Chapter One

Choice, Preference, and Utility

Most people, when they think about microeconomics, think first about the slogan *supply equals demand* and its picture, shown here in Figure 1.1, with a rising supply function intersecting a falling demand function, determining an equilibrium price and quantity.

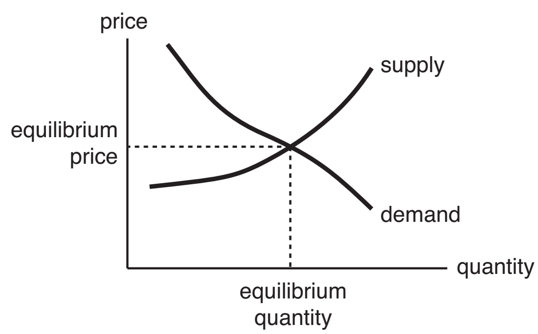

Figure 1.1. Supply equals demand

But before getting to this picture and the concept of an equilibrium, the picture's constituent pieces, the demand and supply functions, are needed. Those functions arise from *choices*, choices by firms and by individual consumers. Hence, microeconomic theory begins with choices. Indeed, the theory not only begins with choices; it remains focused on them for a very long time. Most of this volume concerns modeling the choices of consumers, with some attention paid to the choices of profit-maximizing firms; only toward the end do we seriously worry about equilibrium.

1.1. Consumer Choice: The Basics

The basic story of consumer choice is easily told: Begin with a set X of *possible objects that might be chosen* and an individual, the consumer, who does the choosing. The consumer faces limits on what she might choose, and so we imagine some collection $\mathcal{A}$ of nonempty subsets of X from which the consumer might choose. We let A denote a typical element of $\mathcal{A}$; that is, A is a subset of X. Then the choices of our consumer are denoted by $c(A)$.

The story is that the consumer chooses *one* element of A. Nonetheless, we think of $c(A)$ as a subset of A, not a member or element of A. This allows for the possibility that the consumer is happy with any one of several elements of A, in which case $c(A)$ lists all those elements. When she makes a definite choice of a single element, say x, out of A—when she says, in effect, "I want x and nothing else"—we write $c(A) = \{x\}$, or the singleton set consisting of the single element x. But if she says, "I would be happy with either x or y," then $c(A) = \{x, y\}$.

So far, no restrictions have been put on $c(A)$. But some restrictions are natural. For instance, $c(A) \subseteq A$ seems obvious; we do not want to give the consumer a choice out of A and have her choosing something that is not in A. You might think that we would insist on $c(A) \neq \emptyset$; that is, the consumer makes some choice. But we do not insist on this, at least, not yet. Therefore...

> *A* ***model of consumer choice*** *consists of some set X of possible objects of choice, a collection $\mathcal{A}$ of nonempty subsets of X, and a choice function c whose domain is $\mathcal{A}$ and whose range is the set of subsets of X, with the sole restriction that $c(A) \subseteq A$.*

For instance, we can imagine a world of k commodities, where a *commodity bundle* is a vector $x = (x_1, \ldots, x_k) \in R^k_+$, the positive orthant in k-dimensional Euclidean space. (In this book, *the positive orthant* means all components nonnegative, or $R^k_+ = \{x \in R^k : x \geq 0\}$. *The strict positive orthant*, denoted by R^k_{++}, means elements of R^k all of whose components are strictly positive.) If, say, $k = 3$ and the commodities are (in order) bread, cheese, and salami, the bundle $(3, 0, 0.5)$ means 3 units of bread, no cheese, and 0.5 units of salami, in whatever units we are using. We can also imagine prices p_i for the commodities, so that $p = (p_1, \ldots, p_k)$ is the price vector; for convenience, we assume that all prices are strictly positive, or $p \in R^k_{++}$. And we can imagine that the consumer has some amount of income $y \geq 0$ to spend. Then the consumer's choice problem is to choose some affordable bundle given these prices and her income; that is, a typical set A is a budget set

$$\{x \in R^k_+ : p \cdot x \leq y\}.$$

A model of consumer choice in this context is then a choice function that says which bundles the consumer would be willing to accept, as a function of the prices of the goods p and her level of income y.

This is not much of a model, yet. Economic modeling begins with an assumption that the choices made by the consumer in different situations are somewhat coherent. Imagine, for instance, a customer at a café asking for a cup of coffee and a piece of pie. When told that they have apple and cherry pie, she opts for apple. Then the waiter tells her that they also have peach pie. "If you also have peach," she responds, "I would like cherry pie, please." We want to (and will) assume that choice in different situations is coherent enough to preclude this sort of behavior; we'll formalize this next page, in Definition 1.1b.

This is one sort of coherence. A second is that the consumer's choices are in accord with utility maximization, for some utility function defined on X. That is, there is a function $u : X \to R$, such that for every A,

$$c(A) = \{x \in A : u(x) \geq u(y) \text{ for all } y \in A\}. \tag{1.1}$$

A third sort of coherence involves a *preference relation* over X. A preference relation expresses the consumer's feelings between pairs of objects in X. We denote the preference relation by $\succeq$ and imagine that for every pair x and y from X, the consumer is willing to say that either $x \succeq y$, meaning x is at least as good as y, or not. For any pair x and y, then, one of four mutually exclusive possibilities holds: (1) the consumer says that $x \succeq y$ and that $y \succeq x$; (2) $x \succeq y$ but not $y \succeq x$; (3) $y \succeq x$ but not $x \succeq y$; or (4) neither $x \succeq y$ nor $y \succeq x$. Then, with these preferences in hand, a consumer chooses from a set A precisely those elements of A that are at least as good as everything in A, or

$$c(A) = \{x \in A : x \succeq y \text{ for all } y \in A\}. \tag{1.2}$$

When you look at (most) models in microeconomics that have consumers, consumers make choices, and the choice behavior of the consumer is modeled by either (1) a utility function and the (implicit) assumption that choice from any set A is governed by the rule (1.1) or (2) a preference relation and the (implicit) assumption that choice from any set A is governed by the rule (1.2). (Discrete choice models in econometrics have so-called random utility models, in which choices are stochastic. And in some parts of behavioral economics, you will find models of choice behavior that don't quite fit either of these frameworks. But most models have either utility-maximizing or preference-maximizing consumers.)

The questions before us in this chapter are: How do these different ways of modeling consumer choice compare? If we restrict attention to coherent choice, does one imply the other(s)? Can they be made consistent?

The basic answer is that *under certain coherence assumptions, the three ways of modeling consumer choice are equivalent.* We begin with the case of finite X. (We worry a lot about infinite X later.) To keep matters simple, we make the following assumption for the remainder of this chapter (but see Problems 1.15 and 1.16).

Assumption. *$\mathcal{A}$ is the set of* ***all*** *nonempty subsets of X.*

Two properties of choice functions and two properties of a preference relation must be defined:

Definition 1.1.

a. *A choice function c satisfies* ***finite nonemptiness*** *if $c(A)$ is nonempty for every finite $A \in \mathcal{A}$.*

b. *A choice function c satisfies* ***choice coherence*** *if, for every pair x and y from X and A and B from $\mathcal{A}$, if $x, y \in A \cap B$, $x \in c(A)$, and $y \notin c(A)$, then $y \notin c(B)$.*

c. *A preference relation on X is* ***complete*** *if for every pair x and y from X, either $x \succeq y$ or $y \succeq x$ (or both).*

d. *A preference relation on X is* ***transitive*** *if $x \succeq y$ and $y \succeq z$ implies that $x \succeq z$.*

Some comments about these definitions may be helpful: Concerning a, if X is finite, finite nonemptiness of c means that $c(A)$ is nonempty for all subsets of X. Later in the chapter, the restriction to finite A will have a role to play. Choice coherence is the formalization intended to preclude the apple, cherry, and peach pie vignette: If apple is the (sole) choice out of {apple, cherry}, then cherry cannot be chosen from {apple, cherry, peach}. An equivalent (contrapositive) form for b is: *For every pair x and y from X and A and B from $\mathcal{A}$, if $x, y \in A \cap B$, $x \in c(A)$, and $y \in c(B)$, then $y \in c(A)$ and $x \in c(B)$.*[1]

Proposition 1.2. *Suppose that X is finite.*

a. *If a choice function c satisfies finite nonemptiness and choice coherence, then there exist both a utility function $u : X \to R$ and a complete and transitive preference relation $\succeq$ that produce choices according to c via the formulas (1.1) and (1.2), respectively.*

b. *If a preference relation $\succeq$ on X is complete and transitive, then the choice function it produces via formula (1.2) satisfies finite nonemptiness and choice coherence, and there exists a utility function $u : X \to R$ such that*

$$x \succeq y \text{ if and only if } u(x) \geq u(y). \tag{1.3}$$

c. *Given any utility function $u : X \to R$, the choice function it produces via formula (1.1) satisfies finite nonemptiness and choice coherence, the preference relation it produces via (1.3) is complete and transitive, and the choice function produced by that preference relation via (1.2) is precisely the choice function produced directly from u via (1.1).*

In words, choice behavior (for a finite X) that satisfies finite nonemptiness and choice coherence is equivalent to preference maximization (that is, formula (1.2)) for complete and transitive preferences, both of which are equivalent to utility maximization (via formulas (1.1) and (1.3)). However expressed, whether in terms of choice, preference, or utility, this conglomerate (with the two pairs of assumptions) is the standard model of consumer choice in microeconomics.

A much-used piece of terminology concerns display (1.3), which connects a utility function u and a preference relation $\succeq$. When (1.3) holds, we say that the utility function u *represents* the preference relation $\succeq$.

In terms of economics, Proposition 1.2 is the story of this chapter. Several tasks remain:

1. We prove the proposition.

[1] If it isn't clear to you that this restatement is equivalent to b in the definition, you should verify it carefully. Stated in this alternative form, Mas-Colell, Whinston, and Green (1995) call property b *the weak axiom of revealed preference*, although their setting is a bit different; cf. Problem 1.15. In previous books, I have called property b *Houthakker's Axiom of Revealed Preference*, but I no longer believe this is a correct attribution; the first appearance of this property for choice out of general sets (that is, outside the context of price-and-income-generated budget sets) of which I am aware is Arrow (1959).

