at the same price. (ii) Some bid is above b_i'. In this case bidding $b_i = v$ will yield the exact same outcome: you'll lose to a higher bid. (iii) No bids are above b_i' and some bid b_j^* is in between v_i and b_i'. In this case bidding b_i' will cause you to win in and pay $b_j^* > v_i$ which means that your payoff is negative, while if you would have bid $b_i = v_i$ then you would lose and get nothing. Hence, in cases (i) and (ii) bidding v_i would do as well as bidding b_i', and in case (iii) it would do strictly better, implying that bidding more than your valuation is weakly dominated by actually bidding your valuation. ■

(b) Argue that bidding less than your valuation is weakly dominated by actually bidding your valuation.

Answer: If you put in a bid $b_i = b_i' < v_i$ where v_i is your valuation, then only the three following cases can happen: (i) Some other bid are above In this case bidding $b_i = v_i$ will yield the exact same outcome: you'll lose to a higher bid. (ii) All other bids are below b_i'. In this case bidding $b_i = v_i$ will yield the exact same outcome: you'll win at the same price. (iii) No bids are above v_i and some bid b_j^* is in between b_i' and v_i. In this case bidding b_i' will cause you to lose and get nothing, while if you would have bid $b_i = v_i$ then you would win and get a positive payoff of $v_i - b_j^*$. Hence, in cases (i) and (ii) bidding v_i would do as well as bidding b_i', and in case (iii) it would do strictly better, implying that bidding less than your valuation is weakly dominated by actually bidding your valuation. ■

(c) Use your analysis above to make sense of eBay's recommendation. Would you follow it?

Answer: The recommendation is indeed supported by an analysis of rational behavior.[1]

[1] Those familiar with eBay know about sniping, which is bidding in the last minute. It still is a weakly dominated strategy to bid your valuation at that time, and waiting for the last minute may be a "best response" if you believe other people may respond to an early bid. More on this is discussed in chapter 13.

5.

6. **Roommates**: Two roommates need to each choose to clean their apartment, and each can choose an amount of time $t_i \geq 0$ to clean. If their choices are t and t_j, then player i's payoff is given by $(10 - t_j)t_i - t_i^2$. (This payoff function implies that the more one roommate cleans, the less valuable is cleaning for the other roommate.)

   (a) What is the best response correspondence of each player i?

   Answer: Player i maximizes $(10 - t_j)t_i - t_i^2$ given a belief about t_j, and the first-order optimality condition is $10 - t_j - 2t_i = 0$ implying that the best response is $t_i = \frac{10-t_j}{2}$. ■

   (b) Which choices survive one round of IESDS?

   Answer: The most player i would choose is $t_i = 5$, which is a BR to $t_j = 0$. Hence, any $t_i > 5$ is dominated by $t_i = 5$.[2] Hence, $t_i \in [0, 5]$ are the choices that survive one round of IESDS.

   (c) Which choices survive IESDS?

   Answer: The analysis follows the same ideas that were used for the Cournot duopoly in section 4.2.2. In the second round of elimination, because $t_2 \leq 5$, the best response $t_i = \frac{10-t_j}{2}$ implies that firm 1 will choose $t_1 \geq 2.5$, and a symmetric argument applies to firm 2. Hence, the second round of elimination implies that the surviving strategy sets are $t_i \in [2.5, 5]$ for $i \in \{1, 2\}$. If this process were to converge to an interval, and not to a single point, then by the symmetry between both players, the resulting interval for each firm would be $[t_{\text{min}}, t_{\text{max}}]$ that simultaneously satisfy two equations with two unknowns: $t_{\text{min}} = \frac{10-t_{\text{max}}}{2}$ and $t_{\text{max}} = \frac{10-t_{\text{min}}}{2}$. However, the only solution to these two equations is

[2] This can be shown directly: The payoff from choosing $t_i = 5$ when the opponent is choosing t_j is $v(5, t_j) = (10 - t_j)5 - 25 = 25 - 5t_j$. The payoff from choosing $t_i = 5 + k$ where $k > 0$ when the opponent is choosing t_j is $v(5+k, t_j) = (10-t_j)(5+k)-(5+k)^2 = 25-5t-k^2-t_jk_j$, and because $k > 0$ it follows that $v(5+k, t_j) < v(5, t_j$

.

$t_{\min} = t_{\max} = \frac{10}{3}$. Hence, the unique pair of choices that survive IESDS for this game are $t_1 = t_2 = \frac{10}{3}$. ■

7.

8. Consider the p-Beauty contest presented in section 4.3.5.

(a) Show that if player i believes that everyone else is choosing 20 then 19 is not the only best response for any number of players n.

Answer: If everyone else is choosing 20 and if player i chooses 19 then $\frac{3}{4}$ of the average will be somewhere below 15, and 19 is closer to that number, and therefore is a best response. But the same argument holds for any choice of player i that is between 15 and and 20 regardless of the number of players. (In fact, you should be able to convince yourself that this will be true for any choice of i between 10 and 20.) ■

(b) Show that the set of best response strategies to everyone else choosing the number 20 depends on the number of players n.

Answer: Imagine that $n = 2$. If one player j is choosing 20, then any number s_i between 0 and 19 will beat 20. This follows because the target number ($\frac{3}{4}$ of the average) is equal to $\frac{3}{4} \times \frac{20+s_i}{2} = \frac{15}{2} + \frac{3}{8}s_i$, the distance between 20 and the target number is $\frac{25}{2} - \frac{3}{8}s_i$ (this will always be positive because the target number is less than 20) while the distance between s_i and the target number is $\left|\frac{5}{8}s_i - \frac{15}{2}\right|$. The latter will be smaller than the former if and only if $\left|\frac{5}{8}s_i - \frac{15}{2}\right| < \frac{25}{2} - \frac{3}{8}s_i$, or $-20 < s_i < 20$. Given the constraints on the choices, $BR_i \in \{0, 1, ...19\}$. Now imagine that $n = 5$. The target number is equal to $\frac{3}{4} \times \frac{80+s_i}{5} = 12 + \frac{3}{20}s_i$, the distance between 20 and the target number is $8 - \frac{3}{20}s_i$ while the distance between s_i and the target number is $\left|\frac{17}{20}s_i - 12\right|$. The latter will be smaller than the former if and only if $\left|\frac{17}{20}s - 12\right| < 8 - \frac{3}{20}s$, or $\frac{40}{7} < s_i < 20$. Hence, $BR_i = \{6, 7, ..., 19\}$. You should be able to convince yourself that as $n \to \infty$, if everyone but i chooses 20 then i's best response will converge to $BR_i = \{10, 11, ..., 19\}$. ■

9.

10. **Popsicle stands:** There are five lifeguard towers lined along a beach, where the left-most tower is number 1 and the right most tower is number 5. Two vendors, players 1 and 2, each have a popsicle stand that can be located next to one of five towers. There are 25 people located next to each tower, and each person will purchase a popsicle from the stand that is closest to him or her. That is, if player 1 locates his stand at tower 2 and player 2 at tower 3, then 50 people (at towers 1 and 2) will purchase from player 1, while 75 (from towers 3,4 and 5) will purchase from vendor 2. Each purchase yields a profit of $1.

    (a) Specify the strategy set of each player. Are there any strictly dominated strategies?

    Answer: The strategy sets for each player are $S_i = \{1, 2, ..., 5\}$ where each choice represents a tower. To see whether there are any strictly dominated strategies it is useful to construct the matrix representation of this game. Assume that if a group of people are indifferent between the two places (equidistant) then they will split between the two vendors (e.g., if the vendors are at the same tower then their payoffs will be 62.5 each, while if they are located at towers 1 and 3 then they split the people from tower 2 and their payoffs are 37.5 and 87.5 respectively.) Otherwise they get the people closest to them, so payoffs are:

		Player 2				
		1	2	3	4	5
	1	$62.5, 62.5$	$25, 100$	$37.5, 87.5$	$50, 75$	$62.5, 62.5$
Player 1	2	$100, 25$	$62.5, 62.5$	$50, 75$	$62.5, 62.5$	$75, 50$
	3	$87.5, 37.5$	$75, 50$	$62.5, 62.5$	$75, 50$	$87.5, 37.5$
	4	$75, 50$	$62.5, 62.5$	$50, 75$	$62.5, 62.5$	$100, 25$
	5	$62.5, 62.5$	$50, 75$	$37.5, 87.5$	$25, 100$	$62.5, 62.5$

    Notice that the choices of 1 and 5 are strictly dominated by any other choice for both players 1 and 2. ■

(b) Find the set of strategies that survive Rationalizability.

Answer: Because the strategies 1 and 5 are strictly dominated then they cannot be a best response to any belief (Proposition 4.3). In the reduced game in which these strategies are removed, both strategies 2 and 4 are dominated by 3, and therefore cannot be a best response in this second stage. Hence, only the choice $\{3\}$ is rationalizable. ■

5

Pinning Down Beliefs: Nash Equilibrium

1.

2. A strategy $s^W \in S$ is a **weakly dominant strategy equilibrium** if $s_i^W \in S_i$ is a weakly dominant strategy for all $i \in N$. That is if $v_i(s_i^W, s_{-i}) \geq v_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$. Provide an example of a game for which there is a weakly dominant strategy equilibrium, as well as another Nash equilibrium.

   Answer: Consider the following game:

   Player 2

Player 1		L	R
	U	$1,1$	$1,1$
	D	$1,1$	$2,2$

   In this game, (D, R) is a weakly dominant strategy equilibrium (and of course, a Nash equilibrium), yet (U, L) is a Nash equilibrium that is not a weakly dominant strategy equilibrium. ■

3.

4. **Splitting Pizza:** You and a friend are in an Italian restaurant, and the owner offers both of you an 8-slice pizza for free under the following condition. Each of you must simultaneously announce how many slices you would like; that is, each player $i \in \{1,2\}$ names his desired amount of pizza, $0 \leq s_i \leq 8$ If $s_1 + s_2 \leq 8$ then the players get their demands (and the owner eats any leftover slices). If $s_1 + s_2 > 8$, then the players get nothing. Assume that you each care only about how much pizza you individually consume, and the more the better.

   (a) Write out or graph each player's best-response correspondence.

   Answer: Restrict attention to integer demands (more on continuous demands is below). If player j demands $s_j \in \{0,1,...,7\}$ then i's best response is to demand the complement to 8 slices. If i asks for more then both get nothing while if i asks for less then he is leaving some slices unclaimed. If instead player j demands $s_j = 8$ then player i gets nothing regardless of his request so any demand is a best response. In summary,

$$BR_i(s_j) = \begin{cases} 8 - s_j & \text{if } s_j \in \{0,1,...,7\} \\ \{0,1,...,8\} & \text{if } s_j = 8 \end{cases} .$$

   Note: if the players can ask for amounts that are not restricted to integers then the same logic applies and the best response is

$$BR_i(s_j) = \begin{cases} 8 - s_j & \text{if } s_j \in [0,8) \\ [0,8] & \text{if } s_j = 8 \end{cases} .$$

   ■

   (b) What outcomes can be supported as pure-strategy Nash equilibria?

