

1

Deformation, Stress, and Conservation Laws

In this chapter, we will develop a mathematical description of deformation. Our focus is on relating deformation to quantities that can be measured in the field, such as the change in distance between two points, the change in orientation of a line, or the change in volume of a borehole strain sensor. We will also review the Cauchy stress tensor and the conservation laws that generalize conservation of mass and momentum to continuous media. Last, we will consider constitutive equations that relate the stresses acting on a material element to the resultant strains and/or rates of strain. This necessarily abbreviated review of continuum mechanics borrows from a number of excellent textbooks on the subject to which the reader is referred for more detail (references are given at the end of the chapter).

If all the points that make up a body (say, a tectonic plate) move together without a change in the shape of the body, we will refer to this as *rigid body motion*. On the other hand, if the shape of the body changes, we will refer to this as *deformation*. To differentiate between deformation and rigid body motion, we will consider the relative motion of neighboring points. To make this mathematically precise, we will consider the displacement \mathbf{u} of a point at \mathbf{x} relative to an arbitrary origin \mathbf{x}_0 (Figure 1.1). Note that vector \mathbf{u} appears in boldface, whereas the components of the vector u_i do not.

Taking the first two terms in a Taylor series expansion about \mathbf{x}_0 ,

$$u_i(\mathbf{x}) = u_i(\mathbf{x}_0) + \left. \frac{\partial u_i}{\partial x_j} \right|_{\mathbf{x}_0} dx_j + \dots \quad i = 1, 2, 3, \quad (1.1)$$

in equation (1.1), and in what follows, summation on repeated indices is implied. The first term, $u_i(\mathbf{x}_0)$, represents a rigid body translation since all points in the neighborhood of \mathbf{x}_0 share the same displacement. The second term gives the relative displacement in terms of the gradient of the displacements $\partial u_i / \partial x_j$, or $\nabla \mathbf{u}$ in vector notation. The partial derivatives $\partial u_i / \partial x_j$ make up the *displacement gradient tensor*, a second rank tensor with nine independent components:

$$\begin{bmatrix} \partial u_1 / \partial x_1 & \partial u_1 / \partial x_2 & \partial u_1 / \partial x_3 \\ \partial u_2 / \partial x_1 & \partial u_2 / \partial x_2 & \partial u_2 / \partial x_3 \\ \partial u_3 / \partial x_1 & \partial u_3 / \partial x_2 & \partial u_3 / \partial x_3 \end{bmatrix}. \quad (1.2)$$

The displacement gradient tensor can be separated into symmetric and antisymmetric parts as follows:

$$u_i(\mathbf{x}) = u_i(\mathbf{x}_0) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) dx_j + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) dx_j + \dots \quad (1.3)$$

If the magnitudes of the displacement gradients are small, $|\partial u_i / \partial x_j| \ll 1$, the symmetric and antisymmetric parts of the displacement gradient tensor can be associated with the small strain and rotation tensors, as in equations (1.4) and (1.5). Fortunately, the assumption of small strain and rotation is nearly always satisfied in the study of active crustal deformation, with the important exceptions of deformation within the cores of active fault zones, where finite strain measures are required.

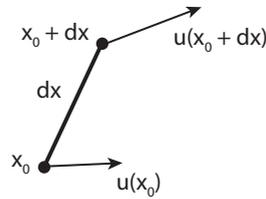


Figure 1.1. Generalized displacement of a line segment \mathbf{dx} . Point \mathbf{x}_0 displaces by amount $\mathbf{u}(\mathbf{x}_0)$, whereas the other end point, $\mathbf{x}_0 + \mathbf{dx}$, displaces by $\mathbf{u}(\mathbf{x}_0 + \mathbf{dx})$.

Define the symmetric part of the displacement gradient tensor, with six independent components, to be the *infinitesimal strain tensor*, ϵ_{ij} :

$$\epsilon_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (1.4)$$

The antisymmetric part of the displacement gradient tensor is defined to be the *rotation*, ω_{ij} , which includes the remaining three independent components:

$$\omega_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right). \quad (1.5)$$

By substituting equations (1.4) and (1.5) into (1.3) and neglecting terms of order $(dx)^2$, we see that the displacement $\mathbf{u}(\mathbf{x})$ is composed of three components—rigid body translation, strain, and rigid body rotation:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathbf{u}(\mathbf{x}_0) + \epsilon \cdot \mathbf{dx} + \omega \cdot \mathbf{dx}, \\ u_i(\mathbf{x}) &= u_i(\mathbf{x}_0) + \epsilon_{ij} dx_j + \omega_{ij} dx_j. \end{aligned} \quad (1.6)$$

We must still show that ϵ_{ij} is sensibly identified with strain and ω_{ij} with rigid body rotation. This will be the subject of the next two sections.

1.1 Strain

To see why ϵ_{ij} is associated with strain, consider a line segment with end points \mathbf{x} and $\mathbf{x} + \mathbf{dx}$ (Figure 1.2). Following deformation, the coordinates of the end points are $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + \mathbf{d}\boldsymbol{\xi}$. The length of the line segment changes from $dl_0 \equiv |\mathbf{dx}|$ to $dl \equiv |\mathbf{d}\boldsymbol{\xi}|$.

