaper design, the recurrence of a basic motif, seemingly repeating itself to infinity, becomes etched in his or her mind as a pattern. *The search for pattern* is indeed the common thread that ties mathematics to art.

The present book has its origin in May 2009, when my good friend Reny Montandon arranged for me to give a talk to the upper mathematics class of the Alte Kantonsschule (Old Cantonal High School) of Aarau, Switzerland. This school has a historic claim to fame: it was here that a 16-year-old Albert Einstein spent two of his happiest years, enrolling there at his own initiative to escape the authoritarian educational system he so much loathed at home. The school still occupies the same building that Einstein knew, although a modern wing has been added next to it. My wife and I were received with great honors, and at lunchtime I was fortunate to meet Eugen Jost.

I had already been acquainted with Eugen's exquisite mathematical artwork through our mutual friend Reny, but to meet him in person gave me special pleasure, and we instantly bonded. Our encounter was the spark that led us to collaborate on the present book. To our deep regret, Reny Montandon passed away shortly before the completion of our book; just one day before his death, Eugen spoke to him over the phone and told him about the progress we were making, which greatly pleased him. Sadly he will not be able to see it come to fruition.

Our book is meant to be enjoyed, pure and simple. Each topic—a theorem, a sequence of numbers, or an intriguing geometric pattern—is explained in words and accompanied by one or more color plates of Eugen's artwork. Most topics are taken from geometry; a few deal with numbers and numerical progressions. The chapters are largely independent of one another, so the reader can choose what he or she likes without affecting the continuity of reading. As a rule we followed a chronological order, but occasionally we grouped together subjects that are related to one another mathematically. I tried to keep the technical details to a minimum, deferring some proofs to the appendix and referring others to external sources (when referring to books already listed in the bibliography, only the author's name and the book's title are given). Thus the book can serve as an informal—and most certainly not complete—survey of the history of geometry.

Our aim is to reach a broad audience of high school and college students, mathematics and science teachers, university instructors, and laypersons who are not afraid of an occasional formula or equation. With this in mind, we limited the level of mathematics to elementary algebra and geometry ("elementary" in the sense that no calculus is used). We hope that our book will inspire the reader to appreciate the beauty and aesthetic appeal of mathematics and of geometry in particular.

Many people helped us in making this book a reality, but special thanks go to Vickie Kearn, my trusted editor at Princeton University Press, whose continuous enthusiasm and support has encouraged us throughout the project; to the editorial and technical staff at Princeton University Press for their efforts to ensure that the book meets the highest aesthetic and artistic standards; to my son Dror for his technical help in typing the script of plate 26 in Hebrew; and, last but not least, to my dear wife Dalia for her steady encouragement, constructive critique, and meticulous proofreading of the manuscript.

PLAYING WITH PATTERNS, NUMBERS, AND FORMS

EUGEN JOST

My artistic life revolves around patterns, numbers, and forms. I love to play with them, interpret them, and metamorphose them in endless variations. My motto is the Pythagorean motto: *Alles ist Zahl* ("All is Number"); it was the title of an earlier project I worked on with my friends Peter Baptist and Carsten Miller in 2008. *Beautiful Geometry* draws on some of the ideas expressed in that earlier work, but its conception is somewhat different. We attempt here to depict a wide selection of geometric theorems in an artistic way while remaining faithful to their mathematical message.

While working on the present book, my mind was often with Euclid: A point is that which has no part; a line is a breadthless length. Notwithstanding that claim, Archimedes drew his broad-lined circles with his finger in the sand of Syracuse. Nowadays it is much easier to meet Euclid's demands: with a few clicks of the mouse you can reduce the width of a line to nothing—in the end there remains only a nonexisting path. It was somewhat awe inspiring to go through the constructions that were invented—or should I say discovered?—by the Greeks more than two thousand years ago.

For me, playing with numbers and patterns always has top priority. That's why I like to call my pictures playgrounds, following a statement by the Swiss Artist Max Bill: "perhaps the goal of concrete art is to develop objects for mental use, just like people created objects for material use." Some illustrations in our book can be looked at in this sense. The onlooker is invited to play: to find out which rules a picture is built on and how the many metamorphoses work, to invent his or her own pictures. In some chapters the relation between text and picture is loose; in others, however, artistic claim stood behind the goal to enlighten Eli's text. Most illustrations were created on the keypad of my computer, but others are acrylics on canvas. Working with Eli was a lot of fun. He is one of those mathematicians that teach you: Mathematics did not fall from heaven; it was invented and found by humans; it is full of stories; it is philosophy, history and culture. I hope the reader will agree.