   Answer: It is easy to see from the best response correspondence that any pair of demands that add up to 8 will be a Nash equilibrium, i.e., $(0,8), (1,7), ..., (8,0)$. However, there is another Nash equilibrium: (8,8) in which both players get nothing. It is a Nash equilibrium because given that each player is asking for 8 slices, the other player gets nothing *regardless* of his request, hence he is indifferent between all of his requests including 8.

Note: The pair $s_j = 8$ and $s_i = s$ where $s \in \{1, 2, ..., 7\}$ is not a Nash equilibrium because even though player i is playing a best response to s_j, player j is not playing a best response to s_i because by demanding 8 player j received nothing, but if he instead demanded $8 - s > 0$ then he would get those amount of slices and get something. ■

5.

6. **Hawk-Dove:** The following game has been widely used in evolutionary biology to understand how "fighting" and "display" strategies by animals could coexist in a population. For a typical Hawk-Dove game there are resources to be gained (i.e. food, mates, territories, etc.) denoted as v. Each of two players can chooses to be aggressive, called "Hawk" (H), or can be compromising, called "Dove" (D). If both players choose H then they split the resources, but loose some payoff from injuries, denoted as k. Assume that $k > \frac{v}{2}$. If both choose D then they split the resources, but engage in some display of power that a display cost d, with $d < \frac{v}{2}$. Finally, if player i chooses H while j chooses D, then i gets all the resources while j leaves with no benefits and no costs.

   (a) Describe this game in a matrix

   Answer:

   Player 2

		H	D
Player 1	H	$\frac{v}{2} - k, \frac{v}{2} - k$	$v, 0$
	D	$0, v$	$\frac{v}{2} - d, \frac{v}{2} - d$

   ■

(b) Assume that $v = 10$, $k = 6$ and $d = 4$. What outcomes can be supported as pure-strategy Nash equilibria?[1]

Answer: The game is:

		Player 2 H	Player 2 D
Player 1	H	$-1, -1$	$10, 0$
	D	$0, 10$	$1, 1$

and the two strategy profiles that can be supported as pure strategy Nash equilibria are (H, D) and (D, H), leading to outcomes $(10, 0)$ and $(0, 10)$ respectively. ■

7.

8. **The n firm Cournot Model:** Suppose there are n firms in the Cournot oligopoly model. Let q_i denote the quantity produced by firm i, and let $Q = q_i + \cdots + q_n$ denote the aggregate production. Let $P(Q)$ denote the market clearing price (when demand equals Q) and assume that inverse demand function is given by $P(Q) = a - Q$ (where $Q < a$). Assume that firms have no fixed cost, and the cost of producing quantity q_i is cq_i (all firms have the same marginal cost, and assume that $c < a$).

(a) Model this as a Normal form game

Answer: The players are $N = \{1, 2, ..., n\}$, each player chooses $q_i \in S$ where the strategy sets are $S_i = [0, \infty)$ for all $i \in N$, and the payoffs of each player are given by,

$$v_i(q_i, q_{-i}) = \begin{cases} (a - \sum_{j=1}^{n} q_j)q_i - cq_i & \text{if } \sum_{j=1}^{n} q_j < a \\ -cq_i & \text{if } \sum_{j=1}^{n} q_j \geq a \end{cases}$$

[1] In the evolutionary biology literature, the analysis performed is of a very different nature. Instead of considering the Nash equilibrium analysis of a static game, the analysis is a dynamic analysis where successful strategies "replicate" in a large population. This analysis is part of a methodology called "evolutionary game theory." For more on this see Gintis (2000).

■

(b) What is the Nash (Cournot) Equilibrium of the game where firms choose their quantities simultaneously?

Answer: Let's begin by assuming that there is a symmetric "interior solution" where each firm chooses the same positive quantity as a Nash equilibrium, and then we will show that this is the only possible Nash equilibrium. Because each firm maximizes

$$v_i(q_i, q_{-i}) = (a - \sum_{j=1}^{n} q_j)q_i - cq_i \ ,$$

the first order condition is

$$a - \sum_{j \neq i} q_j - 2q_i - c = 0 \ ,$$

which yields the best response of player i to be

$$BR_i(q_{-i}) = \frac{a - \sum_{j \neq i} q_j - c}{2} \ .$$

Imposing symmetry in equilibrium implies that all n best response conditions will hold with the same values $q_i^* = q^*$ for all $i \in N$, and can be solved using the best response function as follows,

$$q^* = \frac{a - (n-1)q^* - c}{2} \ ,$$

which yields

$$q^* = \frac{a - c}{n + 1} \ .$$

It is more subtle to show that there cannot be other Nash equilibria. To show this we will show that conditional on whatever is chosen by all but two players, the two players must choose the same amount in a Nash equilibrium. Assume that there is another asymmetric Nash equilibrium in which two players, i and j, choose two different equilibrium quantities $q_i^* \neq q_j^*$. Let $\overline{q} = \sum_{m \neq i,j} q_m^*$ be the sum of all the other equilibrium quantity choices of the players who are not i or l. Because we assumed that this is a Nash equilibrium, the best response function of both i and j must hold simultaneously, that is,

$$q_i^* = \frac{a - \overline{q} - q_j^* - c}{2} \ , \tag{5.1}$$

and

$$q_j^* = \frac{a - \overline{q} - q_i^* - c}{2} \ . \tag{5.2}$$

If we substitute (5.2) into (5.1) we obtain,

$$q_i^* = \frac{a - \overline{q} - \frac{a-\overline{q}-q_i^*-c}{2} - c}{2} \ ,$$

which implies that $q_i^* = \frac{a-\overline{q}-c}{3}$. If we substitute this back into (5.2) we obtain,

$$q_j^* = \frac{a - \overline{q} - \frac{a-\overline{q}-c}{3} - c}{2} = \frac{a - \overline{q} - c}{3} = k_i^* \ ,$$

which contradicts the assumption we started with, that $q_i^* \neq q_j^*$. Hence, the unique Nash equilibrium has all the players choosing the same level $q^* = \frac{a-c}{n+1}$. ■

(c) What happens to the equilibrium price as n approaches infinity? Is this familiar?

Answer: First consider the total quantity in the Nash equilibrium as a function of n,

$$Q^* = nq^* = \frac{n(a-c)}{n+1}$$

and the resulting limit price is

$$\lim_{n\to\infty} P(Q^*) = \lim_{n\to\infty} \left(a - \frac{n(a-c)}{n+1} \right) = c.$$

This means that as the number of firms grow, the Nash equilibrium price will also fall and will approach the marginal costs of the firms as the number of firms grows to infinity. Those familiar with a standard economics class know that in perfect competition price will equal marginal costs, which is what happens here when n approaches infinity. ■

9.

10. **Synergies:** Two division managers can invest time and effort in creating a better working relationship. Each invests $e_i \geq 0$, and if both invest more then both are better off, but it is costly for each manager to invest. In particular, the payoff function for player i from effort levels (e_i, e_j) is $v_i(e_i, e_j) = (a + e_j)e_i - e_i^2$.

    (a) What is the best response correspondence of each player?

    Answer: If player i believes that player j chooses e_j then i's first order optimality condition for maximizing his payoff is,

    $$a + e_j - 2e_i = 0 \ ,$$

    yielding the best response function,

    $$BR_i(e_j) = \frac{a + e_j}{2} \text{ for all } e_j \geq 0.$$

    ■

    (b) In what way are the best response correspondences different from those in the Cournot game? Why?

    Answer: Here the best response function of player i is *increasing* in the choice of player j whereas in the Cournot model it is *decreasing* in the choice of player j. This is because in this game the choices of the two players are strategic complements while in the Cournot game they are strategic substitutes. ■

    (c) Find the Nash equilibrium of this game and argue that it is unique.

    Answer: We solve two equations with two unknowns,

    $$e_1 = \frac{a + e_2}{2} \text{ and } e_2 = \frac{a + e_1}{2},$$

    which yield the solution $e_1 = e_2 = a$. It is easy to see that it is unique because it is the only point at which these two best response functions cross. ■

11.

12. **Asymmetric Bertrand:** Consider the Bertrand game with $c_1(q_1) = q_1$ and $c_2(q_2) = 2q_2$, demand equal to $p = 100 - q$, and where firms must choose prices in increments of one cent. We have seen in section **??** that one possible Nash equilibrium is $(p_1^*, p_2^*) = (1.99, 2.00)$.

    (a) Show that there are other Nash equilibria for this game.

    Answer: Another Nash equilibrium is $(p_1', p_2') = (1.50, 1.51)$.In this equilibrium firm 1 fulfills market demand at a price of 1.50 and has no incentive to change the price in either direction. Firm 2 is indifferent between the current price and any higher price, and strictly prefers it to lower prices. ■

    (b) How many Nash equilibria does this game have?

    Answer: There are 100 Nash equilibria of this game starting with $(p_1, p_2) = (1.00, 1.01)$ and going all the way up with one-cent increases to $(p_1^*, p_2^*) = (1.99, 2.00)$. The same logic explains why each of these is a Nash equilibrium. ■

13.

14. **Negative Ad Campaigns:** Each one of two political parties can choose to buy time on commercial radio shows to broadcast negative ad campaigns against their rival. These choices are made simultaneously. Due to government regulation it is forbidden to buy more than 2 hours of negative campaign time so that each party cannot choose an amount of negative campaigning above 2 hours. Given a pair of choices (a_1, a_2), the payoff of party i is given by the following function: $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$.

    (a) What is the normal form representation of this game?

    Answer: Two players $N = \{1, 2\}$, for each player the strategy space is $S_i = [0, 2]$ and the payoff of player i is given by $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$. ■

(b) What is the best response function for each party?

Answer: Each player maximizes $v_i(a_1, a_2)$ resulting in the first order optimality condition $1 + a_j - 2a_i = 0$ resulting in the best response function,

$$a_i(a_j) = \frac{1 + a_j}{2} .$$

■

(c) What is the pure strategy Nash equilibrium? is it unique?

Answer: Solving the two best response functions simultaneously,

$$a_1 = \frac{1 + a_2}{2} \text{ and } a_2 = \frac{1 + a_1}{2}$$

yields the Nash equilibrium $a_1 = a_2 = 1$, and this is the unique solution to these equations implying that this is the unique equilibrium. ■

(d) If the parties could sign a binding agreement on how much to campaign, what levels would they choose?