2. We consider how (and whether) this proposition extends to infinite X. After all, in the one example we've given, where $X = R^k_+$, we have an infinite X. Most economic applications will have an infinite X.

3. We have so far discussed the binary relation $\succeq$, known as *weak preference*, which is meant to be an expression of "at least as good as." In economic applications, two associated binary relations, *strict preference* ("strictly better than") and *indifference* ("precisely as good as") are used; we explore them and their connection to weak preference.

4. We comment briefly on aspects of the standard model: What if $\mathcal{A}$ does not contain all nonempty subsets of X? What is the empirical evidence for or against the standard model? What alternatives are there to the standard model?

1.2. Proving Most of Proposition 1.2, and More

Parts of Proposition 1.2 are true for all X, finite or not.

Proposition 1.3. *Regardless of the size of X, if $u : X \to R$, then*

a. *the preference relation $\succeq_u$ defined by $x \succeq_u y$ if $u(x) \geq u(y)$ is complete and transitive, and*

b. *the choice function c_u defined by $c_u(A) = \{x \in A : u(x) \geq u(y) \text{ for all } y \in A\}$ satisfies finite nonemptiness and choice coherence.*

Proof. (a) Given any two x and y from X, either $u(x) \geq u(y)$ or $u(y) \geq u(x)$ (since $u(x)$ and $u(y)$ are two real numbers); hence either $x \succeq_u y$ or $y \succeq_u x$. That is, $\succeq_u$ is complete.

If $x \succeq_u y$ and $y \succeq_u z$, then (by definition) $u(x) \geq u(y)$ and $u(y) \geq u(z)$; hence $u(x) \geq u(z)$ (because $\geq$ is transitive for real numbers), and therefore $x \succeq_u z$. That is, $\succeq_u$ is transitive.

(b) Suppose $x, y \in A \cap B$ and $x \in c_u(A)$. Then $u(x) \geq u(y)$. If, moreover, $y \notin c_u(A)$, then $u(z) > u(y)$ for some $z \in A$. But $u(x) \geq u(z)$ since $x \in c_u(A)$ implies $u(x) \geq u(z)$ for all $z \in A$; therefore $u(x) > u(y)$. Since $x \in B$, this immediately implies that $y \notin c_u(B)$, since there is something in B, namely x, for which $u(y) \not\geq u(x)$. This is choice coherence.

If A is a finite subset of X, then $\{r \in R : r = u(x) \text{ for some } x \in A\}$ is a finite set of real numbers. Every finite set of real numbers contains a largest element; that is, some $r^* = u(x^*)$ in the set satisfies $r^* \geq r$ for all the elements of the set. But this says that $u(x^*) \geq u(x)$ for all $x \in A$, which implies that $x^* \in c_u(A)$, and $c_u(A)$ is not empty. ∎

Proposition 1.4. *Regardless of the size of X, if $\succeq$ is a complete and transitive binary relation on X, the choice function $c_{\succeq}$ defined on the set of all nonempty subsets of X by*

$$c_{\succeq}(A) := \{x \in A : x \succeq y \text{ for all } y \in A\}$$

satisfies finite nonemptiness and choice coherence.

Proof. Suppose $x, y \in A \cap B, x \in c_{\succeq}(A)$, and $y \notin c_{\succeq}(A)$. Since $x \in c_{\succeq}(A)$, $x \succeq y$. Since $y \notin c_{\succeq}(A)$, $y \not\succeq z$ for some $z \in A$. By completeness, $z \succeq y$. Since $x \in c_{\succeq}(A)$, $x \succeq z$. I claim that $y \not\succeq x$: Assume to the contrary that $y \succeq x$, then $x \succeq z$ and transitivity of $\succeq$ would imply that $y \succeq z$, contrary to what was assumed. But if $y \not\succeq x$, then since $x \in B$, $y \notin c_{\succeq}(B)$. That is, $c_{\succeq}$ satisfies choice coherence.

I assert that if A is a finite (and nonempty) set, some $x \in A$ satisfies $x \succeq y$ for all $y \in A$ (hence $c_{\succeq}(A)$ is not empty). The proof is by induction[2] on the size of A: if A contains a single element, say, $A = \{x\}$, then $x \succeq x$ because $\succeq$ is complete. Therefore, the statement is true for all sets of size 1. Assume inductively that the statement is true for all sets of size $n-1$ and let A be a set of size n. Take any single element x_0 from A, and let $A' = A \setminus \{x_0\}$. A' is a set of size $n-1$, so there is some $x' \in A'$ such that $x' \succeq y$ for all $y \in A'$. By completeness of $\succeq$, either $x' \succeq x_0$ or $x_0 \succeq x'$. In the first case, $x' \succeq y$ for all $y \in A$, and we are done. In the second case, $x_0 \succeq x_0$ by completeness, and $x_0 \succeq y$ for all $y \in A'$, since $x' \succeq y$, and therefore transitivity of $\succeq$ tells us that $x_0 \succeq y$. Hence, for this arbitrary set of size n, we have produced an element at least as good as every other element. This completes the induction step, proving the result. ∎

Proposition 1.5. *Regardless of the size of X, suppose the choice function c satisfies finite nonemptiness and choice coherence. Define a binary relation $\succeq_c$ on X by*

$$x \succeq_c y \text{ if } x \in c(\{x, y\}).$$

Define a new choice function $c_{\succeq_c}$ by

$$c_{\succeq_c}(A) = \{x \in A : x \succeq_c y \text{ for all } y \in A\}.$$

Then $\succeq_c$ is complete and transitive, $c_{\succeq_c}$ satisfies choice coherence and finite nonemptiness, and for every subset A of X, either

$$c(A) = \emptyset \text{ or } c(A) = c_{\succeq_c}(A).$$

Before proving this, please note an instant corollary: If X is finite and c satisfies finite nonemptiness, then $c(A) \neq \emptyset$ for all $A \subseteq X$, and hence $c(A) = c_{\succeq_c}(A)$ for all A.

Proof of Proposition 1.5. Since c satisfies finite nonemptiness, either $x \in c(\{x, y\})$ or $y \in c(\{x, y\})$; hence either $x \succeq_c y$ or $y \succeq_c x$. That is, $\succeq_c$ is complete.

Suppose $x \succeq_c y$ and $y \succeq_c z$. I assert that choice coherence implies that $x \in c(\{x, y, z\})$. Suppose to the contrary that this is not so. It cannot be that $y \in$

[2] See Appendix 1.

$c(\{x,y,z\})$, for if it were, then x could not be in $c(\{x,y\})$ by choice coherence: Take $A = \{x,y,z\}$ and $B = \{x,y\}$; then $x,y \in A \cap B$, $y \in c(A)$, $x \notin c(A)$, and hence choice coherence implies that $x \notin c(B)$, contrary to our original hypothesis. And then, once we know that $y \notin c(\{x,y,z\})$, choice coherence can be used again to imply that $z \notin c(\{x,y,z\})$: Now $y,z \in \{x,y,z\} \cap \{y,z\}$, and if $z \in c(\{x,y,z\})$, since we know that $y \notin c(\{x,y,z\})$, this would imply $y \notin c(\{y,z\})$, contrary to our original hypothesis. But if x, y, and z are all not members of $c(\{x,y,z\})$, then it is empty, contradicting finite nonemptiness. Hence, we conclude that x must be a member of $c(\{x,y,z\})$. But then choice coherence and finite nonemptiness together imply that $x \in c(\{x,z\})$, for if it were not, z must be in $c(\{x,z\})$, and choice coherence would imply that x cannot be a member of $c(\{x,y,z\})$. Hence we now conclude that $x \in c(\{x,z\})$, which means that $x \succeq_c z$, and $\succeq_c$ is transitive.

Since $\succeq_c$ is complete and transitive, we know from Proposition 1.4 that $c_{\succeq_c}$ satisfies finite nonemptiness and choice coherence.

Now take any set A and any $x \in c(A)$. Let y be any other element of A. By finite nonemptiness and choice coherence, x must be in $c(\{x,y\})$, because, if not, then y is the sole element of $c(\{x,y\})$ and, by choice coherence, x cannot be an element of $c(A)$. Therefore, $x \succeq_c y$. This is true for every member y of A; therefore $x \in c_{\succeq_c}(A)$. That is, $c(A) \subseteq c_{\succeq_c}(A)$.

Finally, suppose $x \in c_{\succeq_c}(A)$ and that $c(A)$ is nonempty. Let x_0 be some member $c(A)$. By the definition of $c_{\succeq_c}$, $x \succeq_c x_0$, which is to say that $x \in c(\{x_0, x\})$. But then $x \notin c(A)$ is a violation of choice coherence. Therefore, $x \in c(A)$, and (assuming $c(A)$ is nonempty) $c_{\succeq_c}(A) \subseteq c(A)$. This completes the proof. ■

1.3. The No-Better-Than Sets and Utility Representations

If you carefully put all the pieces from Section 1.2 together, you see that, to finish the proof of Proposition 1.2, we must show that for finite X, if c satisfies finite nonemptiness and choice coherence, some utility function u gives c via the formula (1.1), and if $\succeq$ is complete and transitive, some utility function u represents $\succeq$ in the sense of (1.3). We will get there by means of an excursion into the no-better-than sets.

Definition 1.6. *For a preference relation $\succeq$ defined on a set X (of any size) and for x a member of X, the **no-better-than** x **set**, denoted* $\mathrm{NBT}(x)$, *is defined by*

$$\mathrm{NBT}(x) = \{y \in X : x \succeq y\}.$$

In words, y is no better than x if x is at least as good as y. We define $\mathrm{NBT}(x)$ for any preference relation $\succeq$, but we are mostly interested in these sets for complete and transitive $\succeq$, in which case the following result pertains.