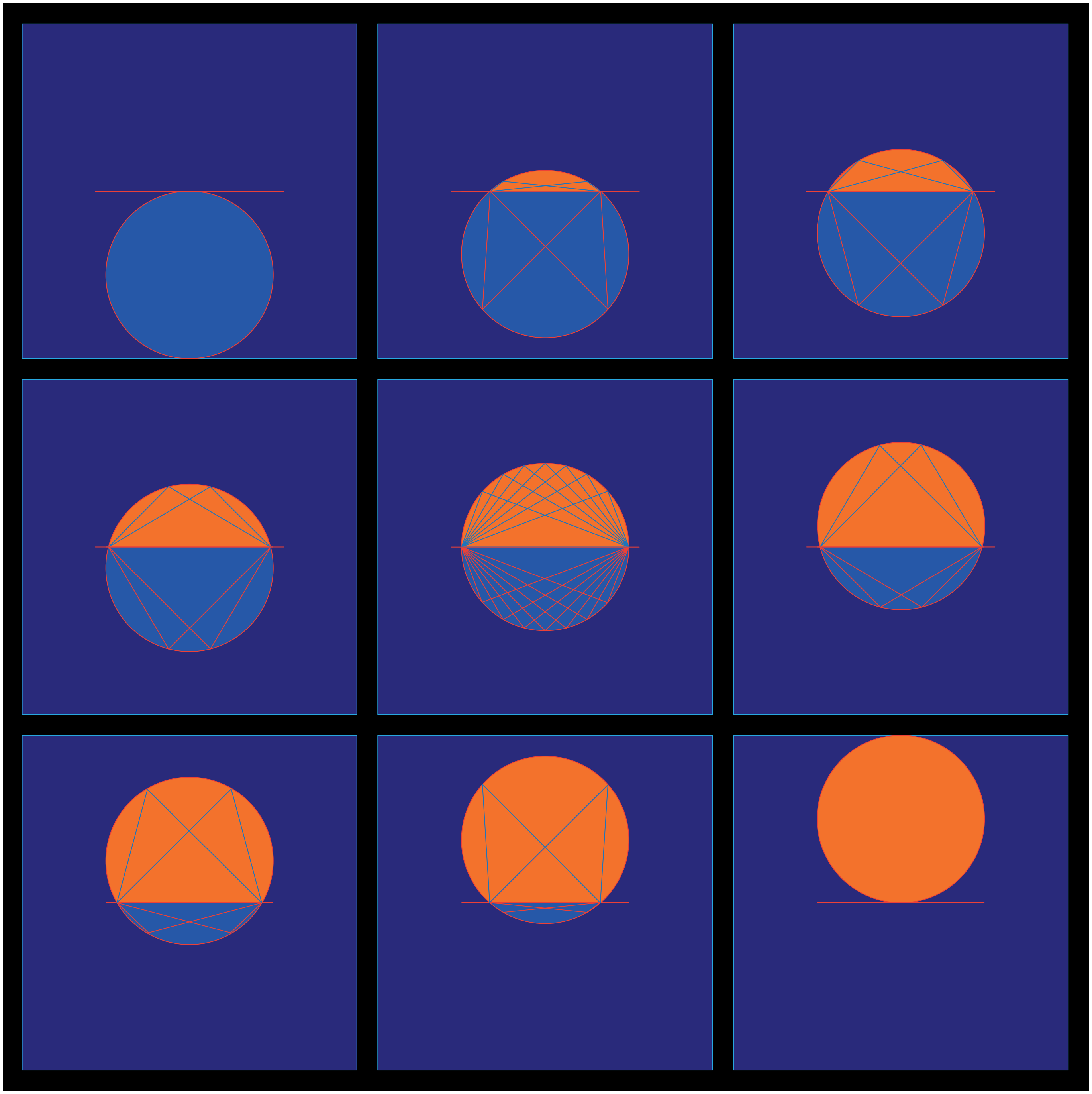


Plate 1. *Sunrise over Miletus*

1

Thales of Miletus

Thales (ca. 624–546 BCE) was the first of the long line of mathematicians of ancient Greece that would continue for nearly a thousand years. As with most of the early Greek sages, we know very little about his life; what we do know was written several centuries after he died, making it difficult to distinguish fact from fiction. He was born in the town of Miletus, on the west coast of Asia Minor (modern Turkey). At a young age he toured the countries of the Eastern Mediterranean, spending several years in Egypt and absorbing all that their priests could teach him.

While in Egypt, Thales calculated the height of the Great Cheops pyramid, a feat that must have left a deep impression on the locals. He did this by planting a staff into the ground and comparing the length of its shadow to that cast by the pyramid. Thales knew that the pyramid, the staff, and their shadows form two similar right triangles. Let us denote by H and h the heights of the pyramid and the staff, respectively, and by S and s the lengths of their shadows (see figure 1.1). We then have the simple equation $H/S = h/s$, allowing Thales to find the value of H from the known values of S, s, and h. This feat so impressed Thales's fellow citizens back home that they recognized him as one of the Seven Wise Men of Greece.

Mathematics was already quite advanced during Thales's time, but it was entirely a practical science, aimed at devising formulas for solving a host of financial, commercial, and engineering problems. Thales was the first to ask not only *how* a particular problem can be solved, but *why*. Not willing to accept facts at face value, he attempted to prove them from fundamental principles. For example, he is credited with demonstrating that the two base angles of an isosceles triangle are equal, as are the two vertical angles formed by a pair of intersecting lines. He also showed that the diameter of a circle cuts it into two equal parts, perhaps by folding over the two halves and observing that they exactly overlapped. His proofs may not stand up to modern standards, but they were a first step toward the kind of deductive mathematics in which the Greeks would excel.