Answer: Both parties would be better off if they can choose not to spend money on negative campaigns. The payoffs for each player from the Nash equilibrium solved in part (c) are $v_i(1, 1) = -1$ while of they agreed not to spend anything they each would obtain zero. This is a variant of the Prisoners' Dilemma. ■

15.

16. **Hotelling's Price Competition:** Imagine a continuum of potential buyers, located on the line segment $[0, 1]$, with uniform distribution. (Hence, the "mass" or quantity of buyers in the interval $[a, b]$ is equal to $b - a$.) Imagine two firms, players 1 and 2 who are located at each end of the interval (player 1 at the 0 point and player 2 at the 1 point.) Each player i can choose its price p_i, and each customer goes to the vendor who offers them the highest value. However, price alone does not determine the value, but distance is important as well. In particular, each buyer who buys the product from player i has

a net value of $v - p_i - d_i$ where d_i is the distance between the buyer and vendor i, and represents the transportation costs of buying from vendor i Thus, buyer $a \in [0,1]$ buys from 1 and not 2 if $v - p_1 - d_1 > v - p_2 - d_2$, and if buying is better than getting zero. (Here $d_1 = a$ and $d_2 = 1 - a$. The buying choice would be reversed if the inequality is reversed.) Finally, assume that the cost of production is zero.

(a) Assume that v is very large so that all the customers will be served by at least one firm, and that some customer $x^* \in [0,1]$ is indifferent between the two firms. What is the best response function of each player?

Answer: Because customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, his indifference implies that

$$v - p_1 - x^* = v - p_2 - (1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1 + p_2 - p_1}{2} \; .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1 + p_2 - p_1}{2} \; , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1 + p_1 - p_2}{2} \; . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1 + p_2 - p_1}{2} \right) p_1$$

which yields the first order condition

$$1 + p_2 - 2p_1 = 0 \; ,$$

implying the best response function

$$p_1 = \frac{1}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{2} + \frac{p_1}{2}.$$

■

(b) Assume that $v = 1$. What is the Nash equilibrium? Is it unique?

Answer: If we use the best response functions calculated in part (a) above then we obtain a unique Nash equilibrium $p_1 = p_2 = 1$, and this implies that $x^* = \frac{1}{2}$ so that each firm gets half the market. However, when $v = 1$ then the utility of customer $x^* = \frac{1}{2}$ is $v - p_1 - \frac{1}{2} = 1 - 1 - \frac{1}{2} = -\frac{1}{2}$, implying that he would prefer not to buy, and by continuity, an interval of customers around x^* would also prefer not to buy. his violated the assumptions we used to calculate the best response functions.[2] So, the analysis in part (a) is invalid when $v = 1$. It is therefore useful to start with the monopoly case when $v = 1$ and see how each firm would have priced if the other is absent. Firm 1 maximizes

$$\max_{p_1} \; (1 - p_1)p_1$$

which yields the solution $p_1 = \frac{1}{2}$ so that everyone in the interval $x \in [0, \frac{1}{2}]$ wished to buy from firm 1 and no other customer would buy. By symmetry, if firm 2 were a monopoly then the solution would be $p_2 = \frac{1}{2}$ so that everyone in the interval $x \in [\frac{1}{2}, 1]$ would buy from firm 2 and no other customer would buy. But this implies that if both firms set their monopoly prices $p_1 = p_2 = \frac{1}{2}$ then each would maximize profits ignoring the other firm, and hence this is the (trivially) unique Nash equilibrium.

■

(c) Now assume that $v = 1$ and that the transportation costs are $\frac{1}{2}d_i$, so that a buyer buys from 1 if and only if $v - p_1 - \frac{1}{2}d_1 > v - p_2 - \frac{1}{2}d_2$ Write the best response function of each player and solve for the Nash

[2] We need $v \geq 1.5$ for customer $x^* = \frac{1}{2}$ to be just indifferent between buying and not buying when $p_1 = p_2 = 1$ All the other customers will strictly prefer buying.

Equilibrium.

Answer: Like in part (a), assume that customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, and he is indifferent between buying from either, so his indifference implies that

$$v - p_1 - \frac{1}{2}x^* = v - p_2 - \frac{1}{2}(1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + p_2 - p_1 \ .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1}{2} + p_2 - p_1 \ , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1}{2} + p_1 - p_2 \ . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + p_2 - p_1 \right) p_1$$

which yields the first order condition

$$\frac{1}{2} + p_2 - 2p_1 = 0 \ ,$$

implying the best response function

$$p_1 = \frac{1}{4} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{4} + \frac{p_1}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \frac{1}{2}$, and

$x^* = \frac{1}{2}$. To verify that this is a Nash equilibrium notice that for customer x^*, the utility form buying from firm 1 is $v - p_1 - \frac{1}{2} = 1 - \frac{1}{2} - \frac{1}{2} = 0$ implying that he is indeed indifferent between buying or not, which in turn implies that every other customer prefer buying over not buying. ■

(d) Following your analysis in (c) above, imagine that transportation costs are αd_i, with $\alpha \in [0, \frac{1}{2}]$. What happens to the Nash equilibrium as $\alpha \to 0$? What is the intuition for this result?

Answer: Using the assumed indifferent customer x^*, his indifference implies that

$$v - p_1 - \alpha x^* = v - p_2 - \alpha(1 - x^*)$$

$$v - p_1 - \alpha x = v - p_2 - \alpha(1 - x)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \ .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \ , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_1 - p_2) \ . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) \right) p_1$$

which yields the first order condition

$$\frac{1}{2} + \frac{p_2}{2\alpha} - \frac{p_1}{\alpha} = 0 \ ,$$

implying the best response function

$$p_1 = \frac{\alpha}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{\alpha}{2} + \frac{p_1}{2}.$$

$$p_2 = \frac{\alpha}{2} + \frac{\frac{\alpha}{2} + \frac{p_2}{2}}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \alpha$, and $x^* = \frac{1}{2}$. From the analysis in (c) above we know that for any $\alpha \in [0, \frac{1}{2})$ customer x^* will strictly prefer to buy over not buying and so will every other customer. We see that as α decreases, so do the equilibrium prices, so that at the limit of $\alpha = 0$ the prices will be zero. The intuition is that the transportation costs d cause firms 1 and 2 to be differentiated, and this "softens" the Bertrand competition between the two firms. When the transportation costs are higher this implies that competition is less fierce and prices are higher, and the opposite holds for lower transportation costs.■

17.

18. **Political Campaigning:** Two candidates are competing in a political race. Each candidate i can spend $s_i \geq 0$ on adds that reach out to voters, which in turn increases the probability that candidate i wins the race. Given a pair of spending choices (s_1, s_2), the probability that candidate i wins is given by $\frac{s_i}{s_1+s_2}$. If neither spends any resources then each wins with probability $\frac{1}{2}$ Each candidate values winning at a payoff of $v > 0$, and the cost of spending s_i is just s_i.

    (a) Given two spend levels (s_1, s_2), write the expected payoff of a candidate i.

    Answer: Player i's payoff function is

    $$v_i(s_1, s_2) = \frac{s_i v}{s_1 + s_2} - s_i \ .$$

    ■

(b) What is the function that represents each player's best response function?

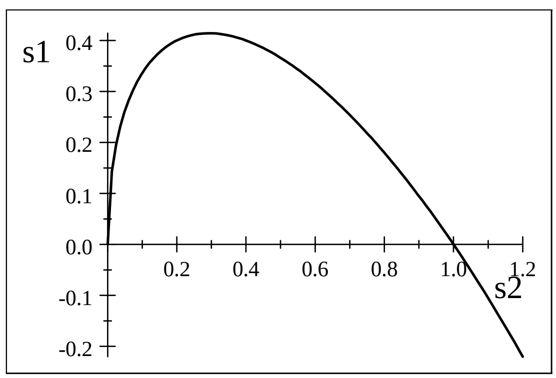
Answer: Player 1 maximizes his payoff $v_1(s_1, s_2)$ shown in (a) above and the first order optimality condition is,

$$\frac{v(s_1+s_2)-s_1v}{(s_1+s_2)^2}-1=0$$

and if we use $s_1(s_2)$ to denote player 1's best response function then it explicitly solves the following equality that is derived from the first-order condition,

$$[s_1(s_2)]^2+2s_1(s_2)s_2+(s_2)^2-vs_2=0\ .$$

Because this is a quadratic equation we cannot write an explicit best response function (or correspondence). However, if we can graph $s_1(s_2)$ as shown in the following figure (the values correspond for the case of $v=1$).



Similarly we can derive the symmetric function for player 2. ■

(c) Find the unique Nash equilibrium.

Answer: The best response functions are symmetric mirror images and have a symmetric solution where $s_1 = s_2$ in the unique Nash equilibrium. We can therefore use any one of the two best response functions and replace both variables with a single variable s,

$$s^2+2s^2+s^2-vs=0\ ,$$

or,

$$s = \frac{v}{4}$$

so that the unique Nash equilibrium has $s_1^* = s_2^* = \frac{v}{4}$.■

(d) What happens to the Nash equilibrium spending levels if v increases?

Answer: It is easy to see from part (c) that higher values of v cause the players to spend more in equilibrium. As the stakes of the prize rise, it is more valuable to fight over it. ■

(e) What happens to the Nash equilibrium levels if player 1 still values winning at v, but player 2 values winning at kv where $k > 1$?

Answer: Now the two best response functions are not symmetric. The best response function of player 1 remains as above, but that of player 2 will now have kv instead of v,

$$(s_1)^2 + 2s_1s_2 + (s_2)^2 - vs_2 = 0 \ . \qquad \text{((BR1))}$$

and

$$(s_2)^2 + 2s_1s_2 + (s_1)^2 - kvs_1 = 0 \ . \qquad \text{((BR2))}$$

Subtracting (BR2) from (BR1) we obtain,

$$ks_1 = s_2,$$

which implies that the solution will no longer be symmetric and, moreover, $s_2 > s_1$, which is intuitive because now player 2 cares more about the prize. Using $ks_1 = s_2$ we substitute for s_2 in (BR1) to obtain,

$$(s_1)^2 + 2k(s_1)^2 + k^2(s_1)^2 - kvs_1 = 0$$

which results in,

$$s_1 = \frac{kv}{1 + 2k + k^2} < \frac{v}{1 + 2k + k^2} < \frac{v}{4}$$

where both inequalities follow from the fact that $k > 1$. From $ks_1 = s_2$ above we have

$$s_2 = \frac{k^2 v}{1 + 2k + k^2} > \frac{k^2 v}{k^2 + 2k^2 + k^2} = \frac{v}{4}$$

where the inequality follows from $k > 1$. ■

6

Mixed Strategies

1.

2. Let σ_i be a mixed strategy of player i that puts positive weight on one strictly dominated pure strategy. Show that there exists a mixed strategy σ'_i that puts no weight on any dominated pure strategy and that dominates σ_i.