Proposition 1.7. *If $\succeq$ is complete and transitive, then* NBT(x) *is nonempty for all x. In particular, $x \in$* NBT(x). *Moreover, $x \succeq y$ if and only if* NBT(y) $\subseteq$ NBT(x), *and if $x \succeq y$ but $y \not\succeq x$, then* NBT(y) *is a proper subset of* NBT(x). *Therefore, the collection of* NBT *sets nest; that is, if x and y are any two elements of X, then either* NBT(x) *is a proper subset of* NBT(y), *or* NBT(y) *is a proper subset of* NBT(x), *or the two are equal.*

This is not hard to prove, so I leave it to you in case you need practice with these sorts of exercises in mathematical theorem proving.

Proposition 1.8. *If X is a finite set and $\succeq$ is complete and transitive, then the function $u : X \to R$ defined by*

$$u(x) = \textit{the number of elements of } \mathrm{NBT}(x)$$

satisfies $u(x) \geq u(y)$ if and only if $x \succeq y$.

Proof. This is virtually a corollary of the previous proposition, but since I failed to give you the proof of that proposition, I spell this one out. Suppose $x \succeq y$. Then by Proposition 1.7, NBT(y) $\subseteq$ NBT(x), so $u(y) \leq u(x)$; that is, $u(x) \geq u(y)$.

Conversely, suppose $u(x) \geq u(y)$. Then there are least as many elements of NBT(x) as there are of NBT(y). But, by Proposition 1.7, these sets nest; hence NBT(y) $\subseteq$ NBT(x). Of course, $y \in$ NBT(y), abd hence $y \in$ NBT(x) so $x \succeq y$. ■

To finish off the proof of Proposition 1.2, we need to produce a utility function u from a choice function c in the case of finite X. Here is one way to do it: Assume X is finite and c is a choice function on X that satisfies finite nonemptiness and choice coherence. Use c to generate a preference relation $\succeq_c$, which is immediately complete and transitive. Moreover, if $c_{\succeq_c}$ is choice generated from $\succeq_c$, we know (since X is finite; hence $c(A)$ is nonempty for every A) that $c_{\succeq_c}$ is precisely c). Use the construction just given to produce a utility function u that represents $\succeq_c$. Because, for any A,

$$c(A) = c_{\succeq_c}(A) = \{x \in A : x \succeq_c y \text{ for all } y \in A\},$$

we know immediately that

$$c(A) = c_{\succeq_c}(A) = \{x \in A : u(x) \geq u(y) \text{ for all } y \in A\}.$$

Done.

Although a lot of what is proved in this section and in Section 1.2 works for any set X, in two places we rely on the finiteness of X.

1. In the proof of Proposition 1.8, if NBT(x) can be an infinite set, defining $u(x)$ to be the number of elements of NBT(x) does not work.

2. In several places, when dealing with choice functions, we had to worry about $c(A) = \emptyset$ for infinite A. We could have added an assumption that $c(A)$ is never empty, but for reasons to be explained, that is a bad idea.

We deal with both these issues in Sections 1.5 and 1.6, respectively, but to help with the exposition, we first take up issues related to preference relations.

1.4. Strict Preference and Indifference

In terms of preferences, the standard theory of choice deals with a complete and transitive binary relation $\succeq$, often called *weak* preference. The statement $x \succeq y$ means that the consumer judges x to be at least as good as y; that is, either x and y are equally good or x is better than y.

For any pair x and y, completeness implies that of the four mutually exclusive possibilities ennumerated in the first paragraph of page 3, one of the first three must hold, namely

1. both $x \succeq y$ and $y \succeq x$, or
2. $x \succeq y$ but not $y \succeq x$, or
3. $y \succeq x$ but not $x \succeq y$.

In case 1, we say that the consumer is *indifferent* between x and y and write $x \sim y$. In case 2, we say that x is *strictly preferred* to y and write $x \succ y$. And in case 3, y is strictly preferred to x, written $y \succ x$.

Proposition 1.9. *Suppose weak preference $\succeq$ is complete and transitive. Then*

a. $x \succ y$ if and only if it is not the case the $y \succeq x$.

*b. Strict preference is **asymmetric**: If $x \succ y$, then it is not the case that $y \succ x$.*

*c. Strict preference is **negatively transitive**: If $x \succ y$, then for any third element z, either $z \succ y$ or $x \succ z$.*

*d. Indifference is **reflexive**: $x \sim x$ for all x.*

*e. Indifference is **symmetric**: If $x \sim y$, then $y \sim x$.*

f. Indifference is transitive: If $x \sim y$ and $y \sim z$, then $x \sim z$.

g. If $x \succ y$ and $y \succeq z$, then $x \succ z$. If $x \succeq y$ and $y \succ z$, then $x \succ z$.

h. Strict preference is transitive: If $x \succ y$ and $y \succ z$, then $x \succ z$.

Proof. Asymmetry of strict preference is definitional: $x \succ y$ if $x \succeq y$ and not $y \succeq x$, either of which implies not $y \succ x$. Indifference is reflexive because $\succeq$ is complete; hence $x \succeq x$ for all x. Indifference is symmetric because the definition of indifference is symmetric. Indifference is transitive because $\succeq$ is transitive: If $x \sim y$ and $y \sim z$, then $x \succeq y$, $y \succeq z$, $z \succeq y$, and $y \succeq x$, and hence $x \succeq z$ and $z \succeq x$, so $x \sim z$. This leaves a, c, g, and h to prove.

For g, if $x \succ y$ then $x \succeq y$. If in addition $y \succeq z$, then $x \succeq z$ by transitivity. Suppose $z \succeq x$. Then by transitivity of $\succeq$, $y \succeq z \succeq x$ implies $y \succeq x$, contradicting the hypothesis that $x \succ y$. Therefore, it is not true that $z \succeq x$, and hence $x \succ z$. The other half is similar.

For h, if $x \succ y$ and $y \succ z$, then $y \succeq z$. Apply part g.

For a, if $x \succ y$, then $x \succeq y$ and not $y \succeq x$ by definition, so in particular not $y \succeq x$. Conversely, not $y \succeq x$ implies $x \succeq y$ by completeness of $\succeq$, and these two together are $x \succ y$ by the definition of $\succ$.

For c, suppose $x \succ y$ but not $z \succ y$. By part a, the second is equivalent to $y \succeq z$, and then $x \succ z$ by part g. ∎

We began with weak preference $\succeq$ and used it to define strict preference $\succ$ and indifference $\sim$. Other textbooks begin with strict preference $\succ$ as the primitive and use it to define weak preference $\succeq$ and indifference $\sim$. While the standard theory is based on a complete and transitive weak preference relation, it could equally well be based on strict preference that is asymmetric and negatively transitive:

Proposition 1.10. *Suppose a binary relation $\succ$ is asymmetric and negatively transitive. Define $\succeq$ by $x \succeq y$ if not $y \succ x$, and define $\sim$ by $x \sim y$ if neither $x \succ y$ nor $y \succ x$. Then $\succeq$ is complete and transitive, and if we defined $\sim'$ and $\succ'$ from $\succeq$ according to the rules given previously, $\sim'$ would be the same as $\sim$, and $\succ'$ would be the same as $\succ$.*

Proving this makes a good exercise and so is left as Problem 1.9.

1.5. Infinite Sets and Utility Representations

This section investigates the following pseudo-proposition:

If $\succeq$ is a complete and transitive binary relation on an arbitrary set X, then some function $u : X \to R$ can be found that represents $\succeq$; that is, such that $x \succeq y$ if and only if $u(x) \geq u(y)$.

Proposition 1.3 tells us the converse: If $\succeq$ is represented by some utility function u, then $\succeq$ must be complete and transitive. But is the pseudo-proposition true? The answer is no, of course; we would not call this a pseudo-proposition if the answer were yes. I do not give the standard counterexample here; it is found in Problem 1.10.

Rather than give the standard counterexample, we look for fixes. The idea is to add some assumptions on preferences or on X or on both together that make the proposition true. The first fix is quite simple.

Proposition 1.11. If $\succeq$ is a complete and transitive binary relation on a ***countable*** set X, then for some function $u : X \to R$, $u(x) \geq u(y)$ if and only if $x \succeq y$.

(A set X is countable if its elements can be enumerated; that is, if there is a way to count them with the positive integers. All finite sets are countable. The set of integers is countable, as is the set of rational numbers. But the set of real numbers is not countable or, in math-speak, is uncountable. Proving this is not trivial.)

Proof. Let $\{x_1, x_2, \ldots\}$ be an enumeration of the set X. Define $d : X \to R$ by $d(x_n) = \left(\frac{1}{2}\right)^n$. Define, for each x,

$$u(x) = \sum_{z \in \mathrm{NBT}(x)} d(z).$$

(The series $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ is absolutely summable, so the potentially infinite sum being taken in the display is well defined. If you are unclear on this, you need to review [I hope it is just a review!] the mathematics of sequences and series.) Suppose $x \succeq y$. Then $\mathrm{NBT}(y) \subseteq \mathrm{NBT}(x)$, so the sum that defines $u(x)$ includes all the terms in the sum that defines $u(y)$ and perhaps more. All the summands are strictly positive, and therefore $u(x) \geq u(y)$.

Conversely, we know that the NBT sets nest, and so $u(x) \geq u(y)$ only if $\mathrm{NBT}(y) \subseteq \mathrm{NBT}(x)$. Therefore $u(x) \geq u(y)$ implies $y \in \mathrm{NBT}(y) \subseteq \mathrm{NBT}(x)$; $y \in \mathrm{NBT}(x)$, and hence $x \succeq y$. ∎

Compare the proofs of Propositions 1.8 and 1.11. In Proposition 1.8, the $u(x)$ is defined to be the size of the set $\mathrm{NBT}(x)$. In other words, we add 1 for every member of $\mathrm{NBT}(x)$. Here, because that might get us into trouble, we add instead terms that sum to a finite number, even if there are (countably) infinitely many of them, making sure that the terms are all strictly positive so that more summands means a bigger sum and so larger utility.

The hard part is to go from countable sets X to uncountable sets. A very general proposition does this for us.