The strain can be defined in terms of the change in the squared length of the segment—that is, $dl^2 - dl_0^2$. The squared length of the segment prior to deformation is

$$dl_0^2 = dx_i dx_i = \delta_{ij} dx_i dx_j, \quad (1.7)$$

where the Kronecker delta $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Similarly, the squared length of the segment following deformation is

$$dl^2 = d\xi_m d\xi_m = \delta_{mn} d\xi_m d\xi_n. \quad (1.8)$$

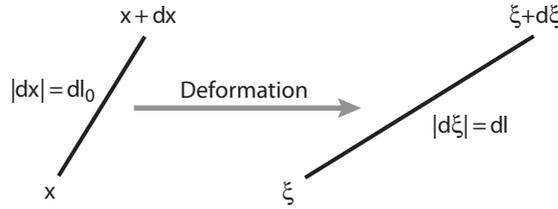


Figure 1.2. Strained line segment. Line segment with endpoints \mathbf{x} and $\mathbf{x} + d\mathbf{x}$ in the undeformed state transforms to a line with endpoints $\boldsymbol{\xi}$ and $\boldsymbol{\xi} + d\boldsymbol{\xi}$ in the deformed state.

We can consider the final coordinates to be a function of the initial coordinates $\xi_i = \xi_i(\mathbf{x})$ or, conversely, the initial coordinates to be a function of the final coordinates $x_i = x_i(\boldsymbol{\xi})$. Adopting the former approach, for the moment, and writing the total differential $d\xi_m$ as

$$d\xi_m = \frac{\partial \xi_m}{\partial x_j} dx_j, \quad (1.9)$$

we find

$$dl^2 = \delta_{mn} \frac{\partial \xi_m}{\partial x_i} \frac{\partial \xi_n}{\partial x_j} dx_i dx_j. \quad (1.10)$$

Thus, the change in the squared length of the line segment is

$$dl^2 - dl_0^2 = \delta_{mn} \frac{\partial \xi_m}{\partial x_i} \frac{\partial \xi_n}{\partial x_j} - \delta_{ij} \quad dx_i dx_j \equiv 2E_{ij} dx_i dx_j. \quad (1.11)$$

E_{ij} , as defined here, is known as the *Green's strain tensor*. It is a so-called *Lagrangian* formulation with derivatives of final position with respect to initial position.

Employing the inverse relationship $x_i = x_i(\boldsymbol{\xi})$, which as described here is an *Eulerian* formulation, yields the so-called *Almansi strain tensor*, e_{ij} :

$$dl^2 - dl_0^2 = \delta_{ij} - \delta_{mn} \frac{\partial x_m}{\partial \xi_i} \frac{\partial x_n}{\partial \xi_j} \quad d\xi_i d\xi_j \equiv 2e_{ij} d\xi_i d\xi_j. \quad (1.12)$$

Note that both equations (1.11) and (1.12) are valid, whether or not $|\partial u_i / \partial x_j| \ll 1$. We do not yet have the strains described in terms of displacements. To do so, note that the displacement is the difference between the initial position and the final position $u_i = \xi_i - x_i$. Differentiating,

$$\frac{\partial \xi_i}{\partial x_j} = \frac{\partial u_i}{\partial x_j} + \delta_{ij}. \quad (1.13)$$

Substituting this into equation (1.11) gives

$$2E_{ij} = \delta_{mn} \left(\frac{\partial u_m}{\partial x_i} + \delta_{mi} \right) \left(\frac{\partial u_n}{\partial x_j} + \delta_{nj} \right) - \delta_{ij}, \quad (1.14)$$

$$= \delta_{mn} \left(\delta_{mi} \frac{\partial u_n}{\partial x_j} + \delta_{nj} \frac{\partial u_m}{\partial x_i} + \frac{\partial u_m}{\partial x_i} \frac{\partial u_n}{\partial x_j} + \delta_{mi} \delta_{nj} \right) - \delta_{ij}. \quad (1.15)$$

Thus, the Green strain is given by

$$E_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} \right). \quad (1.16)$$

The corresponding Almansi (Eulerian) strain is given by

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i} - \frac{\partial u_n}{\partial \xi_i} \frac{\partial u_n}{\partial \xi_j} \right). \quad (1.17)$$

Notice that the third terms in both equations (1.16) and (1.17) are products of gradients in displacement, and are thus second order. These terms are safely neglected as long as the displacement gradients are small compared to unity. Furthermore, in this limit, we do not need to distinguish between derivatives taken with respect to initial or final coordinates. The infinitesimal strain is thus given by equation (1.4).

More generally, the Lagrangian formulation tracks a material element, or a collection of particles. It is an appropriate choice for elasticity problems where the reference state is naturally defined as the unstrained configuration. The Eulerian formulation refers to the properties at a given point in space and is naturally used in fluid mechanics where one could imagine monitoring the velocity, density, temperature, and so on at fixed points in space as the flowing fluid moves by. In contrast, the Lagrangian formulation would track these properties as one moved with the flow. For example, fixed weather stations that measure wind velocity, barometric pressure, and temperature yield an Eulerian description of the atmosphere. Neutrally buoyant weather balloons moving with the wind yield a corresponding Lagrangian description.