Thales's most famous discovery, still named after him, says that from any point on the circumference of a circle, the diameter always subtends a right angle. This was perhaps the first known *invariance* theorem—the fact that in a geometric configuration,

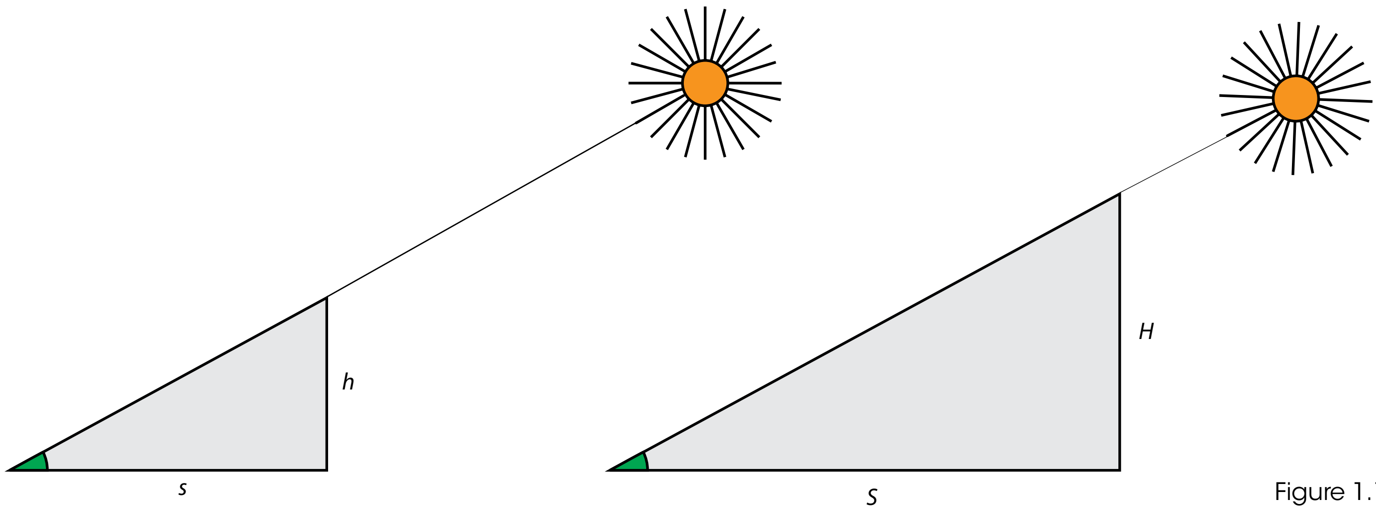


Figure 1.1

some quantities remain the same even as others are changing. Many more invariance theorems would be discovered in the centuries after Thales; we will meet some of them in the following chapters.

Thales's theorem can actually be generalized to any chord, not just the diameter. Such a chord divides the circle into two unequal arcs. Any point lying on the larger of these arcs subtends the chord at a constant angle $\alpha < 90°$; any point on the smaller arc subtends it at an angle $\beta = 180° - \alpha > 90°$.[1] Plate 1, *Sunrise over Miletus*, shows this in vivid color.

NOTE:

1. The converse of Thales's theorem is also true: the locus of all points from which a given line segment subtends a constant angle is an arc of a circle having the line segment as chord. In particular, if the angle is 90°, the locus is a full circle with the chord as diameter.

5

The Pythagorean Theorem I

By any standard, the Pythagorean theorem is the most well-known theorem in all of mathematics. It shows up, directly or in disguise, in almost every branch of it, pure or applied. It is also a record breaker in terms of the number of proofs it has generated since Pythagoras allegedly proved it around 500 BCE. And it is the one theorem that almost everyone can remember from his or her high school geometry class.

Most of us remember the Pythagorean theorem by its famous equation, $a^2+b^2=c^2$. The Greeks, however, thought of it in purely geometric terms, as a relationship between areas. This is how Euclid stated it: *in all right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle*. That is, the area of the square built on the hypotenuse ("the side subtending the right angle") is equal to the combined area of the squares built on the other two sides.

Pythagoras of Samos (ca. 580–ca. 500 BCE) may have been the first to prove the theorem that made his name immortal, but he was not the first to *discover* it: the Babylonians, and possibly the Chinese, knew it at least twelve hundred years before him, as is clear from several clay tablets discovered in Mesopotamia. Furthermore, if indeed he had a proof, it is lost to us. The Pythagoreans did not leave any written records of their discoveries, so we can only speculate what demonstration he gave. There is, however, an old tradition that ascribes to him what became known as the *Chinese proof*, so called because it appeared in an ancient Chinese text dating from the Han dynasty (206 BCE – 220 CE; see figure 5.1). It is perhaps the simplest of the more than 400 proofs that have been given over the centuries.