Proposition 1.12. *Suppose $\succeq$ is a complete and transitive preference relation on a set X. The relation $\succeq$ can be represented by a utility function if and only if some countable subset X^* of X has the property that if $x \succ y$ for x and y from X, then $x \succeq x^* \succ y$ for some $x^* \in X^*$.*

Proof. Suppose X^* exists as described. Enumerate X^* as $\{x_1^*, x_2^*, \ldots\}$ and let $d(x_n^*) = \left(\frac{1}{2}\right)^n$. For each $x \in X$, define

$$U(x) = \sum_{\{x^* \in X^* \cap \mathrm{NBT}(x)\}} d(x^*).$$

If $x \succeq y$, then $\mathrm{NBT}(y) \subseteq \mathrm{NBT}(x)$; hence $\mathrm{NBT}(y) \cap X^* \subseteq \mathrm{NBT}(x) \cap X^*$. The sum defining $u(x)$ is over at least as large a set as the sum defining $u(y)$, and all the summands are positive, so $u(x) \geq u(y)$.

To show the converse, we use the contrapositive: If not $y \succeq x$, then not $u(y) \geq u(x)$. Not $y \succeq x$ is equivalent to $x \succ y$, and not $u(y) \geq u(x)$ is $u(x) > u(y)$. But if $x \succ y$, then there is some x^* in X^* such that $x \succeq x^* \succ y$. Hence x^* is in the sum that defines $u(x)$ but not in the sum that defines $u(y)$. Otherwise, every term in sum defining $u(y)$ is in the sum defining $u(x)$ (see the previous paragraph), and therefore $u(x) > u(y)$.

You may wish to avoid on a first reading the proof that if $\succeq$ is represented by the utility function u, then such a countable set X^* exists. This proof is somewhat technical and filled with special cases.

Let $\{I_n\}$ be an ennumeration of all closed intervals with rational endpoints; that is, each I_n is an interval of the form $[\underline{q}_n, \bar{q}_n]$ where $\bar{q}_n > \underline{q}_n$ are rational numbers. (The set of rational numbers is countable and the cross product of two countable sets is countable.) Let $u(X)$ denote the set of real numbers $\{r \in R : r = u(x) \text{ for some } x \in X\}$. Consider three possibilities:

1. If $u(X) \cap I_n$ is nonempty, pick some single $x \in X$ such that $u(x) \in I_n$ and call this x_n.
2. If $u(X) \cap I_n$ is empty, let $\bar{r}_n = \inf\{r \in u(X) : r > \bar{q}_n\}$. If $u(x) = \bar{r}_n$ for some $x \in X$, choose one such x and call this x_n.
3. If $u(X) \cap I_n$ is empty and $\bar{r}_n \neq u(x)$ for all $x \in X$, then do not bother defining x_n.

Let X^* be the set of all x_n created in cases 1 and 2. Since there are countably many intervals I_n and at most one x_n is produced for each I_n, X^* is a countable set.

Now suppose $x \succ y$ in X. Since u represents $\succeq$, $u(x) > u(y)$. Choose some rational number q in the open interval $(u(y), u(x))$. Let $\bar{r} = \inf\{r \in u(X) : r > q\}$. Clearly, $u(x) \geq \bar{r}$, since $u(x)$ is in the set over which we are taking the infimum. There are two cases:

1. If $u(x) > \bar{r}$, let q' be some rational number such that $u(x) > q' > \bar{r}$, and let n be the index of the interval $[q, q']$. By construction, $u(X) \cap [q, q'] \neq \emptyset$ (you may have to think about that one for a minute); hence there is $x^* \in X^*$, namely x_n, with $u(x^*) \in [q, q']$, which means $u(x) > u(x^*) > u(y)$. Done.
2. If $u(x) = \bar{r}$, then let q' be some rational number such that $q > q' > u(y)$, and let n be the index of the interval $[q', q]$. If $u(X) \cap [q', q] \neq \emptyset$, then there is $x^* \in X^*$ with $u(x) \geq q \geq u(x^*) \geq q' > u(y)$, and therefore $x \succeq x^* \succ y$. Alternatively, if $u(X) \cap [q', q] = \emptyset$, then the interval $[q', q]$ fits into category 2 above, and in particular, there is some $x^* \in X^*$, namely x_n, such that $u(x^*) = \bar{r} = u(x)$. But for this x^*, $u(x) = u(x^*) > u(y)$; hence $x \succeq x^* \succ y$. Once again, done. ■

Proposition 1.12 gives a necessary and sufficient condition that, in addition to $\succeq$ being complete and transitive, provides for a utility representation. This proposition is, therefore, the most general such proposition we can hope for. But general or not, it is not hugely useful, because the condition—the existence of the countable subset X^*—is not very practical. How can you tell, in a particular application, if such a countable subset exists?

For practical purposes, the usual method is to make *topological* assumptions about X and $\succeq$. To illustrate this method, and also to take care of the vast majority of applications you are likely to encounter in a career in economics, I'll specialize to the case where $X = R^k_+$, with the interpretation that there are k commodities and $x \in X$ is a bundle of goods. In this context, the following definition makes sense:

Definition 1.13. *Complete and transitive preferences $\succeq$ on $X = R^k_+$ are **continuous** if, for every pair x and y from X with $x \succ y$, we can find an $\epsilon > 0$ such that for every bundle $x' \in X$ that is less than ϵ distant from x and for every bundle $y' \in X$ that is less than ϵ distant from y, $x' \succ y'$.*

In this definition, the distance between two points is the length of the line segment that joins them; that is, we use Euclidean distance.[3]

The idea is captured by Figure 1.2. If $x \succ y$, then of course $x \neq y$. Denote the distance between them by d. If we take a small enough ϵ, say ϵ equal to 1% of d, then everything within ϵ of x will be very close to being as good as x, and everything within ϵ of y will be very close to being as good (or bad) as y. Since $x \succ y$, if we make the balls small enough, everything in the ball around x should be strictly better than everything in the ball around y.

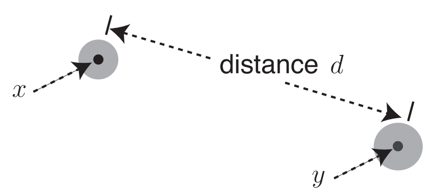

Figure 1.2. Continuity of preferences. Suppose $x \succ y$, and the distance between x and y is d. If preferences are continuous, we can put a ball around x and a ball around y, where you should think of the diameters of the balls being small relative to d, such that for all x' in the ball around x and for all y' in the ball around y, $x' \succ y'$.

This definition of continuity of $\succeq$ provides us with a very nice picture, Figure 1.2, but is neither mathematically elegant nor phrased in way that is useful in proofs of propositions that assume continuous preferences. The next proposition provides some equivalent definitions that are both more elegant and, in many cases, more useful.

[3] This is the first time that the distance between two bundles is mentioned, so to be very explicit: Suppose we are looking at the two bundles (10, 20, 30) and (11, 18, 30) in R^3. The most "natural" way to measure the distance between them is *Euclidean* distance, the square root of the sum of the squares of the distances for each component, or $\sqrt{(11-10)^2 + (20-18)^2 + (30-30)^2} = \sqrt{1+4+0} = \sqrt{5}$. But it is equivalent in terms of all important topological properties, to measure the distance as the sum of absolute values of the differences, component by component—in this case, $|11-10|+|20-18|+|30-30| = 1+2+0 = 3$—or to measure the distance as the maximum of the absolute values of the differences, component by component, or $\max\{|11-10|, |20-18|, |30-30|\} = 2$. For each of these distance measures, two bundles are "close" if and only if they are close in value, component by component; this is what makes these different ways of measuring distance topologically equivalent. It is sometimes useful to have these different ways of measuring distance—so-called norms or metrics—because a particular proposition may be easier to prove using one rather than the others. For more on this, and for many of the real analytic prerequisites of this book, see Appendix 2.

Proposition 1.14. *Continuity of preferences $\succeq$ on R_+^k imply the following, and any one of the following imply that preferences $\succeq$ on R_+^k are continuous. (Therefore, continuity of preferences could equivalently be defined by any one of the following, each of which implies all the others.)*

a. *If $\{x_n\}$ is a sequence from R_+^k with $x_n \succeq y$ for all n, and if $\lim_{n\to\infty} x_n = x$, then $x \succeq y$. If $\{x_n\}$ is a sequence from R_+^k with $y \succeq x_n$ for all n, and if $\lim_{n\to\infty} x_n = x$, then $y \succeq x$.*

b. *If $\{x_n\}$ is a sequence from R_+^k with $\lim_{n\to\infty} x_n = x$, and if $x \succ y$, then for all sufficiently large n, $x_n \succ y$. And if $\lim_{n\to\infty} x_n = x$, and $y \succ x$, then for all sufficiently large n, $y \succ x_n$.*

c. *For all $x \in R_+^k$, the sets* $\mathrm{NBT}(x)$ *and* $\mathrm{NWT}(x) = \{y \in R_+^k \,:\, y \succeq x\}$ *are both closed sets. (NWT is a mnemonic for No Worse Than.)*

d. *For all $x \in R_+^k$, the sets* $\mathrm{SBT}(x) = \{y \in R_+^k \,:\, y \succ x\}$ *and* $\mathrm{SWT}(x) = \{y \in R_+^k \,:\, x \succ y\}$ *are both (relatively, in R_+^k) open sets.*[4] *(SBT is a mnemonic for Strictly Better Than, and SWT stands for Strictly Worse Than.)*

The proof of this proposition is left as an exercise, namely Problem 1.11. Providing the proof is a good diagnostic test for whether you understand concepts of open and closed sets and limits in Euclidean spaces. If you aren't sure that you can provide a proof, you should review these basic topological (or, if you prefer, analytical) concepts until you can prove this proposition; I provide a written-out proof in the *Student's Guide.*

The reason for the definition is probably clear:

Proposition 1.15. *If $X = R_+^k$ and preferences $\succeq$ are complete, transitive, and continuous on X, then $\succeq$ can be represented by a utility function u; that is, $u(x) \geq u(y)$ if and only if $x \succeq y$.*