   Answer: Let player i have L pure strategies $S_i = \{s_{i1}, s_{i2}, ..., s_{iL}\}$ and let s_{ik} be a pure strategy which is strictly dominated by $s_{ik'}$, that is, $v_i(s_{ik'}, s_{-i}) >$ $v_i(s_{ik'}, s_{-i})$ for any strategy profile of i's opponents s_{-i}. Let $\sigma_i = (\sigma_{i1}, \sigma_{i2}, ..., \sigma$ be a mixed strategy that puts some positive weight $\sigma_{ik} > 0$ on s_{ik} and let σ be identical to σ_i except that it puts weight 0 on s_{ik} and diverts that weight over to $s_{ik'}$. That is, $\sigma'_{ik} = 0$ and $\sigma'_{ik'} = \sigma_{ik'} + \sigma_{ik}$, and $\sigma'_{il} = \sigma_{il}$ for all $l \neq k$ and $l \neq k'$. It follows that for all s_{-i},

   $$v_i(\sigma'_i, s_{-i}) = \sum_{l=1}^{L} \sigma'_{il} v_i(s_{il}, s_{-i}) > \sum_{l=1}^{L} \sigma_{il} v_i(s_{il}, s_{-i}) = v_i(\sigma'_i, s_{-i})$$

   because $v_i(s_{ik'}, s_{-i}) > v_i(s_{ik'}, s_{-i})$ and the way in which σ'_i was constructed. Hence, σ_i is strictly dominated by σ'_i. ■

3.

4. **Monitoring:** An employee (player 1) who works for a boss (player 2) can either work (W) or shirk (S), while his boss can either monitor the employee (M) or ignore him (I). Like most employee-boss relationships, if the employee is working then the boss prefers not to monitor, but if the boss is not monitoring then the employee prefers to shirk. The game is represented in the following matrix:

player 1 \ Player 2	M	I
W	$1,1$	$1,2$
S	$0,2$	$2,1$

(a) Draw the best response functions of each player.

Answer: Let p be the probability that player 1 chooses W and q the probability that player 2 chooses M. It follows that $v_1(W,q) > v_1(S,q)$ if and only if $1 > 2(1-q)$, or $q > \frac{1}{2}$, and $v_2(p,M) > v_2(p,I)$ if and only if $p + 2(1-p) > 2p + (1-p)$, or $p < \frac{1}{2}$. It follows that for player 1,

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{2} \\ p \in [0,1] & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

and for player 2,

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{2} \\ q \in [0,1] & \text{if } p = \frac{1}{2} \\ q = 0 & \text{if } p > \frac{1}{2} \end{cases} .$$

Notice that these are identical to the best response functions for the matching pennies game (see Figure 6.3). ■

(b) Find the Nash equilibrium of this game. What kind of game does this game remind you of?

Answer: From the two best response correspondences the unique Nash equilibrium is $(p,q) = (\frac{1}{2}, \frac{1}{2})$ and the game's strategic forces are identical to those in the Matching Pennies game. ■

5.

6. **Declining Industry:** Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices as it must decide whether or not to exit the industry immediately, at the end of this quarter, or at the end of the next quarter. If a firm chooses to exit then its payoff is 0 from that point onward. Every quarter that both firms operate yields each a loss equal to -1, and each quarter that a firm operates alone yields a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter then the payoffs are $(-1, 1)$ because both firms lose -1 in the first quarter and firm 2 gains 2 in the second. The payoff for each firm is the sum of its quarterly payoffs.

   (a) Write down this game in matrix form.

   Answer: Let E denote immediate exit, T denote exit this quarter, and N denote exit next quarter.

		Player 2		
		E	T	N
Player 1	E	$0,0$	$0,2$	$0,4$
	T	$2,0$	$-1,-1$	$-1,1$
	N	$4,0$	$1,-1$	$-2,-2$

   (b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

   Answer: There are no strictly dominated strategies but there is a weakly dominated one: T. To see this note that choosing both E and N with probability $\frac{1}{2}$ each yields the same expected payoff as choosing T against E or N, and a higher expected payoff against T, and hence $\sigma_i = (\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{2}, 0, \frac{1}{2})$ weakly dominates T. The reason there is no strictly dominated strategy is that, starting with σ_i increasing the weight on E causes the mixed strategy to be worse than

T against E, while increasing the weight on N causes the mixed strategy to be worse than T against N, implying it is impossible to find a mixed strategy that strictly dominates T. ■

(c) Find the pure strategy Nash equilibria.

Answer: Because T is weakly dominated, it is suspect of never being a best response. A quick observation should convince you that this is indeed the case: it is never a best response to any of the pure strategies, and hence cannot be part of a pure strategy Nash equilibrium. Removing T from consideration results in the reduced game:

Player 1 \ Player 2	E	N
E	$0, 0$	$0, 4$
N	$4, 0$	$-2, -2$

for which there are two pure strategy Nash equilibria, (E, N) and (N, E) ■

(d) Find the unique mixed strategy Nash equilibrium (hint: you can use your answer to (b) to make things easier.)

Answer: We start by ignoring T and using the reduced game in part (c) by assuming that the weakly dominated strategy T will never be part of a Nash equilibrium. We need to find a pair of mixed strategies, $(\sigma_1(E), \sigma_1(N))$ and $(\sigma_2(E), \sigma_2(N))$ that make both players indifferent between E and N. For player 1 the indifference equation is,

$$0 = 4\sigma_2(E) - 2(1 - \sigma_2(E))$$

which results in $\sigma_2(E) = \frac{1}{3}$, and for player 2 the indifference equation is symmetric, resulting in $\sigma_1(E) = \frac{1}{3}$. Hence, the mixed strategy Nash equilibrium of the original game is $(\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{3}, 0, \frac{2}{3})$ Notice that at this Nash equilibrium, each player is not only indifferent between E and N, but choosing T gives the same expected payoff of zero. However, choosing T with positive probability cannot be part of a mixed strategy Nash equilibrium. To prove this let player 2 play the mixed strategy $\sigma_2 = (\sigma_2(E), \sigma_2(T), \sigma_2(N)) = (\sigma_{2E}, \sigma_{2T}, 1 - \sigma_{2E} - \sigma_{2T})$ The strategy T for player 1 is at least as good as E if and only if,

$$0 \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2E} \geq \frac{1}{3}$. The strategy T for player 1 is at least as good as N if and only if,

$$4\sigma_{2E} - \sigma_{2T} - 2(1 - \sigma_{2E} - \sigma_{2T}) \leq 2\sigma_{2E} - \sigma_{2T} - (1 - \sigma_{2E} - \sigma_{2T})$$

or, $\sigma_{2T} \leq 1 - 3\sigma_{2E}$. But if $\sigma_{2E} \geq \frac{1}{3}$ (when T is as good as E) then $\sigma_{2T} \leq 1 - 3\sigma_{2E}$ reduces to $\sigma_{2T} \leq 0$, which can only hold when $\sigma_{2E} = \frac{1}{3}$ and $\sigma_{2T} = 0$ (which is the Nash equilibrium we found above). A symmetric argument holds to conclude that $(\sigma_i(E), \sigma_i(T), \sigma_i(N)) = (\frac{1}{3}, 0, \frac{2}{3})$ is the unique mixed strategy Nash equilibrium. ■

7.

8. **Market entry:** There are 3 firms that are considering entering a new market. The payoff for each firm that enters is $\frac{150}{n}$ where n is the number of firms that enter. The cost of entering is 62.

   (a) Find all the pure strategy Nash equilibria.

   Answer: The costs of entry are 62 so the benefits of entry must be at least that for a firm to choose to enter. Clearly, if a firm believes the other two are not entering then it wants to enter, and if it believes that the other firms are entering then it would stay out (it would only get 50). If a firm believes that only one other firm is entering then it prefers to enter and get 75. Hence, there are three pure strategy Nash equilibria in which two of the three firms enter and one stays out. ■

   (b) Find the symmetric mixed strategy equilibrium where all three players enter with the same probability.

   Answer: Let p be the probability that a firm enters. In order to be willing to mix the expected payoff of entering must be equal to zero. If a firm enters then with probability p^2 it will face two other entrants and receive $v_i = 50 - 62 = -12$, with probability $(1-p)^2$ it will face no other entrants and receive $v_i = 150 - 62 = 88$, and with probability $2(1-p)p$ it will face one other entrant and receive $v_i = 75 - 62 = 13$ Hence, to be willing to mix the expected payoff must be zero,$p^2 + 1 - p^2$

$$(1-p)^2 88 + 2(1-p)p13 - p^2 12 = 0$$

which results in the quadratic equation $25p^2 - 75p + 44 = 0$, and the relevant solution (between 0 and 1) is $p = \frac{4}{5}$. ■

9.

10. **Continuous all pay auction:** Consider an all-pay auction for a good worth 1 to each of the two bidders. Each bidder can choose to offer a bid from the unit interval so that $S_i = [0, 1]$. Players only care about the expected value they will end up with at the end of the game (i.e., if a player bids 0.4 and expects to win with probability 0.7 then his payoff is $0.7 \times 1 - 0.4$).

    (a) Model this auction as a normal-form game.

    Answer: There are two players, $N = \{1, 2\}$, each has a strategy set $S_i = [0, 1]$, and assuming that the players are equally likely to get the good in case of a tie, the payoff to player i is given by

$$v_i(s_i, s_j) = \begin{cases} 1 - s_i & \text{if } s_i > s_j \\ \frac{1}{2} - s_i & \text{if } s_i = s_j \\ -s_i & \text{if } s_i < s_j \end{cases}$$

    (b) Show that this game has no pure strategy Nash Equilibrium.

    Answer: First, it cannot be the case that $s_i = s_j < 1$ because then each player would benefit from raising his bid by a tiny amount ε in order to win the auction and receive a higher payoff $1 - \varepsilon - s_i > \frac{1}{2} - s_i$. Second, it cannot be the case that $s_i = s_j = 1$ because each player would prefer to bid nothing and receive $0 > -\frac{1}{2}$. Last, it cannot be the case that $s_i > s_j \geq 0$ because then player i would prefer to lower his bid by ε while still beating player j and paying less money. Hence, there cannot be a pure strategy Nash equilibrium. ■

(c) Show that this game cannot have a Nash Equilibrium in which each player is randomizing over a finite number of bids.