The Chinese proof is by dissection. Inside square *ABCD* (figure 5.2) inscribe a smaller, tilted square *KLMN*, as shown in (a). This leaves four congruent right triangles (shaded in the figure). By reassembling these triangles as in (b), we see that the remaining (unshaded) area is the sum of the areas of squares 1 and 2, that is, the squares built on the sides of each of the right triangles.

Elisha Scott Loomis (1852–1940), a high school principal and mathematics teacher from Ohio, collected all the proofs known to him in a classic book, *The Pythagorean Proposition* (first published in 1927, with a second edition in 1940, the year of his death). In it you can find a proof attributed to Leonardo da Vinci, another by James A. Garfield, who would become the twentieth president of the United States, and yet another by Ann Condit, a

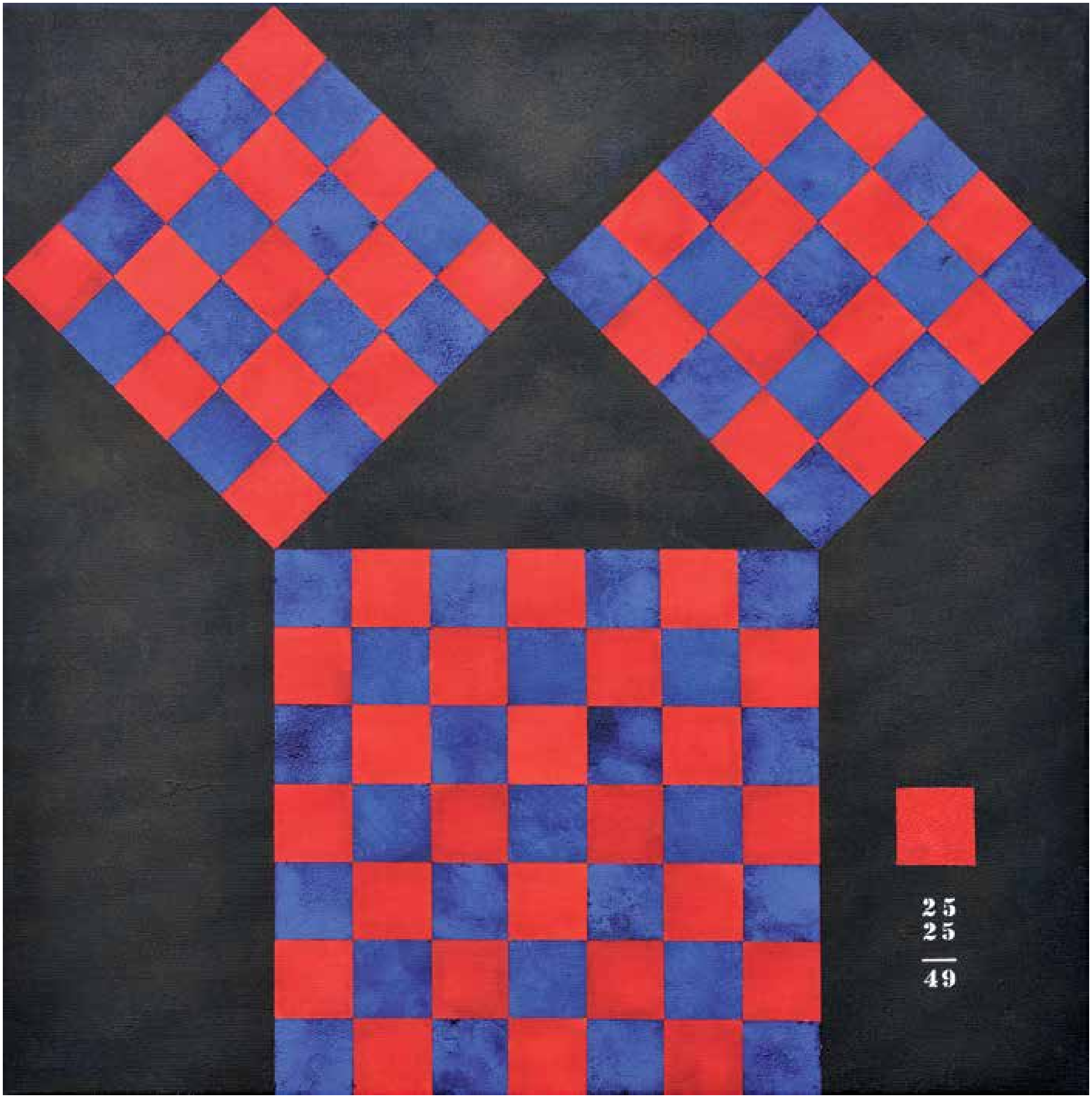


Plate 5. *25 + 25 = 49*

16-year-old high school student from South Bend, Indiana. And of course, there is the most famous proof of them all: Euclid's proof. We will look at it in the next chapter.

Our illustration (plate 5) shows a 45-45-90-degree triangle with squares—or what looks like squares—built on its sides and on the hypotenuse. But wait! Something strange seems to be going on: $5^2+5^2=7^2$, or $25+25=49$! Did anything go wrong? Do we see here an optical illusion? Not really: the illustration is, after all, an artistic rendition of the Pythagorean theorem, not the theorem itself; as such it is not bound by the laws of mathematics. To quote the American artist Josef Albers (1888–1976): "In science, one plus one is always two; in art it can also be three or more."