Proof. The proof consists of showing that there is a countable subset X^* of X that does the trick, in the sense of Proposition 1.12. For instance, let X^* be all bundles $x \in X$ all of whose components are rational numbers. There are countably many of these bundles. Suppose $x \succ y$. Look at the line segment that joins x to y; that is, look at bundles that are convex combinations of x and y, or bundles of the form $ax + (1-a)y$ for $a \in [0,1]$. Let $a_1 = \inf\{a \in [0,1] : ax + (1-a)y \succeq x\}$. It is easy to see that $a_1 > 0$; we can put a ball of some size $\epsilon > 0$ around y such that every bundle in the ball is strictly worse than x, and for small enough a, convex combinations $ax + (1-a)y$ all lie within this ball. Let x_1 denote $a_1 x + (1-a_1)y$; I

[4] The set Y is *relatively open* in another set X if Y is the intersection of X and an open set in the "host space" of X. Since $\succeq$ is assumed to be defined on R_+^k, which is a closed set in R^k, we need the notion of "relatively open" here. It is perhaps worth noting, in addition, that while Definition 1.13 and this proposition are constructed in terms of preferences $\succeq$ defined on R_+^k, they both generalize to binary relations defined on more general sets X. But if you are sophisticated enough to know what I have in mind here, you probably already realized that (and just how far we can push this form of the definition and the proposition).

claim that $x_1 \sim x$. To see this, consider the other two possibilities (both of which entail $a_1 \neq 1$, of course): If $x_1 \succ x$, then there is a ball of positive radius around x_1 such that everything in the ball is strictly preferred to x, but this would mean that for some convex combinations $ax + (1 - a)y$ with $a < a_1$, $ax + (1 - a)y \succeq x$, contradicting the definition of a_1. And if $x \succ x_1$, then a ball of positive radius around x_1 will be such that everything in the ball is strictly worse than x. This ball includes all convex combinations $ax + (1 - a)y$ with a a bit bigger than a_1, again contradicting the definition of a_1.

Since $x_1 \sim x$, $x_1 \succ y$. There is a ball of positive radius around x_1 such that everything in the ball is strictly better than y. This includes convex combinations $ax + (1 - a)y$ that have a slightly smaller than a_1. But by the definition of a_1, all such convex combinations must be strictly worse than x. Therefore, we know for some a_2 less than a_1, and for $x_2 = a_2x + (1 - a_2)y$, $x \succ x_2 \succ y$. Now we are in business. We can put a ball of positive radius around x_2 such that everything in the ball is strictly worse than x, and we can put a ball of positive radius around x_2 such that everything in the ball is strictly better than y. Taking the smaller of these two radii, everything z in a ball of that radius satisfies $x \succ z \succ y$. But any ball of positive radius contains bundles all of whose components are rational; hence some $x^* \in X^*$ satisfies $x \succ x^* \succ y$. Done. ■

This proof uses the original definition of continuity. Can you construct a more elegant proof using one of the alternative characterizations of continuity of preferences given in Proposition 1.14?

Proposition 1.15 says that continuous preferences have a utility representation. We might hope for something more, namely that continuous preferences have a utility representation where the function u is itself continuous. We have not proved this and, in fact, the utility functions that we are producing in this chapter are wildly discontinuous. (See Problem 1.12.) In Chapter 2, we see how to get to the more desirable state of affairs, where continuous preferences have a continuous representation.

1.6. Choice from Infinite Sets

The second difficulty that infinite X poses for the standard theory concerns the possibility that $c(A) = \emptyset$ for infinite sets A. One of the two properties of choice functions that characterize the standard model is that $c(A)$ is nonempty for finite sets A; we could simply require this of all sets A; that is, assume away the problem. But this is unwise: Suppose, for instance, that $X = R^2_+$, and define a utility function u by $u(x) = u((x_1, x_2)) = x_1 + x_2$. Consider the subset of X given by $A = [0, 1) \times [0, 1)$; that is, A is the unit square, but with the north and east edges removed. The set $\{x \in A : u(x) \geq u(y) \text{ for all } y \in A\}$ *is* empty; from this semi-open set, open on the "good" sides, no matter what point you choose, there is something better according to u. If we insisted that c is nonempty valued for all A, we wouldn't be consistent with utility maximization for any strictly increasing utility function, at least for sets

A like the one here.

A different approach is to define choice only for some subsets of X and, in particular, to restrict the domain of c to subsets of X for which it is reasonable to assume that choice is nonempty; then strengthen finite nonemptiness by dropping its restriction to finite sets. See Problem 1.15 for more on this approach.

We can leave things as they are: Proposition 1.5 guarantees that if c satisfies finite nonemptiness and choice coherence, then for infinite A,

$$c(A) = \emptyset \quad \text{or} \quad c(A) = c_{\succeq_c}(A),$$

for $\succeq_c$ defined from c. As long as $c(A)$ is not empty, it gives the "right" answer. But still, it would be nice to know that $c(A)$ is not empty for the appropriate sorts of infinite sets A. For instance, if X is, say, R^k_+ and c generates continuous preferences, $c(A)$ should be nonempty for compact sets A, at least. (Why should this be true? See Proposition 1.19.) And in any setting, suppose we have a set A that contains x and that is a subset of $\text{NBT}(x)$. Then $c(A)$ ought to be nonempty, since it should contain x.

These are nice things to have, but they can't be derived from finite nonemptiness and choice coherence; further assumptions will be needed to have them. To demonstrate this, imagine that c is a well-behaved choice function; it satisfies finite nonemptiness and choice coherence and is nonempty for all the "right" sorts of infinite sets A. Modify c, creating c', by letting $c'(A) = \emptyset$ for an arbitrary collection of infinite sets A. For instance, we could let $c'(A) = \emptyset$ for all compact sets that contain some given x^*, or for all sets A that are countably infinite, or for all sets that contain x^* or are countably infinite but not both. When I say "for an arbitrary collection of infinite sets," I mean "arbitrary." Then c' satisfies finite nonemptiness (of course, since it is identical to c for such arguments) and choice coherence. The latter is quite simple: Suppose $x, y \in A \cap B$, $x \in c'(A)$, and $y \notin c'(A)$. Since $c'(A) \neq \emptyset$, $c'(A) = c(A)$; since c satisfies choice coherence, $y \notin c(B)$. If $c'(B) \neq \emptyset$ then $c'(B) = c(B)$ and hence $y \notin c'(B)$. On the other hand, if $c'(B) = \emptyset$, then $y \notin c'(B)$.

There are lots of assumptions we can add to finite nonemptiness and choice coherence, to ensure that c is well-behaved on infinite sets. But perhaps the most general is the simplest. Begin with a choice function c that satisfies finite nonemptiness and choice coherence. Generate the corresponding preference relation $\succeq_c$. Use that preference relation to generate, for each $x \in X$, $\text{NBT}_{\succeq_c}(x)$, where I've included the subscript $\succeq_c$ to clarify that we are beginning with the choice function c. Then,

Assumption 1.16. *If $x \in A \subseteq \text{NBT}_{\succeq_c}(x)$, $c(A) \neq \emptyset$.*

Let me translate this assumption into words: If faced with a choice from some set A that contains an element x, such that everything in A is revealed to be no better than x when pairwise comparisons are made (that is, $x \in c(\{x, y\})$ for all $y \in A$), then the consumer makes *some* choice out of A. (Presumably that choice includes x, but we do not need to assume this; it will be implied by choice coherence.)

Proposition 1.17. *A choice function c that satisfies finite nonemptiness and choice coherence is identical to choice generated by the preferences it generates—that is, $c \equiv c_{\succeq_c}$—if and only if it satisfies Assumption 1.16.*

Proof. Suppose $c \equiv c_{\succeq_c}$ and A is a set with $x \in A \subseteq \mathrm{NBT}_{\succeq_c}(x)$. Then by the definition of $c_{\succeq_c}$, $x \in c_{\succeq_c}(A)$. Since $c \equiv c_{\succeq_c}$, this implies that $c(A)$ is nonempty. (Therefore, in fact, $c(A) = c_{\succeq_c}(A)$ by Proposition 1.5.) Conversely, suppose c satisfies Assumption 1.16. Take any A. Either $c_{\succeq_c}(A) = \emptyset$ or $\neq \emptyset$. In the first case, $c_{\succeq_c}(A) = c(A) = \emptyset$ by Proposition 1.5. In the second case, let x be any element of $c_{\succeq_c}(A)$. Then $x \in A$ and, by the definition of $c_{\succeq_c}$, $A \subseteq \mathrm{NBT}_{\succeq_c}(x)$. By Assumption 1.16, $c(A)$ is nonempty, and Proposition 1.5 implies that $c(A) = c_{\succeq_c}(A)$. ■

An interesting complement to Assumption 1.16 is the following.

Proposition 1.18. *Suppose that c satisfies finite nonemptiness and choice coherence. If A is such that, for every $x \in A$, $A \not\subseteq \mathrm{NBT}_{\succeq_c}(x)$, then $c(A) = \emptyset$.*

That is, the collection of sets in Assumption 1.16 for which it is assumed a choice is made is the largest possible collection of such sets, if choice is to satisfy finite nonemptiness and choice coherence. The proof is implicit in the proof of Proposition 1.17: If $c(A) \neq \emptyset$, then $c(A) = c_{\succeq_c}(A)$ by Proposition 1.5, and for any $x \in c(A) = c_{\succeq_c}(A)$, it is necessarily the case that $A \subseteq \mathrm{NBT}_{\succeq_c}(x)$.

What about properties such as, c is nonempty valued for compact sets A? Let me state a proposition, although I reserve the proof until Chapter 2 (see, however, Problem 1.13):

Proposition 1.19. *Suppose $X = R^k_+$. Take a choice function c that satisfies finite nonemptiness, choice coherence, and Assumption 1.16. If the preferences $\succeq_c$ generated from c are continuous, then for any nonempty and compact set A, $c(A) \neq \emptyset$.*

1.7. Equivalent Utility Representations

Suppose that $\succeq$ has a utility representation u. What can we say about other possible numerical representations?

Proposition 1.20. *If u is a utility-function representation of $\succeq$ and f is a strictly increasing function with domain and range the real numbers, then v defined by $v(x) = f(u(x))$ is another utility-function representation of $\succeq$.*

Proof. This is obvious: If u and v are related in this fashion, then $v(x) \geq v(y)$ if and only if $u(x) \geq u(y)$. ■

The converse to this is untrue: That is, it is possible that v and u both represent $\succeq$, but there is no strictly increasing function $f : R \to R$ with $v(x) = f(u(x))$ for all x. Instead, we have the following result.