Answer: Assume that a Nash equilibrium involves player 1 mixing between a finite number of bids, $\{s_{11}, s_{12,\ldots,} s_{1K}\}$ where $s_{11} \geq 0$ is the lowest bid, $s_{1K} \leq 1$ is the highest, $s_{1k} < s_{1(k+1)}$ and each bid s_{1k} is being played with some positive probability σ_{1k}. Similarly assume that player 2 is mixing between a finite number of bids, $\{s_{21}, s_{22,\ldots,} s_{2L}\}$ and each bid s_{2l} is being played with some positive probability σ_{2l}. (i) First observe that it cannot be true that $s_{1K} < s_{2L}$ (or the reverse by symmetry). If it were the case then player 2 will win for sure when he bids s_{2L} and pay his bid, while if he reduces his bid by some ε such that $s_{1K} < s_{2L} - \varepsilon$ then he will still win for sure and pay less, contradicting that playing s_{2L} was part of a Nash equilibrium. (ii) Second observe that when $s_{1K} = s_{2L}$ then the expected payoff of player 2 from bidding s_{2L} is

$$
\begin{aligned}
Ev_2 &= \Pr\{s_{1k} < s_{2L}\}(1 - s_{2L}) + \Pr\{s_{1k} = s_{2L}\}(\frac{1}{2} - s_{2L}) \\
&= (1 - \sigma_{1K})(1 - s_{2L}) + \sigma_{1K}(\frac{1}{2} - s_{2L}) \\
&= 1 - s_{2L} - \frac{1}{2}\sigma_K \geq 0
\end{aligned}
$$

where the last inequality follows from the fact that $\sigma_{2L} > 0$ (he would not play it with positive probability if the expected payoff were negative.) Let $s'_{2L} = s_{2L} + \varepsilon$ where $\varepsilon = \frac{1}{4}\sigma_K$. If instead of bidding s_{2L} player 2 bids s'_{2L} then he wins for sure and his utility is

$$
v_2 = 1 - s'_{2L} = 1 - s_{2L} - \frac{1}{4}\sigma_K > 1 - s_{2L} - \frac{1}{2}\sigma_K
$$

contradicting that playing s_{2L} was part of a Nash equilibrium. ■

(d) Consider mixed strategies of the following form: Each player i chooses and interval, $[\underline{x}_i, \overline{x}_i]$ with $0 \leq \underline{x}_i < \overline{x}_i \leq 1$ together with a cumulative distribution $F_i(x)$ over the interval $[\underline{x}_i, \overline{x}_i]$. (Alternatively you can think of each player choosing $F_i(x)$ over the interval $[0, 1]$, with two values $\underline{x}$ and $\overline{x}_i$ such that $F_i(\underline{x}_i) = 0$ and $F_i(\overline{x}_i) = 1$.)

i. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\underline{x}_1 = \underline{x}_2$ and $\overline{x}_1 = \overline{x}_2$.

Answer: Assume not. There are two cases: (a) $\underline{x}_1 \neq \underline{x}_2$: Without loss assume that $\underline{x}_1 < \underline{x}_2$. This means that there are values of $s_1' \in (\underline{x}_1, \underline{x}_2)$ for which $s_1' > 0$ but for which player 1 loses with probability 1. This implies that the expected payoff from this bid is negative, and player would be better off bidding 0 instead. Hence, $\underline{x}_1 = \underline{x}_2$ must hold. (b) $\overline{x}_1 \neq \overline{x}_2$: Without loss assume that $\overline{x}_1 < \overline{x}_2$. This means that there are values of $s_2' \in (\overline{x}_1, \overline{x}_2)$ for which $\overline{x}_1 < s_2' < 1$ but for which player 2 wins with probability 1. But then player 2 could replace s_2' with $s_2'' = s_2' - \varepsilon$ with ε small enough such that $\overline{x}_1 < s_2'' < s_2' < 1$, he will win with probability 1 and pay less than he would pay with s_2'. Hence, $\overline{x}_1 = \overline{x}_2$ must hold. ■

ii. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\underline{x}_1 = \underline{x}_2 = 0$.

Answer: Assume not so that $\underline{x}_1 = \underline{x}_2 = \underline{x} > 0$. This means that when player i bids $\underline{x}$ then he loses with probability 1, and get an expected payoff of $-\underline{x} < 0$. But instead of bidding $\underline{x}$ player i can bid 0 and receive 0 which is better than $-\underline{x}$, implying that $\underline{x}_1 = \underline{x}_2 = \underline{x} > 0$ cannot be an equilibrium. ■

iii. Using the above, argue that if two such strategies are a mixed strategy Nash equilibrium then both players must be getting an expected payoff of zero.

Answer: As proposition 6.1 states, if a player is randomizing between two alternatives then he must be indifferent between them. Because both players are including 0 in the support of their mixed strategy, their payoff from 0 is zero, and hence their expected payoff from any choice in equilibrium must be zero. ■

iv. Show that if two such strategies are a mixed strategy Nash equilibrium then it must be that $\overline{x}_1 = \overline{x}_2 = 1$.

Answer: Assume not so that $\overline{x}_1 = \overline{x}_2 = \overline{x} < 1$. From (iii) above the expected payoff from any bid in $[0, \overline{x}]$ is equal to zero. If one of the players deviates from this strategy and choose to bid $\overline{x}+\varepsilon < 1$ then he will win with probability 1 and receive a payoff of $1-(\overline{x}+\varepsilon) > 0$ contradicting that $\overline{x}_1 = \overline{x}_2 = \overline{x} < 1$ is an equilibrium. ■

v. Show that $F_i(x)$ being uniform over $[0, 1]$ is a symmetric Nash equilibrium of this game.

Answer: Imagine that player 2 is playing according to the proposed strategy $F_2(x)$ uniform over $[0, 1]$. If player 1 bids some value $s_1 \in [0, 1]$ then his expected payoff is

$$\Pr\{s_1 > s_2\}(1-s_1)+\Pr\{s_1 < s_2\}(-s_1) = s_1(1-s_1)+(1-s_1)(-s_1) = 0$$

implying that player 1 is willing to bid any value in the $[0, 1]$ interval, and in particular, choosing a bid according to $F_1(x)$ uniform over $[0, 1]$. Hence, this is a symmetric Nash equilibrium. ■

11.

12. **The Tax Man:** A citizen (player 1) must choose whether or not to file taxes honestly or whether to cheat. The tax man (player 2) decides how much effort to invest in auditing and can choose $a \in [0, 1]$, and the cost to the tax man of investing at a level a is $c(a) = 100a^2$. If the citizen is honest then he receives the benchmark payoff of 0, and the tax man pays the auditing costs without any benefit from the audit, yielding him a payoff of $(-100a^2)$. If the citizen cheats then his payoff depends on whether he is caught. If he is caught then his payoff is (-100) and the tax man's payoff is $100 - 100a^2$. If he is not caught then his payoff is 50 while the tax man's payoff is $(-100a^2)$. If the

citizen cheats and the tax man chooses to audit at level a then the citizen is caught with probability a and is not caught with probability $(1-a)$.

(a) If the tax man believes that the citizen is cheating for sure, what is his best response level of a?

Answer: The tax man maximizes $a(100-100a^2)+(1-a)(0-100a^2) = 100a-100a^2$. The first-order optimality condition is $100-200a=0$ yielding $a=\frac{1}{2}$. ■

(b) If the tax man believes that the citizen is honest for sure, what is his best response level of a?

Answer: The tax man maximizes $-100a^2$ which is maximized at $a=0$ ■

(c) If the tax man believes that the citizen is honest with probability p what is his best response level of a as a function of p?

Answer: The tax man maximizes $p(-100a^2)+(1-p)(100a-100a^2) = 100(1-p)a-100a^2$. The first-order optimality condition is $100(1-p)-200a=0$, yielding the best response function $a^*(p)=\frac{1-p}{2}$. ■

(d) Is there a pure strategy Nash equilibrium of this game? Why or why not?

Answer: There is no pure strategy Nash equilibrium. To see this, consider the best response of player 1 who believes that player 2 chooses some level $a \in [0,1]$. His payoff from being honest is 0 while his payoff from cheating is $a(-100)+(1-a)50 = 50-150a$. Hence, he prefers to be honest if and only if $0 \geq 50-150a$, or $a \geq \frac{1}{3}$. Letting $p^*(a)$ denote the best response correspondence of player 1 as the probability that he is honest, we have that

$$p^*(a) = \begin{cases} 1 & \text{if } a > \frac{1}{3} \\ [0,1] & \text{if } a = \frac{1}{3} \\ 0 & \text{if } a < \frac{1}{3} \end{cases}$$

and it is easy to see that there are no values of a and p for which both players are playing mutual best responses. ■

(e) Is there a mixed strategy Nash equilibrium of this game? Why or why not?

Answer: From (d) above we know that player 1 is willing to mix if and only if $a = \frac{1}{3}$, which must therefore hold true in a mixed strategy Nash equilibrium. For player 2 to be willing to play $a = \frac{1}{3}$ we use his best response from part (c), $\frac{1}{3} = \frac{1-p}{2}$, which yields, $p = \frac{1}{3}$. Hence, the unique mixed strategy Nash equilibrium has player 1 being honest with probability $\frac{1}{3}$ and player 2 choosing $a = \frac{1}{3}$. ■

Part III

Dynamic Games of Complete Information

7

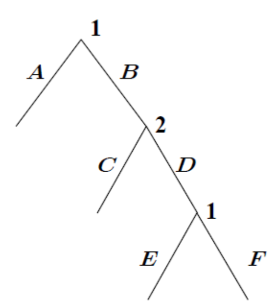
Preliminaries

1.

2. **Strategies and equilibrium:** Consider a two player game in which player 1 can choose A or B. The game ends if he chooses A while it continues to player 2 if he chooses B. Player 2 can then choose C or D, with he game ending after C and continuing again with player 1 after D. Player 1 then can choose E or F, and the game ends after each of these choices.

   (a) Model this as an extensive form game tree. Is it a game of perfect or imperfect information?

   Answer:



   This game is a game of perfect information. ■

(b) How many terminal nodes does the game have? How many information sets?

Answer: The game has 4 terminal nodes (after choices A, C, E and F) and 3 information sets (one for each player. ■

(c) How many pure strategies does each player have?

Answer: Player 1 has 4 pure strategies and player 2 has 2. ■

(d) Imagine that the payoffs following choice A by player 1 are $(2,0)$, following C by player 2 are $(3,1)$, following E by player 1 are $(0,0)$ and following F by player 1 are $(1,2)$. What are the Nash equilibria of this game? Does one strike you as more "appealing" than the other? If so, explain why.