Geodetic observing stations are fixed to the solid earth, suggesting that a Lagrangian formulation is sensible. Furthermore, constitutive equations, which refer to a parcel of material, are Lagrangian. However, the familiar Cauchy stress, to be defined shortly, is inherently Eulerian. (Dahlen and Tromp 1998 give a more complete discussion of these issues.) These distinctions turn out not to be important as long as the initial stresses are of the same order as, or smaller than, the incremental stresses (and any rotations are not much larger than the strains). For the most part in this book, we will restrict our attention to infinitesimal strains and will not find it necessary to distinguish between Lagrangian and Eulerian formulations. In the earth, however, the initial stresses, at least their isotropic components, are likely to be large, and a more thorough accounting of the effects of this prestress is required. This arises in chapter 9, where we will consider the role of gravitation in the equilibrium equations.

An important measure of strain is the change in the length of an arbitrarily oriented line segment. In practice, linear extension is measured by a variety of extensometers, ranging from strain gauges in laboratory rock mechanics experiments to field extensometers and laser strainmeters. To the degree that the strain is spatially uniform, geodimeters, or Electronic Distance Meters (EDM) measure linear extension. Consider the change in length of a reference line \mathbf{dx} , with initial length l_0 and final length l . In the small strain limit, we have

$$\begin{aligned} l^2 - l_0^2 &= (l + l_0)(l - l_0) = 2\epsilon_{ij} dx_i dx_j, \\ 2l_0(l - l_0) &\simeq 2\epsilon_{ij} dx_i dx_j, \\ \frac{\Delta l}{l_0} &\simeq \epsilon_{ij} \widehat{dx}_i \widehat{dx}_j, \end{aligned} \quad (1.18)$$

where \widehat{dx}_i are the components of a unit vector in the direction of the baseline \mathbf{dx} . The change in length per unit length of the baseline \mathbf{dx} is given by the scalar product of the strain tensor

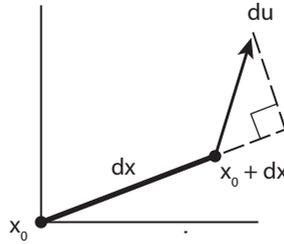


Figure 1.3. Extension and rotation of a line segment. The extension is the dot product of the relative displacement \mathbf{du} with the unit vector in the direction of the line segment $\widehat{\mathbf{dx}} = \mathbf{dx}/|\mathbf{dx}|$. The rotation of the line segment is the cross product of the relative displacement with the same unit vector.

and the unit vectors $\widehat{\mathbf{dx}}$. This can also be written as $\widehat{\mathbf{dx}} \cdot \boldsymbol{\epsilon} \cdot \widehat{\mathbf{dx}}$. This result has a simple geometric interpretation. The change in length of the baseline is the projection of the relative displacement vector \mathbf{du} onto the baseline direction $\Delta l = du_i \widehat{dx}_i$ (Figure 1.3). In an irrotational field, the relative displacement between nearby points is given by $du_i = \epsilon_{ij} dx_j$ (equation [1.6]). Thus, $\Delta l = \epsilon_{ij} \widehat{dx}_i dx_j$. Dividing both sides by l_0 yields the last equation in equation (1.18).

Geodimeters, which measure distance with great accuracy, can be used to determine the change in distance Δl between two benchmarks separated by up to several tens of kilometers. If the baseline lies in a horizontal plane, $\mathbf{dx}^T = (dx_1, dx_2, 0)$, and the strain is spatially uniform between the two end points, the extensional strain is given by

$$\frac{\Delta l}{l_0} = \epsilon_{11} \cos^2 \alpha + 2\epsilon_{12} \cos \alpha \sin \alpha + \epsilon_{22} \sin^2 \alpha, \quad (1.19)$$

where α is the angle between the x_1 axis and the baseline. Equation (1.19) shows how the elongation varies with orientation. In the small strain limit, we can relate the rate of line length change to the *strain rate* by differentiating both sides of equation (1.19) with respect to time; to first-order changes in orientation, $\partial\alpha/\partial t$ can be neglected. The exact relationship corresponding to equation (1.18) valid for finite strain is $(1/l)dl/dt = \widehat{d\xi} \cdot D \cdot \widehat{d\xi}$, (e.g., Malvern 1969), where the *rate of deformation tensor* is defined as

$$D_{ij} \equiv \frac{1}{2} \left(\frac{\partial v_i}{\partial \xi_j} + \frac{\partial v_j}{\partial \xi_i} \right), \quad (1.20)$$

where \mathbf{v} is the instantaneous particle velocity with components v_i . The rate of deformation tensor, in which derivatives are taken with respect to the current configuration is not generally equivalent to the time derivative of a strain tensor in which derivatives are taken with respect to the initial configuration. The distinction is negligible, however, if one takes the initial configuration to be that at the time of the first geodetic measurement, not an unrealizable, undeformed state, so that generally, $|\partial u_i / \partial x_j| \ll 1$.