Proposition 1.21. *The functions u and v are two utility-function representations of weak preferences $\succeq$ if and only if there is a function $f : R \to R$ that is strictly increasing on the set $\{r \in R : r = u(x)$ for some $x \in X\}$ such that $v(x) = f(u(x))$ for all $x \in X$. Moreover, the function f can be taken to be nondecreasing if we extend its range to $R \cup \{-\infty, \infty\}$.*

Problem 1.14 asks you to prove this.

These results may seem technical only, but they make an important economic point. Utility, at least as far as representing weak preferences is concerned, is purely ordinal. To compare utility differences, as in $u(x)-u(y) > u(y)-u(z)$, and conclude that "x is more of an improvement over y than y is over z," or to compare the utility of a point to some cardinal value, as in $u(x) < 0$, and conclude that "x is worse than nothing," makes no sense.

1.8. Commentary

This ends the mathematical development of the standard models of choice, preference, and utility. But a lot of commentary remains.

The standard model as positive theory

At about this point (if not earlier), many students object to utility maximization. "No one," this objection goes, "chooses objects after consulting some numerical index of goodness. A model that says that consumers choose in this fashion is a bad description of reality and therefore a bad foundation for any useful social science."

Just because consumers don't actively maximize utility doesn't mean that the model of utility-maximizing choice is a bad descriptive or positive model. To suppose that individuals act *as if* they maximize utility is not the same as supposing that they consciously do so. We have proven the following: If choice behavior satisfies finite nonemptiness and the choice coherence, then (as long as something is chosen) choice behavior is *as if* it were preference driven for some complete and transitive weak preference relation. And if the set of objects for which choice is considered is countable or if revealed preferences are continuous, then preference-driven choice is *as if* it were done to maximize a numerical index of goodness.

Utility maximization is advanced as a descriptive or positive model of consumer choice. Direct falsification of the model requires that we find violations of nonemptiness or choice coherence. If we don't, then utility maximization is a perfectly fine as-if model of the choices that are made.

Incomplete data about choice

Unhappily, when we look at the choices of real consumers, we do see some violations of choice coherence and nonemptiness (or, when we ask for preference judgments, of completeness and transitivity). So the standard model *is* empirically falsified. We will discuss this unhappy state of affairs momentarily.

But another problem should be discussed first. The assertion of two paragraphs ago fails to recognize the empirical limitations that we usually face. By this I mean, to justify utility maximization as a model of choice, we need to check the consumer's choice function for every subset A of X, and for each A we need to know all of $c(A)$. (But see Problem 1.15 for a slight weakening of this.) In any real-life situation, we will observe (at best) $c(A)$ for finitely many subsets of X, and we will probably see something less than this; we will probably see for each of a finite number of subsets of X *one* element out of $c(A)$; namely, the object chosen. We won't know if there are other, equally good members of A.

To take seriously the model of utility maximization as an empirically testable model of choice, we must answer the question: Suppose we see $c(A)$, or even one element from $c(A)$, for each of a finite number of subsets A of X. When are these data consistent with utility maximization?

The answer to this question at the level of generality of this chapter is left to you to develop; see Problem 1.16. In Chapter 4, we will provide an answer to a closely related problem, where we specialize to the case of consumer demand given a budget constraint.

Now for the bigger question: In the data we see, how does the model do? What criticisms can be made of it? What does it miss, by how much, and what repairs are possible? Complete answers to these questions would take an entire book, but I can highlight several important categories of empirical problems, criticisms, and alternatives.

Framing

In the models we have considered, the objects or consumption bundles x are presented abstractly, and it is implicitly assumed that the consumer knows x when she sees it. In real life, the way in which we present an object to the consumer can influence how she perceives it and (therefore) what choices she makes. If you find this hard to believe, answer the following question, which is taken from Kahneman and Tversky (1979):

> As a doctor in a position of authority in the national government, you've been informed that a new flu epidemic will hit your country next winter and that this epidemic will result in the deaths of 600 people. (Either death or complete recovery is the outcome in each case.) There are two possible vaccination programs that you can undertake, and doing one precludes doing the other. Program A will save 400 people with certainty. Program B will save no one with probability $1/3$ and 600 with probability $2/3$. Would you choose Program A or Program B?

Formulate an answer to this question, and then try:

> As a doctor in a position of authority in the national govenment, you've been informed that a new flu epidemic will hit your country next winter. To fight this epidemic, one of two possible vaccination programs is to be chosen, and undertaking one program precludes attempting the other. If Program X is

adopted, 200 people will die with certainty. Under Program Y, there is a 2/3 chance that no one will die, and a 1/3 chance that 600 will die. Would you choose Program X or Program Y?

These questions are complicated by the fact that they involve some uncertainty, the topic of Chapter 5. But they make the point very well. Asked of health-care professionals, the modal responses to this pair of questions were: Program A is strictly preferred to B, while Program X is worse than Y. To be clear, the modal health-care professional strictly preferred A to B and strictly preferred X to Y. The point is that Program A is precisely Program X in terms of outcomes, and Programs B and Y are the same. They sound different because Programs A and B are phrased in terms of saving people, while X and Y are phrased in terms of people dying. But within the context of the whole story, A is X and B is Y. Yet (by the modal response) A is better than B, and X is worse than Y. Preference judgments certainly depend on frame.

The way bundles are framed can affect how they are perceived and can influence the individual's cognitive processes in choosing an alternative. Choice coherence rules out the following sort of behavior: A consumer chooses apple pie over cherry if those are the only two choices, but chooses cherry when informed that peach is also available. Ruling this out seems sensible—the ruled-out behavior is silly—but change the objects and you get a phenomenon that is well known to (and used by) mail-order marketers. When, in a mail-order catalog, a consumer is presented with the description of an object, the consumer is asked to choose between the object and her money. To influence the consumer to choose the object, the catalog designer will sometimes include on the same page a slightly better version of the object at a much higher price, or a very much worse version of the object at a slightly lower price. The idea is to convince the consumer, who will compare the different versions of the object, that one is a good deal, and so worthy of purchase. Of course, this strikes directly at choice coherence.

The point is simple: When individuals choose, and when they make pairwise preference judgments, they do so using various processes of perception and cognition. When the choices are complex, individuals simplify, by focusing (for example) on particularly salient features. Salience can be influenced by the frame: how the objects are described; what objects are available; or (in the case of pairwise comparisons) how the two objects compare. This leads to violations of choice coherence in the domain of choice, and intransitivities when consumers make pairwise preference judgments.

Indecision

Indecision attacks a different postulate of the standard model: finite nonemptiness or, in the context of preference, completeness. If asked to choose between 3 cans of beer and 10 bottles of wine or 20 cans of beer and 6 bottles of wine, the consumer might be unable to make a choice; in terms of preferences, she may be unable to say that either bundle is as good or better than the other.

An alternative to the standard model allows the consumer the luxury of indecision. In terms of preferences, for each pair of objects x and y the consumer is assumed to choose one (and only one) of four alternatives:

x is better than y or y is better than x or

x and y are equally good or I can't rank them.

In such a case, expressed strict preference and expressed indifference are taken as primitives, and (it seems most natural) weak preference $\succeq$ is defined not as the absence of strict preference but instead as the union of expressed strict preference and expressed indifference. In the context of such a model, transitivity of strict preference and reflexivity of expressed indifference seem natural, transitivity of expressed indifference is a bit problematic, and negative transitivity of strict preference is entirely problematic: The whole point of this alternative theory is that the consumer is allowed to say that 4 cans of beer and 11 bottles of wine is strictly better than 3 and 10, but both are incomparable to 20 cans of beer and 6 bottles of wine. In terms of choice functions, we would allow $c(A) = \emptyset$—"a choice is too hard"—even for finite sets A, although we could enrich the theory by having another function b on the set of subsets of X, the *rejected set* function, where for any set A, $b(A)$ consists of all elements of A for which something else in A is strictly better.

Inconsistency and probabilistic choice

It is not unknown, empirically, for a consumer to be offered a (hypothetical) choice between x and y and indicate that she will take x, and later to be offered the same hypothetical choice and indicate that she prefers y. This can be an issue of framing or anchoring; something in the series of questions asked of the consumer changes the way she views the relative merits of x and y. Or it can be a matter of indecision; she is not really sure which she prefers and, if forced to make a choice, she does so inconsistently. Or it could be simple inconsistency. Whatever it is, it indicates that when we observe the choice behavior of real consumers, their choices may be stochastic. The standard model assumes that a consumer's preferences are innate and unchanging, which gives the strong coherence or consistency of choice (as we vary the set A) that is the foundation of the theory. Perhaps a more appropriate model is one where we suppose that a consumer is more likely or less likely to choose a particular object depending on how highly she values it "innately," but she might choose an object of lower "utility" if the stars are in the right alignment or for some other essentially random reason.

To deal empirically with the choices of real consumers, one needs a model in which there is uncertainty in how they choose—how can you fit a model that assumes rigid consistency and coherence to data that do not exhibit this?—the likelihood functions just do not work—and so, especially in the context of discrete choice models, microeconomists have developed so-called random utility or probabilistic-choice models. In these models, choice in different contexts exhibits coherence or consistency statistically, but choices in specific instances may, from the perspective of the standard model, appear inconsistent.

The determinants of preference

The standard model makes no attempt to answer the question, Where do preferences come from? Are they something innate to the individual, given (say) genetically? Or are they a product of experience? And if they are a product of experience, is that experience primarily social in character? Put very baldly, does social class determine preferences?

These questions become particularly sharp in two contexts that we reach in this book. The first concerns dynamic choice. If the consumer's experiences color her preferences for subsequent choices, having a model of how this happens is important for models of how the consumer chooses through time. This is true whether her earlier choices are made in ignorance of the process or, more provocatively, if her earlier choices take into account the process. We will visit this issue briefly in Chapter 7, when we discuss dynamic choice theory; it arises very importantly in the context of cooperation and trust in dynamic relationships (and is scheduled for discussion in that context in Volume 3).