Answer: We can write down the matrix form of this game as follows (xy denotes a strategy for player 1 where $x \in \{A, B\}$ is what he does in his first information set and $y \in \{E, F\}$ in his second one),

Player 1 \ Player 2	C	D
AE	$\overline{2,0}$	$\underline{\overline{2,0}}$
AF	$\overline{2,0}$	$\underline{\overline{2,0}}$
BE	$\underline{\overline{3,1}}$	$0,0$
BF	$\underline{3,1}$	$\overline{1,2}$

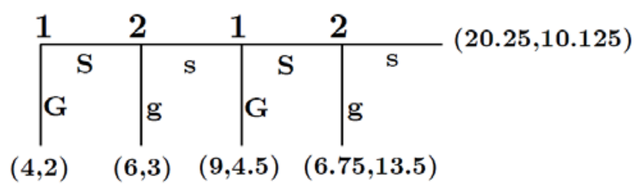
It's easy to see that there are three pure strategy Nash equilibria: $(AE, D), (AF, D)$ and (BE, C). The equilibria $(AE, D), (AF, D)$ are Pareto dominated by the equilibrium (BE, C), and hence it would be tempting to argue that (BE, C) is the more "appealing" equilibrium. As we will see in Chapter 8 it is actually (AF, D) that has properties that are more appealing (sequential rationality). ■

3.

4. **Centipedes**: Imagine a two player game that proceeds as follows. A pot of money is created with \$6 in it initially. Player 1 moves first, then player 2, then player 1 again and finally player 2 again. At each player's turn to move, he has two possible actions: grab (G) or share (S). If he grabs, he gets $\frac{2}{3}$ of the current pot of money, the other player gets $\frac{1}{3}$ of the pot and the game ends. If he shares then the size of the current pot is multiplied by $\frac{3}{2}$ and the next player gets to move. At the last stage in which player 2 moves, if he chooses share then the pot is still multiplied by $\frac{3}{2}$, player 2 gets $\frac{1}{3}$ of the pot and player 1 gets $\frac{2}{3}$ of the pot.

   (a) Model this as an extensive form game tree. Is it a game of perfect or imperfect information?

   Answer:



   This is a game of perfect information. Note that we draw the game from left to right (which is the common convention for "centipede games" of this sort.) We use capital letters for player 1 and lower case for player 2. ■

   (b) How many terminal nodes does the game have? How many information sets?

   Answer: The game has five terminal nodes and four information sets. ■

   (c) How many pure strategies does each player have?

   Answer: Each player has four pure strategies (2 actions in each of his 2 information sets). ■

   (d) Find the Nash equilibria of this game. How many outcomes can be supported in equilibrium?

Answer: Using the convention of xy to denote a strategy of player where he chooses x in his first information set and y in his second, we can draw the following matrix representation of this game,

Player 2

Player 1	gg	gs	sg	ss
GG	$\overline{\underline{4,2}}$	$\overline{\underline{4,2}}$	$\overline{4,2}$	$\overline{4,2}$
GS	$\overline{\underline{4,2}}$	$\overline{\underline{4,2}}$	$\overline{4,2}$	$\overline{4,2}$
SG	$\overline{3,6}$	$\overline{3,6}$	$\underline{9,4.5}$	$9,4.5$
SS	$3,6$	$3,6$	$\overline{6.75,13.5}$	$\underline{20.25,10.125}$

We see that only one outcome can be supported as a Nash equilibrium: player 1 grabs immediately and the players' payoffs are $(4,2)$. ■

(e) Now imagine that at the last stage in which player 2 moves, if he chooses to share then the pot is equally split among the players. Does your answer to part (d) above change?

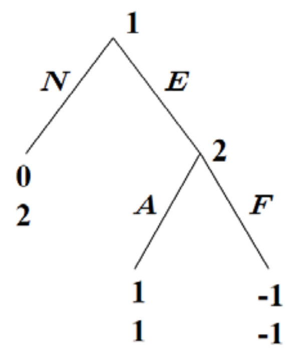
Answer: The answer does change because the payoffs from the pair of strategies (SS, ss) changes from $(20.25, 10.125)$ to $(15.1875, 15.1875)$ in which case player 2's best response to SS will be ss, and player 1's best response to ss remains SS, so that (SS, ss) is another Nash equilibrium in which they split 30.375 equally (the previous Nash equilibria are still equilibria). ■

5.

6. **Entering an Industry:** A firm (player 1) is considering entering an established industry with one incumbent firm (player 2). Player 1 must choose whether to enter or to not enter the industry. If player 1 enters the industry then player 2 can either accommodate the entry, or fight the entry with a price war. Player 1's most preferred outcome is entering with player 2 not fighting, and his least preferred outcome is entering with player 2 fighting. Player 2's most preferred outcome is player 1 not entering, and his least preferred outcome is player 1 entering with player 2 fighting.

(a) Model this as an extensive form game tree (choose payoffs that represent the preferences).

Answer:



(b) How many pure strategies does each player have?

Answer: Each player has two pure strategies. ■

(c) Find all the Nash equilibria of this game.

Answer: There are two Nash equilibria which can be seen in the matrix,

		Player 2	
		A	F
Player 1	N	$0, 2$	$0, 2$
	E	$1, 1$	$-1, -1$

Both (N, F) and (E, A) are Nash equilibria of this game. ■

7.

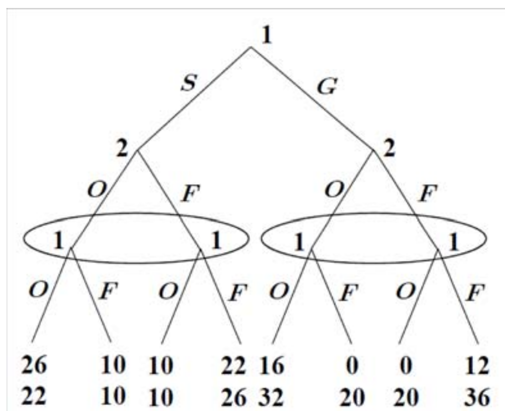
8. **Brothers**: Consider the following game that proceeds in two steps: In the first stage one brother (player 2) has two \$10 bills and can choose one of two options: he can give his younger brother (player 1) \$20, or give him one of the \$10 bills (giving nothing is inconceivable given the way they were raised.) This money will be used to buy snacks at the show they will see, and each

one dollar of snack yields one unit of payoff for a player who uses it. The show they will see is determined by the following "battle of the sexes" game:

Player 1 \ Player 2	O	F
O	16,12	0,0
F	0,0	12,16

(a) Present the *entire* game in extensive form (a game tree).

Answer: Let the choices of player 1 first be S for spliting the \$20 and G for giving it all away. The entire game will have the payoffs from the choice of how to split the money added to the payoffs from the Battle of the Sexes part of the game as follows,



Because the latter is simultaneous, it does not mater which player moves after player 1 as long as the last player cannot distinguish between the choice of the player who moves just before him. ■

(b) Write the (pure) strategy sets for both players.

Answer: Both players can condition their choice in the Battle of the Sexes game on the initial split/give choice of player 1. For player 2, $S_2 = \{OO, OF, FO, FF\}$ where $s_2 = xy$ means that player 2 chooses $x \in \{O, F\}$ after player 1 chose S while player 2 chooses $y \in \{O, F\}$ after player 1 chose G. For player 1, however, even though he chooses first between S or G, he must specify his action for each information set even if he knows it will not happen (e.g., what he will do following S even when he plans to play G). Hence, he has 8 pure strategies, $S_1 = \{SOO, SOF, SFO, SFF, GOO, GOF, GFO, GFF\}$ where $s_1 = xyz$ means that player 1 first chooses $x \in \{S, G\}$ and then chooses $y \in \{O, F\}$ if he played S and $z \in \{O, F\}$ if he played G. ■

(c) Present the *entire* game in one matrix.

Answer: This will be a 8×4 matrix as follows,

Player 2

Player 1		OO	OF	FO	FF
	SOO	$26, 22$	$26, 22$	$10, 10$	$10, 10$
	SOF	$26, 22$	$26, 22$	$10, 10$	$10, 10$
	SFO	$10, 10$	$10, 10$	$22, 26$	$22, 26$
	SFF	$10, 10$	$10, 10$	$22, 26$	$22, 26$
	GOO	$16, 32$	$0, 20$	$16, 32$	$0, 20$
	GOF	$0, 20$	$12, 36$	$0, 20$	$12, 36$
	GFO	$16, 32$	$0, 20$	$16, 32$	$0, 20$
	GFF	$0, 20$	$12, 36$	$0, 20$	$12, 36$

(d) Find the Nash equilibria of the *entire* game (pure and mixed strategies).

Answer: First note that for player 1, mixing equally between SOO and SFO will strictly dominate the four strategies GOO, GOF, GFO and GFF. Hence, we can consider the reduced 4×4 game,

Player 2

Player 1		OO	OF	FO	FF
	SOO	$\overline{\underline{26, 22}}$	$\overline{\underline{26, 22}}$	$10, 10$	$10, 10$
	SOF	$\overline{\underline{26, 22}}$	$\overline{\underline{26, 22}}$	$10, 10$	$10, 10$
	SFO	$10, 10$	$10, 10$	$\overline{\underline{22, 26}}$	$\overline{\underline{22, 26}}$
	SFF	$10, 10$	$10, 10$	$\overline{\underline{22, 26}}$	$\overline{\underline{22, 26}}$

The simple overline-underline method shows that we have eight pure strategy Nash equilibria, four yielding the payoffs $(26, 22)$ and the other four yielding $(22, 26)$. Because of each players indifference between the ways in which the payoffs are reached, there are infinitely many mixed strategies that yield the same payoffs. For example, any profile where player 1 mixes between SOO and SOF and where player 2 mixes between OO and OF will be a Nash equilibrium that yields $(26, 22)$. Similarly, any profile where player 1 mixes between SFO and SFF and where player 2 mixes between FO and FF will be a Nash equilibrium that yields $(22, 26)$. There is, however, one more class of mixed strategy Nash equilibria that are similar to the one found in section 6.2.3. To see this, focus on an even simpler game where we eliminate the duplicate payoffs as follows,

Player 1	Player 2: OO	Player 2: FF
SOO	$\underline{\overline{26}}, \underline{22}$	$10, 10$
SFF	$10, 10$	$\overline{22}, \underline{26}$

which preserve the nature of the game. For player 1 to be indifferent between SOO and SFF it must be that player 2 chooses OO with probability q such that

$$26q + 10(1 - q) = 10q + 22(1 - q)$$

which yields $q = \frac{3}{7}$. Similarly, for player 2 to be indifferent between OO and FF it must be that player 1 chooses SOO with probability p such that

$$22p + 10(1 - p) = 10p + 26(1 - p)$$

which yields $p = \frac{4}{7}$. Hence, we found a mixed strategy Nash equilibrium that results in each player getting an expected payoff of $26 \times \frac{3}{7} + 10 \times \frac{4}{7} = 16\frac{6}{7}$. Notice, however, that player 1 is always indifferent between SOO and SOF, as well as between SFO and SFF so there are infinitely many ways to achieve this kind of mixed strategy, and similarly for player 2 because of his indifference between OO and OF as well as FO and FF. ■

9.

10.