If the rate of elongation is measured on a number of baselines with different orientations α_i , it is possible to determine the components of the average strain-rate tensor by least squares solution of the following equations:

$$\begin{bmatrix} (1/l_i)dl_i/dt \\ \vdots \end{bmatrix} = \begin{bmatrix} \cos^2 \alpha_i & 2 \cos \alpha_i \sin \alpha_i & \sin^2 \alpha_i \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \dot{\epsilon}_{11} \\ \dot{\epsilon}_{12} \\ \dot{\epsilon}_{22} \end{bmatrix}, \quad (1.21)$$

where the superimposed dot indicates differentiation with respect to time.

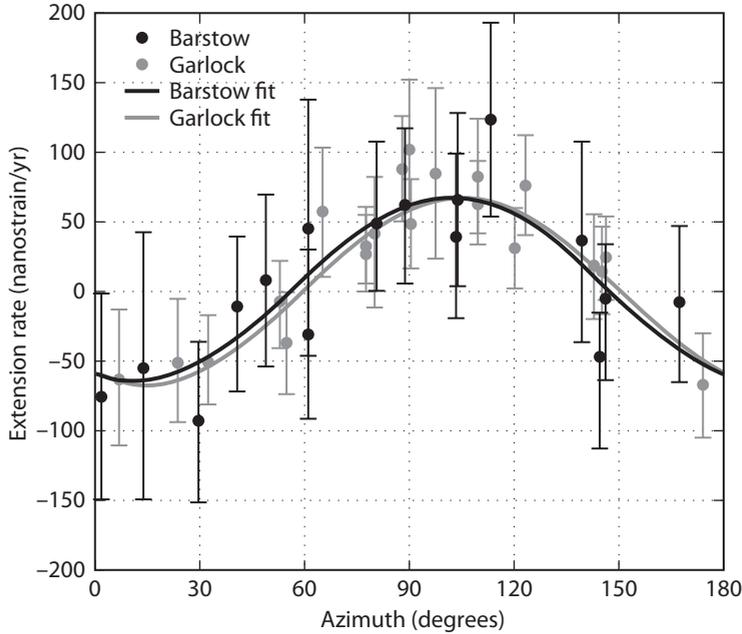


Figure 1.4. Linear extension rate as a function of azimuth for precise distance measurements from the Eastern California Shear Zone. Data from two networks are shown. The Garlock network is located at the eastern end of the Garlock fault. The estimated strain-rate fields for the two networks (shown by the continuous curves) are very similar. From Savage et al. (2001).

In Figure 1.4, linear extension rates vary as a function of azimuth in a manner consistent with a uniform strain-rate field, within rather large errors given the magnitude of the signal. These data were collected in the Southern California Shear Zone using precise distance measuring devices. Data from the U.S. Geological Survey (USGS) Geodolite program, which measured distance repeatedly using an Electronic Distance Meter (EDM), were analyzed by Johnson et al. (1994) to determine the average strain rate in southern California. Figure 1.5 shows the principal strain-rate directions and magnitudes for different polygons assuming that the strain rate is spatially uniform within each polygon.

Borehole dilatometers measure volume change, or dilatational strain, in the earth's crust (Figure 1.6). The volume change of an element is related to the extension in the three coordinate directions. Let V_0 be the initial volume and V be the final volume, and define the *dilatation* as the change in volume over the initial volume $\Delta \equiv (V - V_0)/V_0$. If x_i represent the sides of a small cubical element in the undeformed state, then $V_0 = x_1 x_2 x_3$. Similarly, if ξ_i represent the sides in the deformed state, then $V = \xi_1 \xi_2 \xi_3$. The final dimensions are related to the undeformed dimensions by $\xi_1 = x_1(1 + \epsilon_{11})$, $\xi_2 = x_2(1 + \epsilon_{22})$, and $\xi_3 = x_3(1 + \epsilon_{33})$, so that

$$\Delta = \frac{x_1 x_2 x_3 (1 + \epsilon_{11})(1 + \epsilon_{22})(1 + \epsilon_{33}) - x_1 x_2 x_3}{x_1 x_2 x_3}. \quad (1.22)$$

To first order in strain, this yields $\Delta = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{ii} = \text{trace}(\epsilon)$, which can be shown to be independent of the coordinate system. In other words, the dilatation is an *invariant* of the strain tensor. Note further that $\epsilon_{ij} = \partial u_i / \partial x_j$, so volumetric strain is equivalent to the divergence of the displacement field $\Delta = \nabla \cdot \mathbf{u}$.