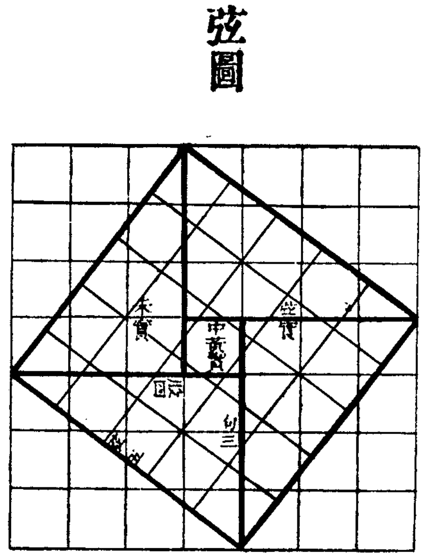
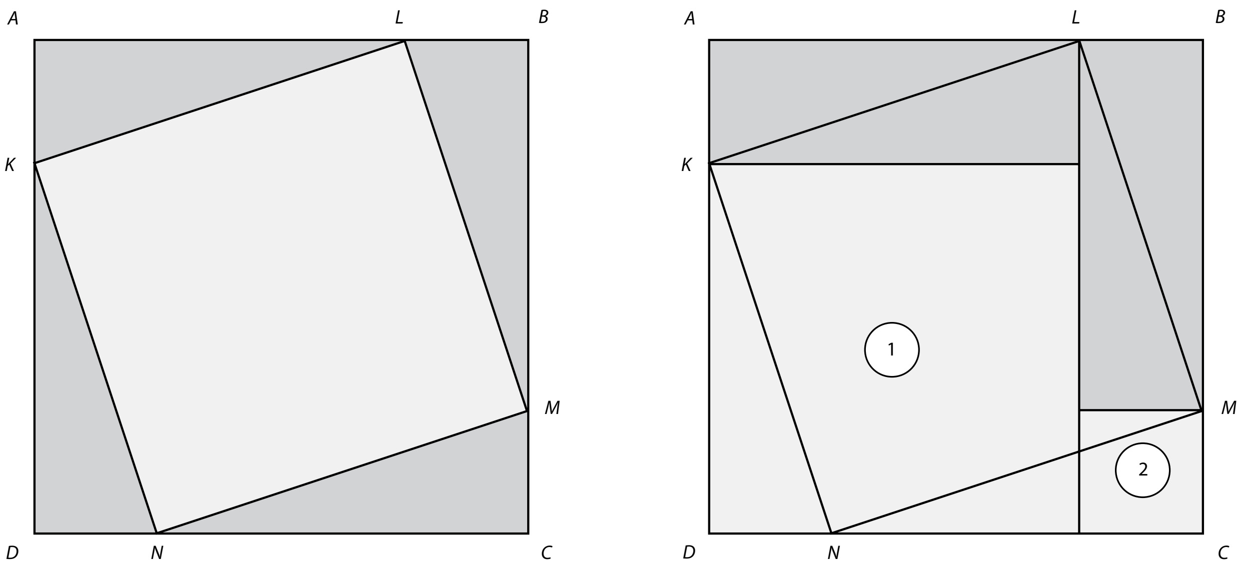


Figure 5.1. Joseph Needham, *Science and Civilisation in China*, courtesy of Cambridge University Press, Cambridge, UK.



(a) (b)

Figure 5.2

20

Fibonacci Numbers

Almost anyone with the slightest interest in mathematics will be familiar with the name Fibonacci. Leonardo of Pisa—he later adopted the name Fibonacci (son of Bonacci)—was born in Pisa around 1170, the son of a wealthy merchant. Pisa at that time was an important commercial center, serving both Christian Europe and Moslem Middle East and North Africa. Fibonacci was thus acquainted with the newly invented Hindu-Arabic numeration system, with the numerals (or "ciphers") 0 through 9 as its centerpiece. Convinced that this system was superior to the cumbersome Roman numerals, he wrote a book entitled *Liber Abaci* ("Book of the Calculation," sometimes translated as "Book of the Abacus"), in which he advocated the new system and explained its operation. Published in 1202, it became an instant bestseller and was in no small measure responsible for the acceptance of the new system by European merchants and, eventually, by most of the learned world.

So it is ironic that Fibonacci's name is remembered today not for the main thrust of his influential book—promoting the Hindu-Arabic numeration system—but for a little problem he posed in it, perhaps as a recreational exercise. The problem deals with the number of offsprings a hypothetical pair of rabbits can produce, assuming that a pair becomes productive from the second month on and gives birth to a new pair every subsequent month. This leads to the sequence of numbers 1, 1, 2, 3, 5, 8, 13, 21, 34, . . ., in which each number from the third term on is the sum of its two predecessors.[1] The Fibonacci sequence, as it became known, grows very fast: the tenth member is 55, the twentieth is 6,765, and the thirtieth is 832,040. In his famous problem Fibonacci asked how many rabbits will there be after one year. The answer is 144, the twelfth Fibonacci number.