8

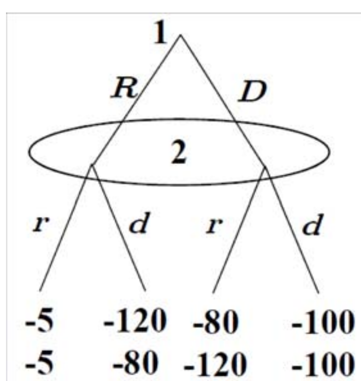
Credibility and Sequential Rationality

1.

2. **Mutually Assured Destruction (revisited):** Consider the game in section **??**.

   (a) Find the mixed strategy equilibrium of the war stage game and argue that it is unique.

   Answer: The war-game in the text has a weakly dominated Nash equilibrium (D, d) and hence does not have an equilibrium in which any player is mixing. This exercise should have replaced the war-stage game with the following game:



The subgame we called the war-stage game is given in the following matrix:

Player 2

Player 1		r	d
	R	$-5, -5$	$-120, -80$
	D	$-80, -120$	$-100, -100$

Let player 1 choose R with probability p and player 2 choose r with probability q. For player 2 to be indifferent it must be that $\frac{4}{19}(-5) + (1 - \frac{4}{19})(-120) = -\frac{1820}{19}$

$$p(-5) + (1 - p)(-120) = p(-80) + (1 - p)(-100)$$

and the solution is $p = \frac{4}{19}$. By symmetry, for player 1 to be indifferent it must be that $q = \frac{4}{19}$. Hence, $(p, q) = (\frac{4}{19}, \frac{4}{19})$ is the unique mixed strategy Nash equilibrium of this subgame with expected payoffs of $(v_1, v_2) = (-95.78, -95.79)$. ■

(b) What is the unique subgame perfect equilibrium that includes the mixed strategy you found above?

Answer: Working backward, player 2 would prefer to choose B over N and player 1 would prefer E over I.

3.

4. **The Industry Leader:** Three oligopolists operate in a market with inverse demand given by $P(Q) = a - Q$, where $Q = q_1 + q_2 + q_3$, and q_i is the quantity produced by firm i. Each firm has a constant marginal cost of production, c and no fixed cost. The firms choose their quantities dynamically as follows: (1) Firm 1, who is the industry leader, chooses $q_1 \geq 0$; (2) Firms 2 and 3 observe q_1 and then simultaneously choose q_2 and q_3 respectively.

   (a) How many proper subgames does this dynamic game have? Explain Briefly.

   Answer: There are infinitely many proper subgames because every quantity choice of payer 1 results in a proper subgame. ■

(b) Is it a game of perfect or imperfect information? Explain Briefly.

Answer: This is a game of imperfect information because players 2 and 3 make their choice without observing each other's choice first. ■

(c) What is the subgame perfect equilibrium of this game? Show that it is unique.

Answer: first we solve for the Nash equilibrium of the simultaneous move stage in which players 2 and 3 make their choices as a function of the choice made first by player 1. Given a choice of q_1 and a belief about q_3, player 2 maximizes

$$\max_{q_2} \ (a - (q_1 + q_2 + q_3) - c)q_2$$

which leads to the first order condition

$$a - q_1 - q_3 - c - 2q_2 = 0$$

yielding the best response function

$$q_2 = \frac{a - q_1 - q_3 - c}{2} \ ,$$

and symmetrically, the best response function of player 3 is

$$q_3 = \frac{a - q_1 - q_2 - c}{2}.$$

Hence, following any choice of q_1 by player 1, the unique Nash equilibrium in the resulting subgame is the solution to the two best response functions, which yields

$$q_2^*(q_1) = q_3^*(q_1) = \frac{a - c - q_1}{3} \ .$$

Moving back to player 1's decision node, he will choose q_1 knowing that q_2 and q_3 be be chosen using the best response function above, and hence player 1 maximizes,

$$\max_{q_1} \ (a - (q_1 + \frac{a - c - q_1}{3} + \frac{a - c - q_1}{3}) - c)q_1$$

which leads to the first order equation

$$\frac{1}{3}(a - c - 2q_1) = 0$$

resulting in a unique solution $q_1 = \frac{a-c}{2}$. Hence, the unique subgame perfect equilibrium dictates that $q_1^* = \frac{a-c}{2}$, and $q_2^*(q_1) = q_3^*(q_1) = \frac{a-c-q_1}{3}$ ■

(d) Find a Nash equilibrium that is not a subgame perfect equilibrium.

Answer: There are infinitely many Nash equilibria of the form "if player 1 plays q_1' then players 2 and 3 play $q_2^*(q_1') = q_3^*(q_1') = \frac{a-c-q_1'}{3}$, and otherwise they play $q_2 = q_3 = a$." In any such Nash equilibrium, players 2 and 3 are playing a Nash equilibrium on the equilibrium path (following q_1') while they are flooding the market and casing the price to be zero off the equilibrium path. One example would be $q_1' = 0$. In this case, following $q_1' = 0$ the remaining two players play the duopoly Nash equilibrium, and player 1 gets zero profits. If player 1 were to choose any positive quantity, his belief is that players 2 and 3 will flood the market and he will earn $-cq_1 < 0$, so he would prefer to choose $q_1' = 0$ given those beliefs. Of course, the threats of players 2 and 3 are not sequentially rational, which is the reason that this Nash equilibrium is not a subgame perfect equilibrium. ■

5.

6. **Investment in the Future:** Consider two firms that play a Cournot competition game with demand $p = 100 - q$, and costs for each firm given by $c_i(q_i) = 10q_i$. Imagine that before the two firms play the Cournot game, firm 1 can invest in cost reduction. If it invests, the costs of firm 1 will drop to $c_1(q_1) = 5q_1$. The cost of investment is $F > 0$. Firm 2 does not have this investment opportunity.

   (a) Find the value F^* for which the unique subgame perfect equilibrium involves firm 1 investing.

Answer: If firm 1 does not invest then they are expected to play the Cournot Nash equilibrium where both firms have costs of $10q_i$. Each firm solves,

$$\max_{q_i} \ (100 - (q_i + q_j) - 10)q_i$$

which leads to the first order condition

$$90 - q_j - 2q_i = 0$$

yielding the best response function

$$q_i(q_j) = \frac{90 - q_j}{2} \ ,$$

and the unique Cournot Nash equilibrium is $q_1 = q_2 = 30$ with profits $v_1 = v_2 = 900$. If firm 1 does invest then for firm 1 the problem becomes

$$\max_{q_1} \ (100 - (q_1 + q_2) - 5)q_1$$

which leads to the best response function

$$q_1(q_2) = \frac{95 - q_2}{2} \ .$$

For firm 2 the best response function remains the same as solved earlier with costs $10q_2$, so the unique Cournot Nash equilibrium is now solved using both equations,

$$q_1 = \frac{95 - \frac{90 - q_1}{2}}{2} \ ,$$

which yields $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$, and profits are $v_1 = 1,111\frac{1}{9}$ while $v_2 = 802\frac{7}{9}$. Hence, the increase in profits from the equilibrium with investment for player 1 are $F^* = 1,111\frac{1}{9} - 900 = 211\frac{1}{9}$, which is the most that player 1 would be willing to pay for the investment anticipating that they will play the Cournot Nash equilibrium after any choice of player 1 regarding investment. If $F < F^*$ then the unique subgame perfect equilibrium is that first, player 1 invests, then they players choose $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$, and if player 1 did not invest the payers choose $q_1 = q_2 = 30$ (Note that if $F > F^*$ then the unique subgame perfect equilibrium is that first, player 1 does not invest, then they players choose $q_1 = q_2 = 30$, and if player 1 did invest the payers choose $q_1 = \frac{100}{3}$, $q_2 = \frac{85}{3}$.) ■

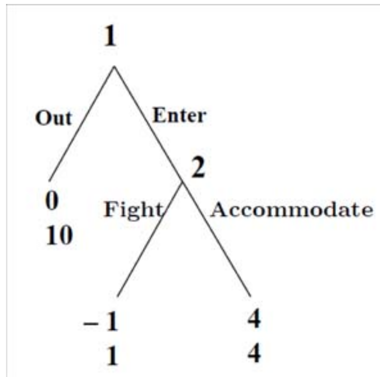
(b) Assume that $F > F^*$. Find a Nash equilibrium of the game that is not subgame perfect.

Answer: We construct a Nash equilibrium in which player 1 will invest despite $F > F^*$. Player 2's strategy will be, play $q_2 = \frac{85}{3}$ if player 1 invests, and $q_2 = 100$ if he does not invest. With this belief, if player 1 does not invest then he expects the price to be 0, and his best response is $q_1 = 0$ leading to profits $v_1 = 0$. If he invests then his best response to $q_2 = \frac{85}{3}$ is $q_1 = \frac{100}{3}$, which together are a Nash equilibrium in the Cournot game after investment. For any $F < 1,111\frac{1}{9}$ this will lead to positive profits, and hence, for $F^* < F < 1,111\frac{1}{9}$ the strategy of player 2 described above, together with player 1 choosing to invest, play $q_1 = \frac{100}{3}$ if he invests and $q_1 = 0$ if he does not is a Nash equilibrium. It is not subgame perfect because in the subgame following no investment, the players are not playing a Nash equilibrium. ■

7.

8. **Entry Deterrence 1:** NSG is considering entry into the local phone market in the Bay Area. The incumbent S&P, predicts that a price war will result if NSG enters. If NSG stays out, S&P earns monopoly profits valued at $10 million (net present value, or NPV of profits), while NSG earns zero. If NSG enters, it must incur irreversible entry costs of $2 million. If there is a price war, each firm earns $1 million (NPV). S&P always has the option of accommodating entry (i.e., not starting a price war). In such a case, both firms earn $4 million (NPV). Suppose that the timing is such that NSG first has to choose whether or not to enter the market. Then S&P decides whether to "accommodate entry" or "engage in a price war." What is the subgame perfect equilibrium outcome to this sequential game? (Set up a game tree.)

Answer: Letting NSG be player 1 and S&P be player 2,



Backward induction implies that player 2 will Accommodate, and player 1 will therefore enter. Hence, the unique subgame perfect equilibrium is (Enter,Accommodate). ■

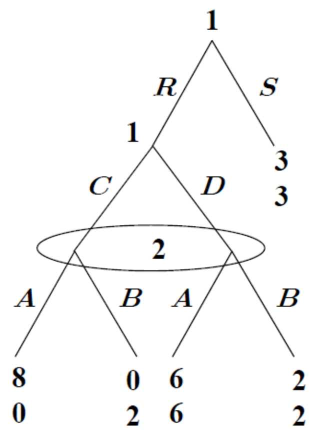
9.