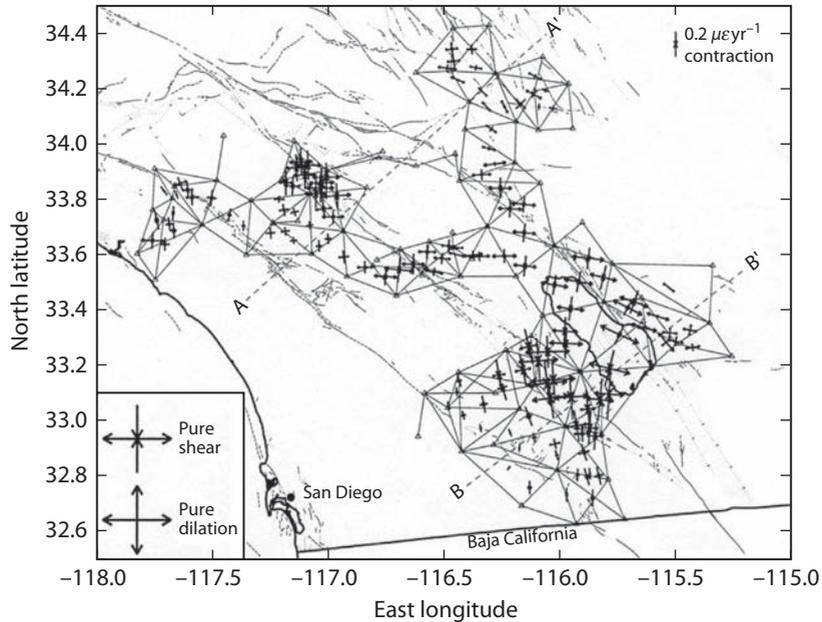


Figure 1.5. Strain-rate distribution in southern California determined from the rate of linear extension measured by the U.S. Geological Survey Geodolite project. Maximum and minimum extension rates are average for the different quadrilaterals. Scale bar is located in the upper-right corner. From Johnson et al. (1994).

We have seen that the three normal strains ϵ_{ij} , $i = j$ represent stretching in the three coordinate directions. The off-diagonal components ϵ_{ij} , $i \neq j$, or shear strains, represent a change in shape. It can be shown that the shear strain is equal to the change in an initially right angle. In section 1.2, you will see how horizontal angle measurements determined by geodetic triangulation can be used to measure crustal shear strain.

1.1.1 Strains in Curvilinear Coordinates

Due to the symmetry of a particular problem, it is often convenient to express the strains in cylindrical polar or spherical coordinates. We present the results without derivation; details are given in, for example, Malvern (1969). In cylindrical polar coordinates, the strains are

$$\begin{aligned}
 \epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\
 \epsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \\
 \epsilon_{zz} &= \frac{\partial u_z}{\partial z}, \\
 \epsilon_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \\
 \epsilon_{rz} &= \frac{1}{2} \left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right), \\
 \epsilon_{z\theta} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right).
 \end{aligned} \tag{1.23}$$

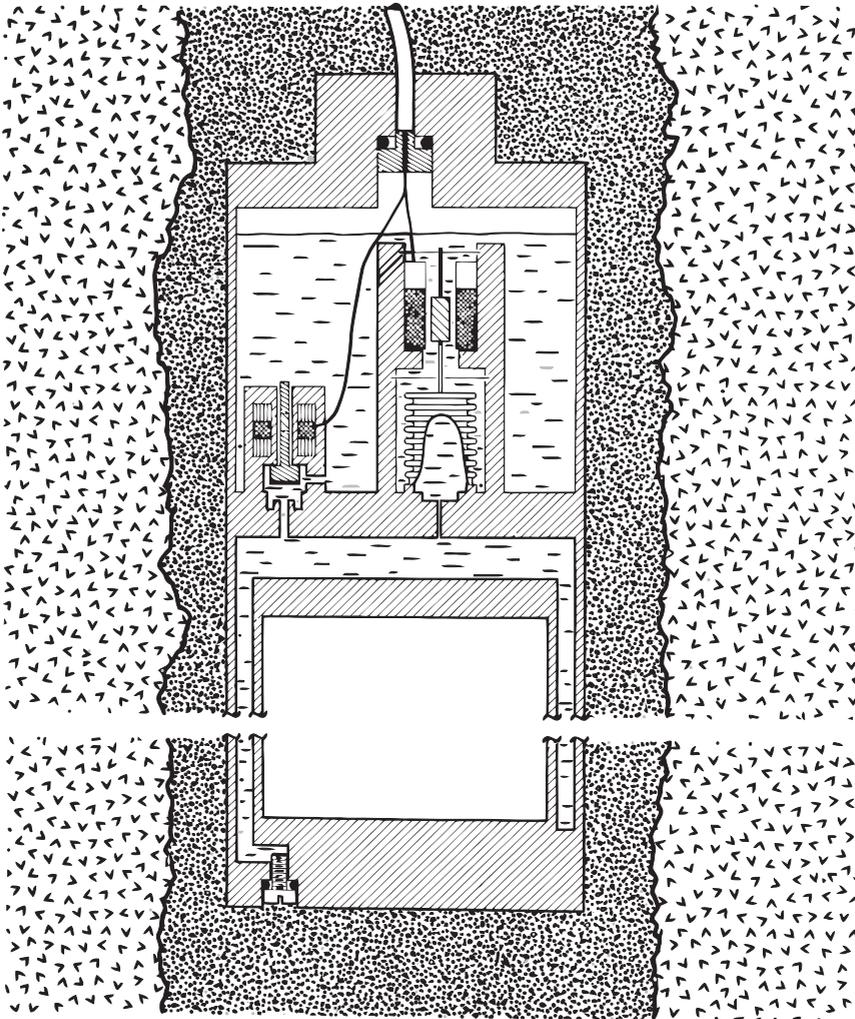


Figure 1.6. Cross section of a Sacks-Evertson borehole dilatometer. The lower sensing volume is filled with a relatively incompressible fluid. The upper reservoir is partially filled with a highly compressible, inert gas and is connected to the sensing volume by a thin tube and bellows. The bellows is attached to a displacement transducer that records the flow of fluid into or out of the sensing volume. As the instrument is compressed, fluid flows out of the sensing volume. The output of the displacement transducer is calibrated to strain by comparing observed solid earth tides to theoretically predicted tides. After Agnew (1985).