Fibonacci hardly could have anticipated the stir his little puzzle would create. The sequence enjoys numerous properties—so many, in fact, that a scholarly journal, the *Fibonacci Quarterly*, is entirely devoted to it. Fibonacci numbers seem to appear where you least expect them. For example, the seeds of a sunflower are arranged in two systems of spirals, one winding clockwise, the other, counterclockwise. The number of spirals in each system is always a Fibonacci number, typically 34 one way and 55 the other (see figure 20.1), with occasional higher numbers. Smaller Fibonacci numbers also show up in the scales arrangement of pinecones and the leaf patterns of many plants.

Plate 20. *Girasole*

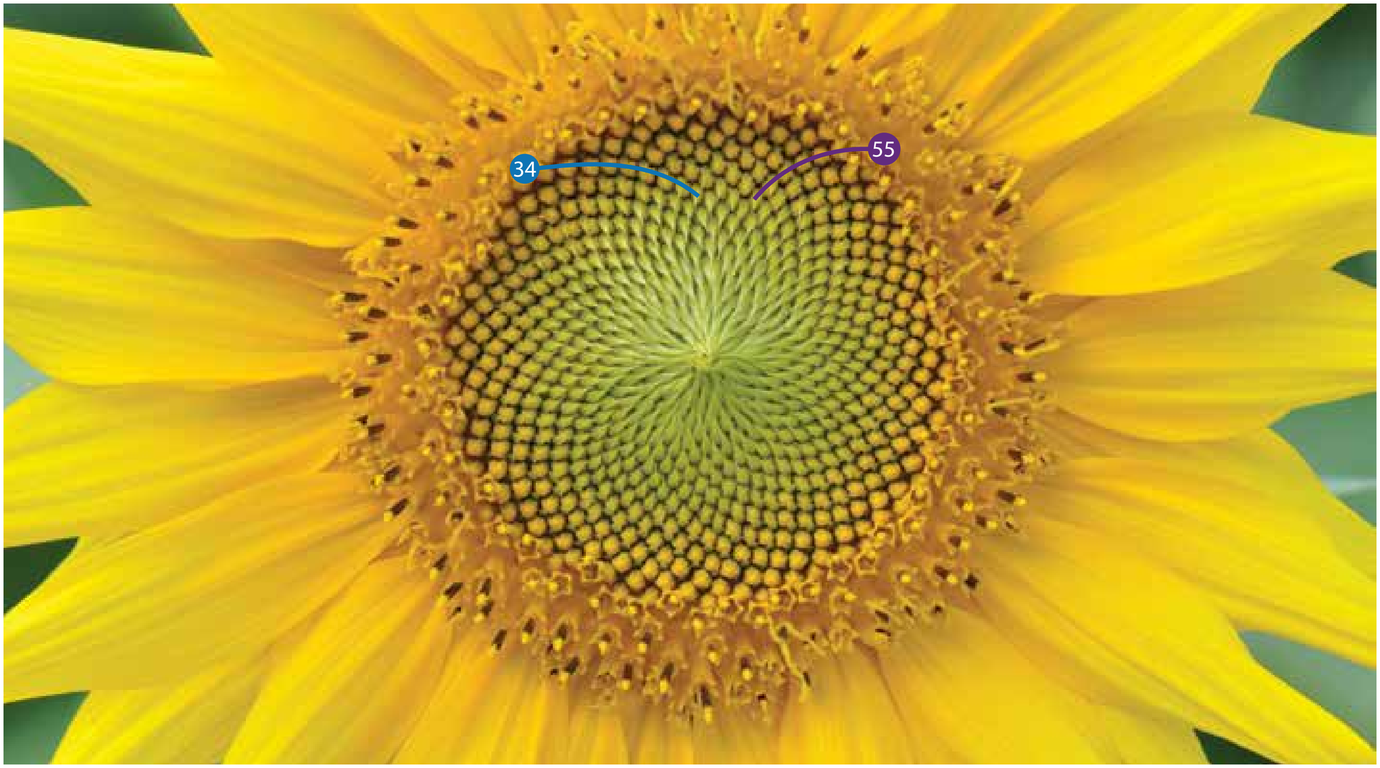


Figure 20.1

Among the purely mathematical properties of the Fibonacci numbers, we mention here just one: the sum of the first n members of the sequence is always equal to the next-to-next member, minus 1; that is,

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$

For example, the sum of the first 8 Fibonacci numbers is the tenth number minus 1: $1+1+2+3+5+8+13+21=54=55-1$. You can use this fact to surprise your friends by asking them to find the sum of, say, the first 10 Fibonacci numbers. Most likely they will start by adding the terms one by one, a process that will take some time. But knowing that the twelfth Fibonacci number is 144, you can outdo them by announcing the answer, 143, while they are still doing their sums. It always works! (See the appendix for a proof.)