10. **Playing it safe:** Consider the following dynamic game: Player 1 can choose to play it safe (denote this choice by S), in which case both he and player 2 get a **payoff of 3 each**, or he can risk playing a game with player 2 (denote this choice by R). If he chooses R, then they play the following **simultaneous move** game:

		Player 2 A	Player 2 B
player 1	C	$8, 0$	$0, 2$
player 1	D	$6, 6$	$2, 2$

(a) Draw a game tree that represents this game. How many proper subgames does it have?

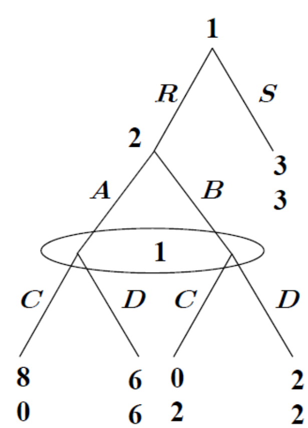
Answer:



The game has two proper subgames: the whole game and the subgame starting at the node where 1 chooses between C and D. ■

(b) Are there other game trees that would work? Explain briefly.

Answer: Yes - it is possible to have player 2 move after 1's initial move, and then have player 1 with an information set as follows:



(c) Construct the matrix representation of the normal form of this dynamic game.

Answer: The game can be represented by the following matrix,

		Player 2	
		A	B
Player 1	RC	$8,0$	$0,2$
	RD	$6,6$	$2,2$
	SC	$3,3$	$3,3$
	SD	$3,3$	$3,3$

■

(d) Find all the Nash and subgame perfect equilibria of the dynamic game.

Answer: It is easy to see that there are two pure strategy Nash equilibria: (SC, B) and (SD, B). It follows immediately that there are infinitely many mixed strategy Nash equilibria in which player 1 is mixing between SC and SD in any arbitrary way and player 2 chooses B. It is also easy to see that following a choice of R, there is no pure strategy Nash equilibrium in the resulting subgame. To find the mixed strategy Nash equilibrium in that subgame, let player 1 choose RC with probability p and RD with probability $(1-p)$, and let player 2 choose A with probability q. For player 2 to be indifferent it must be that

$$p(0) + (1-p)(6) = p(2) + (1-p)(2)$$

and the solution is $p = \frac{2}{3}$. Similarly, for player 1 to be indifferent it must be that

$$q(8) + (1-q)(0) = q(6) + (1-q)(2)$$

and the solution is $q = \frac{1}{2}$. Hence, $(p, q) = (\frac{1}{2}, \frac{3}{5})$ is a mixed strategy Nash equilibrium of the subgame after player 1 chooses R, yielding expected payoffs of $(v_1, v_2) = (4, 2)$. In any subgame perfect equilibrium the players will have to play this mixed strategy equilibrium following R, and because $4 > 3$ player 1 will prefer R over S. Hence, choosing R followed by the mixed strategy computed above is the unique subgame perfect equilibrium. ■

11.

12. **Agenda Setting:** An agenda-setting game is described as follows. The "issue space" (set of possible policies) is an interval $X = [0, 5]$. An Agenda Setter (player 1) proposes an alternative $x \in X$ against the status quo $q = 4$ After player 1 proposes x, the Legislator (player 2) observes the proposal and selects between the proposal x and the status quo q. Player 1's most preferred policy is 1, and for any final policy $y \in X$, his payoff is given by

$$v_1(y) = 10 - |y - 1|,$$

where $|y - 1|$ denotes the absolute value of $(y - 1)$. Player 2's most preferred policy is 3, and for any final policy $y \in X$, her payoff is given by

$$v_2(y) = 10 - |y - 3|.$$

That is, each player prefers policies that are closer to their most preferred policy.

(a) Write the game down as a normal form game. Is this a game of perfect or imperfect information?

Answer: There are two players, $i \in \{1, 2\}$ with strategy sets $S_1 = X = [0, 5]$ and $S_2 = \{A, R\}$ where A denotes accepting the proposal $x \in X$ and R means rejecting it and adopting the status quo $q = 4$. The payoffs are given by

$$v_1(s_1, s_2) = \begin{cases} 10 - |s_1 - 1| & \text{if } s_2 = A \\ 7 & \text{if } s_2 = R \end{cases}$$

and

$$v_2(s_1, s_2) = \begin{cases} 10 - |s_1 - 3| & \text{if } s_2 = A \\ 9 & \text{if } s_2 = R \end{cases}$$

(b) Find a subgame perfect equilibrium of this game. Is it unique?

Answer: Player 2 can guarantee himself a payoff of 9 by choosing R

implying that his best response is to choose A if and only if $10-|s_1-3| \geq 9$, which will hold for any $s_1 \in [2,4]$. Player 1 would like to have an alternative adopted that is closest to 1, which implies that his best response to player 2's sequentially rational strategy is to choose $s_1 = 2$ This is the unique subgame perfect equilibrium which results in the payoffs of $(v_1, v_2) = (9, 9)$. ■

(c) Find a Nash equilibrium that is not subgame perfect. Is it unique? If yes, explain. If not, show all the Nash equilibria of this game.

Answer: One Nash equilibrium is where player 2 adopts the strategy "I will reject anything except $s_1 = 3$." If player 1 chooses $s_1 = 3$ then his payoff is 8, while any other choice of s_1 is expected to yield player 1 a payoff of 7. Hence, player 1s best response to player 2's proposed strategy is indeed to choose $s_1 = 3$ and the payoffs from this Nash equilibrium are $(v_1, v_2) = (8, 10)$. Since player 2 can guarantee himself a payoff of 9, there are infinitely many Nash equilibria that are not subgame perfect and that follow a similar logic: player 2 adopts the strategy "I will reject anything except $s_1 = x$" for some value $x \in (2, 4)$ Player 1 would strictly prefer the adoption of x over 4, and hence would indeed propose x, and player 2 would accept the proposal. For $x = 4$ both players are indifferent so it would also be supported as a Nash equilibrium. ■

13.

14. **Hyperbolic Discounting:** Consider the three period example of a player with hyperbolic discounting described in section 8.3.4 with $\ln(x)$ utility in each of the three periods and with discount factors $0 < \delta < 1$ and $0 < \beta < 1$

(a) Solve the optimal choice of player 2, the second period self, as a function of his budget K_2, δ and β.

Answer: Player 2's optimization problem is given by

$$\max_{x_2} \; v_2(x_2, K - x_2) = \ln(x_2) + \beta\delta \ln(K - x_2),$$

for which the first order condition is

$$\frac{dv_2}{dx_2} = \frac{1}{x_2} - \frac{\beta\delta}{K_2 - x_2} = 0 \ ,$$

which in turn implies that player 2's best response function is,

$$x_2(K_2) = \frac{K_2}{\beta\delta + 1} \ ,$$

which leaves $x_3 = K_2 - x_2(K_2) = \frac{\beta\delta K_2}{\beta\delta+1}$ for consumption in the third period. ■

(b) Solve the optimal choice of player 1, the first period self, as a function of K, δ and β.

Answer: Player 1 decides how much to allocate between his own consumption and that of player 2 taking into account that $x_2(K_2) = \frac{K_2}{\beta\delta+1}$ hence player 1 solves the following problem,

$$\max_{x_1} v_1(x_1, \frac{K - x_1}{\beta\delta + 1}, \frac{\beta\delta(K - x_1)}{\beta\delta + 1}) = \ln(x_1) + \beta\delta \ln(\frac{K - x_1}{\beta\delta + 1}) + \beta\delta^2 \ln(\frac{\beta\delta(K}{\beta\delta}$$

for which the first order condition is,

$$\frac{dv_1}{dx_1} = \frac{1}{x_1} - \frac{\beta\delta}{K - x_1} - \frac{\beta\delta^2}{K - x_1} = 0 \ ,$$

which in turn implies that player 1's best response function is,

$$x_1(K) = \frac{K}{\beta\delta + \beta\delta^2 + 1} \ .$$

■

15.

16. **The Value of Commitment:** Consider the three period example of a player with hyperbolic discounting described in section 8.3.4 with $\ln(x)$ utility in each of the three periods and with discount factors $\delta = 1$ and $\beta = \frac{1}{2}$. We solved the optimal consumption plan of a sophisticated player 1.

(a) Imagine that an external entity can enforce any plan of action that player 1 chooses in $t = 1$ and will prevent player 2 from modifying it. What is the plan that player 1 would choose to enforce?

Answer: Player 1 wants to maximize,

$$\max_{x_2,x_3} v(K - x_2 - x_3, x_2, x_3) = \ln(K - x_2 - x_3) + \beta\delta \ln(x_2) + \beta\delta^2 \ln(x$$
$$= \ln(K - x_2 - x_3) + \frac{1}{2}\ln(x_2) + \frac{1}{2}\ln(x_3)$$

when $\beta = \frac{1}{2}$ and $\delta = 1$. The two fist order conditions are,

$$\frac{\partial v}{\partial x_2} = -\frac{1}{K - x_2 - x_3} + \frac{1}{2x_2} = 0 \ ,$$

and,

$$\frac{\partial v}{\partial x_3} = -\frac{1}{K - x_2 - x_3} + \frac{1}{2x_3} = 0 \ .$$

Solving these two equations yields the solution

$$x_2 = x_3 = \frac{K}{4},$$

and using $x_1 = K - x_2 - x_3$ gives,

$$x_1 = \frac{K}{2} \ .$$

Thus, player 1 would choose to enforce $x_1 = \frac{K}{2}$ and $x_2 = x_3 = \frac{K}{4}$. ■

(b) Assume that $K = 90$. Up to how much of his initial budget K will player 1 be willing to pay the external entity in order to enforce the plan you found in part (a)?

Answer: If the external entity does not enforce the plan, then from the analysis on pages 168-169 we know that player 2 will choose $x_2 = \frac{K}{3} = 30$ and $x_3 = \frac{K}{6} = 15$, and player 1 will choose $x_1 = \frac{K}{2} = 45$. The discounted value of the stream of payoffs for player 1 from this outcome is therefore,

$$\ln(45) + \frac{1}{2}\ln(30) + \frac{1}{2}\ln(15) \approx 6.86 \ .$$

If, however, player 1 can have the plan in part (a) above enforced then his discounted value of the stream of payoffs is

$$\ln(45) + \frac{1}{2}\ln(22.5) + \frac{1}{2}\ln(22.5) \approx 6.92\ .$$

We can therefore solve for the amount m of budget $K = 90$ that player 1 would be willing to give up which is found by the following equality,

$$\ln(45 - m) + \frac{1}{2}\ln(22.5) + \frac{1}{2}\ln(22.5) = 6.86\ ,$$

which yields $m \approx 2.63$. Hence, player 1 will be willing to give up to 2.63 of his initial budget $K = 90$ in order to enforce the plan $x_2 = x_3 = \frac{K}{4} = 22.5$. ■