These questions are also important to so-called welfare analysis, which we meet in Chapter 8. Roughly, a set of institutions will be "good" if they give consumers things they (the consumers) prefer. Those who see preferences as socially determined often balk at such judgments, especially if, as is sometimes supposed, members of an oppressed class have socially determined tastes or preferences that lead them to prefer outcomes that are "objectively" bad. In this book, we follow the principles of standard (western, or capitalist, or neoclassical) economics, in which the tastes and preferences of the individual consumer are sovereign and good outcomes are those that serve the interests of individual consumers, as those consumers subjectively perceive their own interests. But this is not the only way one can do economics.

The range of choices as a value

To mention a final criticism of the standard model, some economists (perhaps most notably, the Nobel Laureate Amartya Sen) hold that standard theory is too ends-oriented and insufficiently attentive to process, in the following sense: In the standard theory, suppose $x \in c(A)$. Then the individual is equally well off if given a choice from A as if she is simply given x without having the opportunity to choose. But is this correct? If individuals value being able to choose, and there is ample psychological evidence that they do (although there is also evidence that too much choice becomes bad), it might be sensible to use resources to widen the scope of choice available to the individual, even if this means that the final outcome chosen is made a bit worse evaluated purely as an outcome.

I call the standard model by that name because it is indeed the standard, employed by most models in microeconomics. The rise of behavioral economics and the development of random-choice models in empirical work make this less true than it was, say, a decade ago. But still, most models in microeconomics have utility-maximizing or preference-maximizing consumers. Certainly, except in a very few and brief instances, that is what is assumed in the remainder of this book.

My point, then, in raising all these caveats, criticisms, and possible alternatives to the standard model is not to indicate where we are headed. Instead, it is to remind you that the standard model starts with a number of assumptions about human choice behavior, assumptions that are not laws of nature. Too many economists learn the standard model and then invest in it a quasi-religious aura that it does not deserve. Too many economists get the idea that the standard model defines "rational" behavior and any alternative involves irrational behavior, with all the pejorative affect that the adjective "irrational" can connote. The standard model is an extremely useful model. It has and continues to generate all manner of interesting insights into economic (and political, and other social) phenomena. But it is just a model, and when it is time to abandon it, or modify it, or enrich it, one should not hesitate to do so.

Bibliographic Notes

The material in this chapter lies at the very heart of microeconomics and, as such, has a long, detailed, and in some ways controversial history. Any attempt to provide bibliographic references is bound to be insufficient. "Utility" and "marginal utility" were at first concepts advanced as having cardinal significance—the units mean something concrete—but then theory and thought evolved to the position that (more or less) is taken here: Choice is primitive; choice reveals preference; and utility maximization is solely a theoretician's convenient mathematical construct for modeling coherent choice and/or preference maximization. If you are interested in this evolution, Robbins (1998) is well worth reading. Samuelson (1947) provides a classic statement of where economic thought "wound up." Samuelson's development is largely in the context of consumer choice in perfect markets, subject to a budget constraint; that is, more germane to developments in Chapters 4 and 11. As I mentioned within the text of the chapter, to the best of my knowledge, Arrow originated what I have called "choice coherence" and its connection to preference orderings in the abstract setting of this chapter; this was done while writing Arrow (1951a), although the specific results were published in Arrow (1959).

Problems

Most problems associated with the material of this chapter involve proving propositions or constructing counterexamples. Therefore, these problems will give you a lot of drill on your theorem-proving skills. If you have never acquired such skills, most of these problems will be fairly tough. But don't be too quick to give up. (Reminder: Solutions to problems marked with an asterisk [such as *1.1] are provided in the *Student's Guide*, which you can access on the web at the URL http://www.microfoundations1.stanford.edu/student.)

■ *1.1. A friend of mine, when choosing a bottle of wine in a restaurant, claims that he always chooses as follows. First, he eliminates from consideration any bottle that costs more than \$40.

Then he counts up the number of bottles of wine still under consideration (price \$40 or less) on the wine list that come from California, from France, from Italy, from Spain, and from all other locations, and he chooses whichever of these five categories is largest. If two or more categories are tied for largest number, he chooses California if it is one of the leaders, then France, Italy, and Spain, in that order. He says he does this because the more bottles of wine there are on the list, the more likely it is that the restaurant has good information about wines from that country. Then, looking at the geographical category selected, he compares the number of bottles of white, rosé, and red wine in that category that cost \$40 or less, and picks the type (white/rosé/red) that has the most entries. Ties are resolved: White first, then red. He rationalizes this the same way he rationalized geographical category. Finally, he chooses the most expensive bottle (less than or equal to \$40) on the list of the type and geographical category he selected. If two or more are tied, he doesn't care which he gets.

Assume every bottle of wine on any wine list can be uniquely described by its price, place of origin, and color (one of white/rosé/red). The set of all wine bottles so described (with prices \$40 or less) is denoted by X, which you may assume is finite. (For purposes of this problem, the same bottle of wine selling for two different prices is regarded as two distinct elements of X.) Every wine list my friend encounters is a nonempty subset A of X. (He never dines at a restaurant without a wine list.)

The description above specifies a choice function c for all the nonempty subsets of X, with $c(A) \neq \emptyset$ for all nonempty A. (You can take my word for this.) Give an example showing that this choice function doesn't satisfy choice coherence.

■ 1.2. Two good friends, Larry and Moe, wish to take a vacation together. All the places they might go on vacation can be described as elements x of some given finite set X.

Taken as individuals, Larry and Moe are both standard sorts of homo economicus. Specifically, each, choosing singly, would employ a choice function that satisfies finite nonemptiness and choice coherence. Larry's choice function is c_{Larry}, and Moe's is c_{Moe}.

To come to a joint decision, Larry and Moe decide to construct a "joint choice function" c^* by the rule

$$c^*(A) = c_{\text{Larry}}(A) \cup c_{\text{Moe}}(A), \text{ for all } A \subseteq X.$$

That is, they will be happy as a pair with any choice that either one of them would make individually.

Does c^* satisfy finite nonemptiness? Does c^* satisfy choice coherence? To answer each of these questions, you should either provide a proof or a counterexample.

■ *1.3. Disheartened by the result (in Problem 1.2) of their attempt to form a joint choice function, Larry and Moe decide instead to work with their preferences. Let $\succeq_{\text{Larry}}$ be Larry's (complete and transitive) preferences constructed from c_{Larry}, and let $\succeq_{\text{Moe}}$ be Moe's. For their "joint" preferences $\succeq^*$, they define

$$x \succeq^* y \quad \text{if} \quad x \succeq_{\text{Larry}} y \text{ or } x \succeq_{\text{Moe}} y.$$

In words, as a pair they weakly prefer x to y if either one of them does so. Prove that $\succeq^*$ is complete. Show by example that it need not be transitive.

■ 1.4. What is the connection (if any) between c^* from Problem 1.2 and $\succeq^*$ from Problem 1.3?

■ 1.5. Amartya Sen suggests the following two properties for a choice function c:

$$\text{If } x \in c(A) \text{ and } x \in B \subseteq A, \text{ then } x \in c(B). \qquad (\alpha)$$

$$\text{If } y \in B \text{ and, for } B \subseteq A, y \in c(A), \text{ then } c(B) \subseteq c(A). \qquad (\beta).$$

Paraphrasing Sen, (α) says "If the best soccer player in the world is Brazilian, he must be the best soccer player from Brazil." And (β) says: "If the best soccer player in the world is Brazilian, then every best soccer player from Brazil must be one of the best soccer players in the world."

Suppose (for simplicity) that X is finite. Show that choice coherence and finite nonemptiness imply (α) and (β) and, conversely, that (α) and (β) together imply choice coherence.

■ *1.6. Suppose $X = R_+^k$ for some $k \geq 2$, and we define $x = (x_1, \ldots, x_k) \succeq y = (y_1, \ldots, y_k)$ if $x \geq y$; that is, if for each $i = 1, \ldots, k$, $x_i \geq y_i$. (This is known as the Pareto ordering on R_+^k; it plays an important role in the context of social choice theory in Chapter 8.)

(a) Show that $\succeq$ is transitive but not complete.

(b) Characterize $\succ$ defined from $\succeq$ in the usual fashion; that is, $x \succ y$ if $x \succeq y$ and not $y \succeq x$. Is $\succ$ asymmetric? Is $\succ$ negatively transitive? Prove your assertions.

(c) Characterize $\sim$ defined from $\succeq$ in the usual fashion; that is, $x \sim y$ if $x \succeq y$ and $y \succeq x$. Is $\sim$ reflexive? Symmetric? Transitive? Prove your assertions.

■ *1.7. Suppose that $X = R_+^3$, and we define weak preference by $x \succeq y$ if for at least two out of the three components, x gives as much of the commodity as does y. That is, if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then $x \succeq y$ if $x_i \geq y_i$ for two (or three) out of $i = 1, 2, 3$.

(a) Prove that this expression of weak preference is complete but not transitive.

(b) Define strict preference from these weak preferences by the usual rule: $x \succ y$ if $x \succeq y$ but not $y \succeq x$. Show that this rule is equivalent to the following alternative: $x \succ y$ if x gives strictly more than y in at least two components. Is $\succ$ asymmetric? Is $\succ$ negatively transitive?

(Hint: Before you start on the problem, figure out what it means if y is *not* weakly preferred to x in terms of pairwise comparison of the components of x and y. Once you have this, the problem isn't too hard.)

■ 1.8. Prove Proposition 1.7.

■ 1.9. Prove Proposition 1.10.

■ *1.10. Consider the following preferences: $X = [0,1] \times [0,1]$, and $(x_1, x_2) \succeq (x'_1, x'_2)$ if either $x_1 > x'_1$ or if $x_1 = x'_1$ and $x_2 \geq x'_2$. These are called *lexicographic* preferences, because they work something like alphabetical order; to rank any two objects, the first component (letter) of each is compared, and only if those first components agree are the second components considered. Show that this preference relation is complete and transitive but does not have a numerical representation.