In spherical coordinates, the strains are given by

$$\begin{aligned}\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\ \epsilon_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \\ \epsilon_{\phi\phi} &= \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{\cot \theta}{r} u_\theta,\end{aligned}$$

$$\begin{aligned}
\epsilon_{r\theta} &= \frac{1}{2} \left[\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right], \\
\epsilon_{r\phi} &= \frac{1}{2} \left[\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right], \\
\epsilon_{\theta\phi} &= \frac{1}{2} \left[\frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} - \frac{\cot \phi}{r} u_\phi \right].
\end{aligned} \tag{1.24}$$

1.2 Rotation

There are only three independent components in the rotation tensor as defined by equation (1.5), so the infinitesimal rotation may be represented as a vector $\boldsymbol{\Omega}$:

$$\Omega_k \equiv -\frac{1}{2} e_{kij} \omega_{ij}, \tag{1.25}$$

where e_{ijk} is the permutation tensor. e_{ijk} vanishes if any index is repeated, is equal to +1 for e_{123} or for any cyclic permutation of the indices, and is equal to -1 for e_{321} or for any cyclic permutation of the indices. Because it is a rank-three tensor, the permutation tensor is easily distinguished from the Almansi strain. From the definition of ω_{ij} (1.5), we can write the rotation $\boldsymbol{\Omega}$ as

$$\Omega_k = \frac{1}{2} \left[-\frac{1}{2} e_{kij} \frac{\partial u_i}{\partial x_j} + \frac{1}{2} e_{kij} \frac{\partial u_j}{\partial x_i} \right]. \tag{1.26}$$

Now $-e_{kij} = e_{kji}$, so (noting that i and j are dummy indices)

$$\Omega_k = \frac{1}{2} \left[\frac{1}{2} e_{kji} \frac{\partial u_i}{\partial x_j} + \frac{1}{2} e_{kij} \frac{\partial u_j}{\partial x_i} \right], \tag{1.27}$$

$$\Omega_k = \frac{1}{2} e_{kij} \frac{\partial u_j}{\partial x_i}, \tag{1.28}$$

$$\boldsymbol{\Omega} = \frac{1}{2} (\nabla \times \mathbf{u}). \tag{1.29}$$

That is, the rotation vector is half the curl of the displacement field.

Equation (1.25) can be inverted to write ω_{ij} in terms of Ω_k . To do so, we make use of the so-called e - δ identity:

$$e_{kmn} e_{kij} = \delta_{mi} \delta_{nj} - \delta_{mj} \delta_{ni}. \tag{1.30}$$

Multiplying both sides of equation (1.25) by e_{mnk} :

$$\begin{aligned}
e_{mnk} \Omega_k &= -\frac{1}{2} e_{mnk} e_{kij} \omega_{ij}, \\
&= -\frac{1}{2} (\delta_{mi} \delta_{nj} \omega_{ij} - \delta_{mj} \delta_{ni} \omega_{ij}), \\
&= -\frac{1}{2} (\omega_{mn} - \omega_{nm}) = -\omega_{nm}.
\end{aligned} \tag{1.31}$$

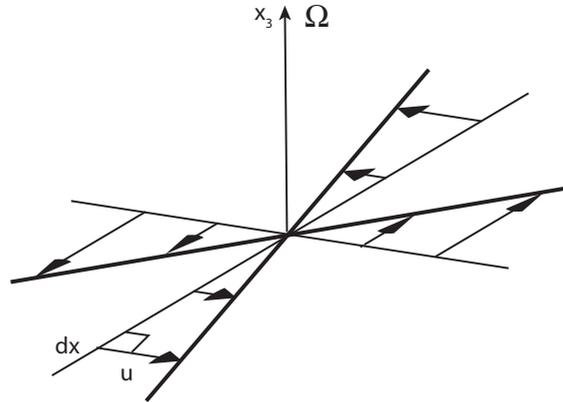


Figure 1.7. Rotational deformation.

We can now see clearly the meaning of the term $\omega_{ij}dx_j$ in the Taylor series expansion (1.6) for the displacements

$$du_i = \omega_{ij}dx_j = -e_{ijk}\Omega_k dx_j = e_{ikj}\Omega_k dx_j, \quad (1.32)$$

which in vector notation is $\mathbf{du} = \boldsymbol{\Omega} \times \mathbf{dx}$. This is illustrated in Figure 1.7.