Perhaps most surprising of all is a discovery made in 1611 by Johannes Kepler: divide any member of the sequence by its immediate predecessor. As you do this with ever-increasing numbers, the ratios seem to converge to a fixed number, a limit:

$$\frac{2}{1}=2,\ \frac{3}{2}=1.5,\ \frac{5}{3}=1.666\ldots,\ \frac{8}{5}=1.6,$$

$$\frac{13}{8}=1.625,\ \frac{21}{13}=1.615\ldots,\ \ldots$$

This limit, about 1.618, turns out to be one of the most famous numbers in mathematics, nearly the equal in status to π and e. Its exact value is $(1+\sqrt{5})/2$. It came to be known as the *golden ratio* (*sectio aura* in Latin), and it holds the secret for constructing the regular pentagon, as we will see in chapter 22.

Plate 20, *Girasole*, shows a series of squares, each of which, when adjoined to its predecessor, forms a rectangle. Starting with a black square of unit length, adjoin to it its white twin, and you get a 2×1 rectangle. Adjoin to it the green square, and you get a 3×2 rectangle. Continuing in this manner, you get rectangles whose dimensions are exactly the Fibonacci numbers. The word *Girasole* ("turning to the sun" in Italian) refers to the presence of these numbers in the spiral arrangement of the seeds of a sunflower—a truly remarkable example of mathematics at work in nature.

NOTE:

1. The sequence is sometimes counted with 0 as the first member. It can also be extended to negative numbers: . . . 5, –3, 2, –1, 1, 0, 1, 1, 2, 3, 5, ...

26

Squaring the Circle

At first glance, the circle may seem to be the simplest of all geometric shapes and the easiest to draw: take a string, hold down one end on a sheet of paper, tie a pencil to the other end, and swing it around—a simplified version of the compass. But first impressions can be misleading: the circle has proved to be one of the most intriguing shapes in all of geometry, if not the most intriguing of them all.

How do you find the area of a circle, when its radius is given? You instantly think of the formula $A = \pi r^2$. But what exactly is that mysterious symbol π? We learn in school that it is approximately 3.14, but its *exact* value calls for an endless string of digits that never repeat in the same order. So it is impossible to find the exact area of a circle numerically. But perhaps we can do the next best thing—construct, using only straightedge and compass, a square equal in area to that of a circle?

This problem became known as *squaring the circle*—or simply the *quadrature* problem—and its solution eluded mathematicians for well over two thousand years. The ancient Egyptians came pretty close: In the Rhind Papyrus, a collection of 84 mathematical problems dating back to around 1800 BCE, there is a statement that the area of a circle is equal to the area of a square of side $8/9$ of the circle's diameter. Taking the diameter to be 1 and equating the circle's area to that of the square, we get $\pi(1/2)^2 = (8/9)^2$, from which we derive a value of π equal to $256/81 \approx 3.16049$—within 0.6 percent of the true value. However, as remarkable as this achievement is, it was based on "eyeballing," not on an exact geometric construction.

In the Bible (I Kings 7:23) we find the following verse: "And he made a molten sea, ten cubits from one brim to the other; it was round all about . . . and a line of thirty cubits did compass it round about." In this case, "he" refers to King Solomon, and the "molten sea" was a pond that adorned the entrance to the Holy Temple in Jerusalem. Taken literally, this would imply that $\pi = 3$, and the quadrature of the circle would have become a simple task! A great deal of commentary has been written on this one verse (it also appears, with a slight change, in II Chronicles 4:2), but that would take us outside the realm of geometry. Plate 26.1, *π = 3*, quotes this famous verse in its original Hebrew; to read it, start at the red dot and proceed counterclockwise all the way around.

Numerous attempts have been made over the centuries to solve the quadrature problem. Many careers were spent on this task—all in vain. The defini-