■ *1.11. Prove Proposition 1.14.

■ 1.12. Propositions 1.12 and 1.15 guarantee that continuous preferences on R^k_+ have a utility representation. This problem aims to answer the question, Does the construction of the utility representation implicit in the proofs of these two propositions provide a continuous utility function? (The answer is no, and the question really is, What sort of utility function is produced?) Consider the following example: Let $X = [0, 1]$ (not quite the full positive orthant, but the difference won't be a problem), and let preferences be given by $x \succeq y$ if $x \geq y$. The proof of Proposition 1.12 requires a countable subset X^*; so take for this set the set of rational numbers, enumerated in the following order:

$$\left\{0,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{5},\ \frac{2}{5},\ \ldots\right\}.$$

First prove that this set X^* suits; that is, if $x \succ y$, then $x \succeq x^* \succ y$ for some x^* from X^*. Then to the best of your ability, draw and/or describe the function u produced by the proof of Proposition 1.15. This function u is quite discontinuous; can you find a continuous function v that represents $\succeq$?

■ *1.13. (This problem should only be attempted by students who were enchanted by their course on real analysis.) Proposition 1.19 states that if preferences $\succeq_c$ generated from choice function c are continuous on $X = R^k_+$ and if c satisfies finite nonemptiness, choice coherence, and Assumption 1.16, then $c(A) \neq \emptyset$ for all compact sets A. In Chapter 2, this is going to be an easy corollary of a wonderful result known as Debreu's Theorem, which shows that continuous preferences can always be represented by a continuous function; with Debreu's Theorem in hand, proving Proposition 1.19 amounts to remembering that continuous functions on nonempty and compact sets attain their supremum. (Well, not quite. I've included Assumption 1.16 here for a reason. What is that reason?) But suppose we try to prove Proposition 1.19 without Debreu's Theorem. One line of attack is to enlist Proposition 1.14: If preferences are continuous, then for every x, the set $\{y \in X : x \succ y\}$ is (relatively, in X) open. Use this to prove Proposition 1.19.

■ 1.14. Concerning Proposition 1.21, suppose throughout that u and v are two utility representations of (complete and transitive) preference relations $\succeq_u$ and $\succeq_v$ on a given set X.

(a) Show that if $f : R \to R$ is such that $v(x) = f(u(x))$ for all $x \in X$ and if f is strictly increasing on $u(X)$, then $\succeq_u$ and $\succeq_v$ are identical.

(b) Show that if $f : R \to R$ is such that $v(x) = f(u(x))$ for all x in X and if $\succeq_u$ and $\succeq_v$ are identical, then f is strictly increasing on $u(X)$.

(c) Suppose that $X = [0, \infty)$, $v(x) = x$, and

$$u(x) = \begin{cases} x, & \text{for } x \leq 1, \text{ and} \\ x + 1, & \text{for } x > 1. \end{cases}$$

Show that if $f : R \to R$ is such that $v(x) = f(u(x))$ for all x, then f cannot be a strictly increasing function on all of R.

(d) Suppose that $\succeq_u$ and $\succeq_v$ are the same. For each $r \in R$, define $X_r = \{x : u(x) \leq r\}$ and $f(r) = \sup\{v(x) : x \in X_r\}$. Prove that f composed with u is v (that is, $f(u(x)) = v(x)$ for all $x \in X$) and that f is strictly increasing on $u(X)$. Prove that f is nondecreasing on all of R. Why, in the statement of Proposition 1.21, does it talk about how f might have to be extended real-valued (that is, $f(r) = \pm\infty$)?

■ 1.15. As we observed on page 16, one approach to the "problem" that choice on some subsets of a set X might be infinite is to restrict the domain of the choice function c to a collection $\mathcal{A}$ of subsets of X where it is reasonable to assume that $c(A) \neq \emptyset$ for all $A \in \mathcal{A}$. So suppose, for a given set X, we have a collection of nonempty subsets of X, denoted $\mathcal{A}$, and a choice function $c : \mathcal{A} \to 2^X \setminus \emptyset$ with the usual restriction that $c(A) \subseteq A$. Note that we just assumed that $c(A) \neq \emptyset$ for all $A \in \mathcal{A}$! Suppose that c satisfies choice coherence, and suppose that $\mathcal{A}$ contains all one-, two-, and three-element subsets of X. Prove: *For every pair $x, y \in X$, define $x \succeq_c y$ if $x \in c(\{x, y\})$. Then for every $A \in \mathcal{A}$, $c(A) = \{x \in A : x \succeq_c y$ for all $y \in A\}$.* In words, as long as c satisfies choice coherence and $\mathcal{A}$ contains all the one-, two-, and three-element sets (and possibly others in addition), choice out of any $A \in \mathcal{A}$ is choice according to the preferences that are revealed by choice from the one- and two-element subsets of X.[5]

[5] With reference to footnote 1, this is how Mas-Collel et al. tackle the connection between choice and preference.

■ *1.16. Proposition 1.5 provides the testable restrictions of the standard model of preference-driven choice for finite X; it takes a violation of either finite nonemptiness or choice coherence to reject the theory. But this test requires tht we have all the data provided by $c(\cdot)$; that is, we know $c(A)$ in its entirety for every nonempty subset of X.

Two problems arise if we really mean to test the theory empirically. First, we will typically have data on $c(A)$ for only some subsets of X. Second, if $c(A)$ contains more than one element, we may only get to see one of those elements at a time; we see what the consumer chooses in a particular instance, not everything she would conceivably have been happy to choose.

(a) Show that the second of these problems can reduce the theory to a virtual tautology: Assume that when we see $x \in A$ chosen from A, this doesn't preclude the possibility that one or more $y \in A$ with $y \neq x$ is just as good as x. Prove that in this case, no data that we see (as long as the consumer makes a choice from every set of objects) ever contradict the preference-based choice model. (This is a trick question. If you do not see the trick quickly, and you will know if you do, do not waste a lot of time on it.)

(b) Concerning the first problem, suppose that, for some (but not all) subsets $A \subseteq X$, we observe all of $c(A)$. Show that these partial data about the function c may satisfy choice coherence and still be inconsistent with the standard preference-based choice model. (Hint: Suppose X has three elements and you only see $c(A)$ for all two-element subsets of X.)

(c) Continue to suppose that we know $c(A)$ for some but not all subsets of X. Specifically, suppose that we are given data on $c(A)$ for a *finite* collection of subsets of X, namely for $A_1, \ldots, A_n$ for some finite integer n. From these data, define

$$x \succeq^r y \text{ if } x \in c(A_k) \text{ and } y \in A_k, \text{ for some } k = 1, \ldots, n, \text{ and}$$

$$x \succ^r y \text{ if } x \in c(A_k) \text{ and } y \notin c(A_k) \text{ for some } k = 1, \ldots, n.$$

The superscript r is a mnemonic for "revealed." Note that $x \succ^r y$ implies $x \succeq^r y$.

Definition 1.22. *The data $\{c(A_k); k = 1, 2, \ldots, n\}$ violate the **Simple Generalized Axiom of Revealed Preference** (or **SGARP**), if there exists a finite set $\{x_1, \ldots, x_m\} \subseteq X$ such that $x_i \succeq^r x_{i+1}$ for $i = 1, \ldots, m-1$ and $x^m \succ^r x^1$. The data satisfy SGARP if no such set can be produced.*

Proposition 1.23. *If the data $\{c(A_k); k = 1, 2, \ldots, n\}$ violate SGARP, then no complete and transitive $\succeq$ gives rise (in the usual fashion) to these data. If the data satisfy SGARP, then a complete and transitive $\succeq$ can be produced to rationalize the data.*

Prove Proposition 1.23. (This is neither easy nor quick. But it is important for things we do in Chapter 4, so you should at least read through the solution to this problem that is provided in the *Student's Guide*.)

■ 1.17. In this problem, we consider an alternative theory to the standard model, in which the consumer is unable/unwilling to make certain preference judgments. We desire a theory along the following lines: There are two primitive relations that the consumer provides, strict preference $\succ$ and positive indifference $\sim$. The following properties are held to be desirable in this theory:

1. $\succ$ is asymmetric and transitive;
2. $\sim$ is reflexive, symmetric, and transitive;
3. if $x \succ y$ and $y \sim z$, then $x \succ z$; and
4. if $x \sim y$ and $y \succ z$, then $x \succ z$.

For all parts of this problem, assume that X, the set on which $\succ$ and $\sim$ are defined, is a finite set.

(a) Prove that 1 through 4 imply: If $x \succ y$, then neither $y \sim x$ nor $x \sim y$.

(b) Given $\succ$ and $\sim$ (defined for a finite set X) with the four properties listed, construct a weak preference relationship $\succeq$ by $x \succeq y$ if $x \succ y$ or $x \sim y$. Is this weak preference relationship complete? Is it transitive?

(c) Suppose we begin with a primitive weak preference relationship $\succeq$ and define $\succ$ and $\sim$ from it in the usual manner: $x \succ y$ if $x \succeq y$ and not $y \succeq x$, and $x \sim y$ if $x \succeq y$ and $y \succeq x$. What properties must $\succeq$ have so that $\succ$ and $\sim$ so defined have properties 1 through 4?

(d) Suppose we have a function $U : X \rightarrow R$ and we define $x \succ y$ if $U(x) > U(y) + 1$ and $x \sim y$ if $U(x) = U(y)$. That is, indifferent bundles have the same utility; to get strict preference, there must be a "large enough" utility difference between the two bundles. Do $\succ$ and $\sim$ so constructed from U have any/all of the properties 1 through 4?

(e) Suppose we have $\succ$ and $\sim$ satisfying 1 through 4 for a finite set X. Does there exist a function $U : X \rightarrow R$ such that $U(x) = U(y)$ if and only if $x \sim y$ and $U(x) > U(y) + 1$ if and only if $x \succ y$? To save you the effort of trying to prove this, I will tell you that the answer is no, in general. Provide a counterexample.

(f) (Good luck.) Can you devise an additional property or properties for $\succ$ and $\sim$ such that we get precisely the sort of numerical representation described in part d? (This is quite difficult; you may want to ask your instructor for a hint.)