Consider now how an arbitrarily oriented baseline rotates during deformation. This could represent the change in orientation of a tiltmeter or the change in orientation of a geodetic baseline determined by repeated triangulation surveys. We will denote the rotation of a baseline as Θ and recognize that the rotation of a line segment is different from the rotational component of the deformation $\boldsymbol{\Omega}$. Define the rotation of a line segment Θ as

$$\Theta \equiv \frac{\mathbf{dx} \times \mathbf{d\xi}}{dI_o^2} \quad (1.33)$$

(Agnew 1985). Note that this definition is reasonable since in the limit that the deformation gradients are small, the magnitude of Θ is

$$|\Theta| = \frac{|dI||dI_o|}{dI_o^2} \sin \Theta \sim \Theta, \quad (1.34)$$

where Θ is the angle between \mathbf{dx} and $\mathbf{d\xi}$. In indicial notation, equation (1.33) is written as

$$\Theta_i = \frac{e_{ijk}dx_j d\xi_k}{dx_n dx_n}. \quad (1.35)$$

As before, we write the final position as the initial position plus the displacement $\xi_i = x_i + u_i$, so

$$d\xi_k = \frac{\partial \xi_k}{\partial x_m} dx_m = \frac{\partial u_k}{\partial x_m} + \delta_{km} dx_m. \quad (1.36)$$

Substituting equation (1.36) into (1.35) yields

$$\begin{aligned}
 dx_n dx_n \Theta_i &= e_{ijk} dx_j \frac{\partial u_k}{\partial x_m} + \delta_{km} dx_m, \\
 &= e_{ijk} \frac{\partial u_k}{\partial x_m} dx_j dx_m + e_{ijk} dx_j dx_k, \\
 &= e_{ijk} (\epsilon_{km} + \omega_{km}) dx_j dx_m.
 \end{aligned} \tag{1.37}$$

Note that the term $e_{ijk} dx_j dx_k$ is the cross product of \mathbf{dx} with itself and is therefore zero. We have also expanded the displacement gradients as the sum of strain and rotation, so finally

$$\Theta_i = e_{ijk} (\epsilon_{km} + \omega_{km}) \widehat{dx}_j \widehat{dx}_m, \tag{1.38}$$

where as before the \widehat{dx}_j are unit vectors.

Consider the rotational term in equation (1.38):

$$\begin{aligned}
 e_{ijk} \omega_{km} \widehat{dx}_j \widehat{dx}_m &= -e_{ijk} e_{kmn} \Omega_n \widehat{dx}_j \widehat{dx}_m, \\
 &= -(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \Omega_n \widehat{dx}_j \widehat{dx}_m, \\
 &= \Omega_i - \Omega_j \widehat{dx}_j \widehat{dx}_i,
 \end{aligned} \tag{1.39}$$

where in the first equation we have made use of equation (1.31), and in the last that $\widehat{dx}_m \widehat{dx}_m = 1$.

Thus, the general expression for the rotation of a line segment (1.38) can be written as

$$\Theta_i = e_{ijk} \epsilon_{km} \widehat{dx}_j \widehat{dx}_m + \Omega_i - \Omega_j \widehat{dx}_j \widehat{dx}_i, \tag{1.40}$$

or in vector form as

$$\Theta = \widehat{\mathbf{dx}} \times (\epsilon \cdot \widehat{\mathbf{dx}}) + \Omega - (\Omega \cdot \widehat{\mathbf{dx}}) \widehat{\mathbf{dx}}. \tag{1.41}$$

The strain component of the rotation has a simple interpretation. Recall from Figure 1.3 that the elongation is equal to the dot product of the relative displacement and a unit vector parallel to the baseline. The rotation is the cross product of the unit vector and the relative displacement $du_i = \epsilon_{ij} \widehat{dx}_j$.

To understand the dependence of Θ on Ω geometrically, consider two end-member cases when the strain vanishes. In the first case, the rotation vector is normal to the baseline. In this case, $\Omega \cdot \widehat{\mathbf{dx}} = 0$, so $\Theta = \Omega$; the rotation of the line segment is simply Ω (Figure 1.8). (Notice from the right-hand rule for cross products that the sign of the rotation due to strain is consistent with the sign of Ω .) If, on the other hand, the rotation vector is parallel to the line segment, then $(\Omega \cdot \widehat{\mathbf{dx}}) \widehat{\mathbf{dx}} = |\Omega| \widehat{\mathbf{dx}} = \Omega$ so that $\Theta = \mathbf{0}$. When the rotation vector is parallel to the baseline, the baseline does not rotate at all (Figure 1.8). For intermediate cases, $\mathbf{0} < \Theta < \Omega$.

Consider the vertical-axis rotation of a horizontal line segment $\mathbf{dx}^T = (dx_1, dx_2, 0)$. In this case, equation (1.40) reduces to

$$\Theta_3 = \epsilon_{12} (\cos^2 \alpha - \sin^2 \alpha) + (\epsilon_{22} - \epsilon_{11}) \cos \alpha \sin \alpha + \Omega_3, \tag{1.42}$$

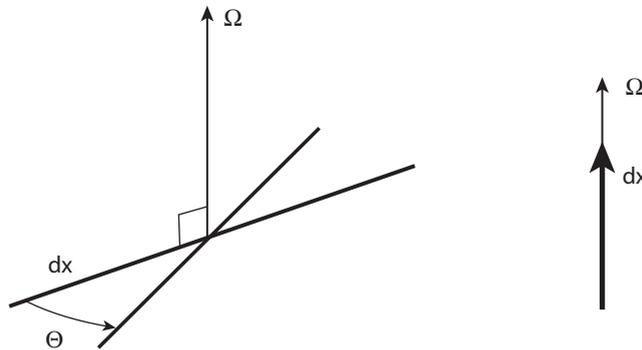


Figure 1.8. Rotation of a line segment. In the left figure, the rotation vector is perpendicular to the baseline, and $\Theta = \Omega$. In the right figure, the rotation vector is parallel to the baseline, and $\Theta = 0$.