Plate 26.1. $\pi = 3$

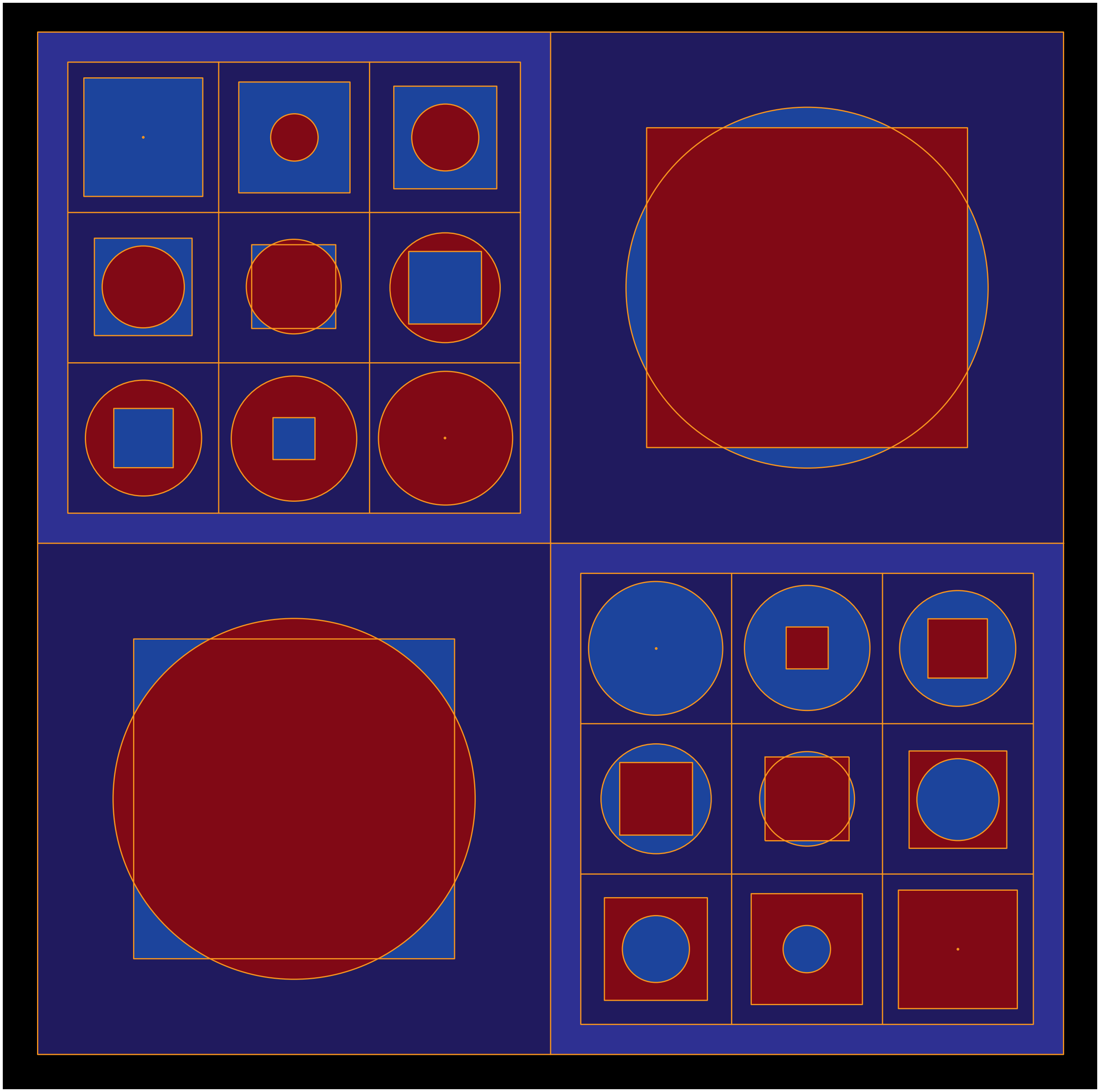


Plate 26.2. *Metamorphosis of a Circle*

tive solution—a negative one—came only in 1882, when Carl Louis Ferdinand von Lindemann (1852–1939) proved that the task cannot be done—it is impossible to square a circle with Euclidean tools. Actually, Lindemann proved something different: that the number π, the constant at the heart of the quadrature problem, is transcendental. A *transcendental number* is a number that is not the solution of a polynomial equation with integer coefficients. A number that is not transcendental is called *algebraic*. All rational numbers are algebraic; for example, $3/5$ is the solution of the equation $5x-3=0$. So are all square roots, cubic roots, and so on; for example, $\sqrt{2}$ is the positive solution of $x^2-2=0$, and $\sqrt[3]{2-\sqrt{5}}$ is one solution of $x^6-4x^3-1=0$. The name *transcendental* has nothing mysterious about it; it simply implies that such numbers transcend the realm of algebraic (polynomial) equations.

Now it had already been known that if π turned out to be transcendental, this would at once establish that the quadrature problem cannot be solved. Lindemann's proof of the transcendence of π therefore settled the issue once and for all. But settling the issue is not the same as putting it to rest; being the most famous of the three classical problems, we can rest assured that the "circle squarers" will pursue their pipe dream with unabated zeal, ensuring that the subject will be kept alive forever.

Plate 26.2, *Metamorphosis of a Circle*, shows four large panels. The panel on the upper left contains nine smaller frames, each with a square (in blue) and a circular disk (in red) centered on it. As the squares decrease in size, the circles expand, yet the sum of their areas remains constant. In the central frame, the square and circle have the same area, thus offering a computer-generated "solution" to the quadrature problem. In the panel on the lower right, the squares and circles reverse their roles, but the sum of their areas is still constant. The entire sequence is thus a metamorphosis from square to circle and back.

Of course, Euclid would not have approved of such a solution to the quadrature problem, because it does not employ the Euclidean tools—a straightedge and compass. It does, instead, employ a tool of far greater power—the computer. But this power comes at a price: the circles, being generated pixel by pixel like a pointillist painting, are in reality not true circles, only simulations of circles.[1] As the old saying goes, "there's no free lunch"—not even in geometry.

NOTE:

1. The very first of the 23 definitions that open Euclid's *Elements* defines a point as "that which has no part." And since all objects of classical geometry—lines, circles, and so on—are made of points, they rest on the subtle assumption that Euclidean space is continuous. This, of course, is not the case with computer space, where Euclid's dimensionless point is replaced by a pixel—small, yet of finite size—and space between adjacent pixels is empty, containing no points.