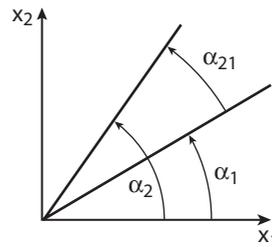


Figure 1.9. Angle α_{21} between two direction measurements α_1 and α_2 .

where α is the angle between the x_1 axis and the baseline. Note that the rotation Θ_3 is equivalent to the change in orientation $d\alpha$. With geodetic triangulation, surveyors were able to determine the angle between three observation points with a *theodolite*. Triangulation measurements were widely made during the late nineteenth and early twentieth centuries before the advent of laser distance measuring devices. Surveying at different times, it was possible to determine the change in the angle. Denoting the angle formed by two baselines oriented at α_1 and α_2 as $\alpha_{21} = \alpha_2 - \alpha_1$ (Figure 1.9), then

$$\delta\alpha_{21} = \delta\alpha_2 - \delta\alpha_1 = \epsilon_{12} (\cos 2\alpha_2 - \cos 2\alpha_1) + \frac{\epsilon_{22} - \epsilon_{11}}{2} (\sin 2\alpha_2 - \sin 2\alpha_1). \quad (1.43)$$

Note that the rotation Ω_3 is common to both $\delta\alpha_1$ and $\delta\alpha_2$ and thus drops out of the difference. The change in the angle thus depends only on the strain field. Defining engineering shear strains in terms of the tensor strains as $\gamma_1 = (\epsilon_{11} - \epsilon_{22})$ and $\gamma_2 = 2\epsilon_{12}$:

$$\delta\alpha_{21} = \frac{1}{2}\gamma_2 (\cos 2\alpha_2 - \cos 2\alpha_1) - \frac{1}{2}\gamma_1 (\sin 2\alpha_2 - \sin 2\alpha_1). \quad (1.44)$$

Notice from Figure 1.10 that γ_1 is a north-south compression and an east-west extension, equivalent to right-lateral shear across planes trending 45 degrees west of north. γ_2 is right-lateral shear across east-west planes.

Determination of shear strain using the change in angles measured during a geodetic survey was first proposed by Frank (1966). Assuming that the strain is spatially uniform over some

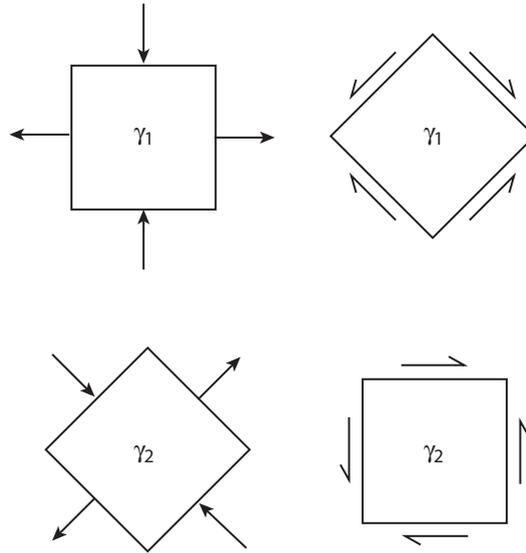


Figure 1.10. Definition of engineering shear strains.

area, it is possible to collect a series of angle changes in a vector of observations related to the two shear strains:

$$\begin{bmatrix} \delta\alpha_{ij} \\ \vdots \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \sin 2\alpha_i - \sin 2\alpha_j & \frac{1}{2} \cos 2\alpha_i - \cos 2\alpha_j \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}. \quad (1.45)$$

Given a system of equations such as this, the two shear strains are easily estimated by least squares. As expected, the angle changes are insensitive to areal strain. A significant advantage of *trilateration* using geodimeters, which measure distance, over triangulation, which measures only angles, is that with trilateration it became possible to determine areal strain as well as shear strain.

Savage and Burford (1970) used Frank's method to determine the strain in small quadrilaterals that cross the San Andreas fault near the town of Cholame, California. The triangulation network is shown in Figure 1.11. The quads are designated by a letter (A to X) from southwest to northeast across the fault. The shear strain estimates with error bars are shown in Figure 1.12. The data show a positive γ_1 centered over the fault, as expected for a locked right-lateral fault. The γ_2 distribution shows negative values on the southwest side of the fault and positive values on the northeast side of the fault. Savage and Burford (1970) showed that these strains were caused by the 1934 Parkfield earthquake, which ruptured the San Andreas just northwest of the triangulation network. The data are discussed in more detail by Segall and Du (1993).

1.3 Stress

While this text focuses on deformation parameters that can be measured by modern geophysical methods (Global Positioning System [GPS], strainmeters, Interferometric Synthetic Aperture Radar [InSAR], and so on), it is impossible to develop physically consistent models of earthquake and volcano processes that do not involve the forces and stresses acting within the earth. It will be assumed that the reader has been introduced to the concept of stress in

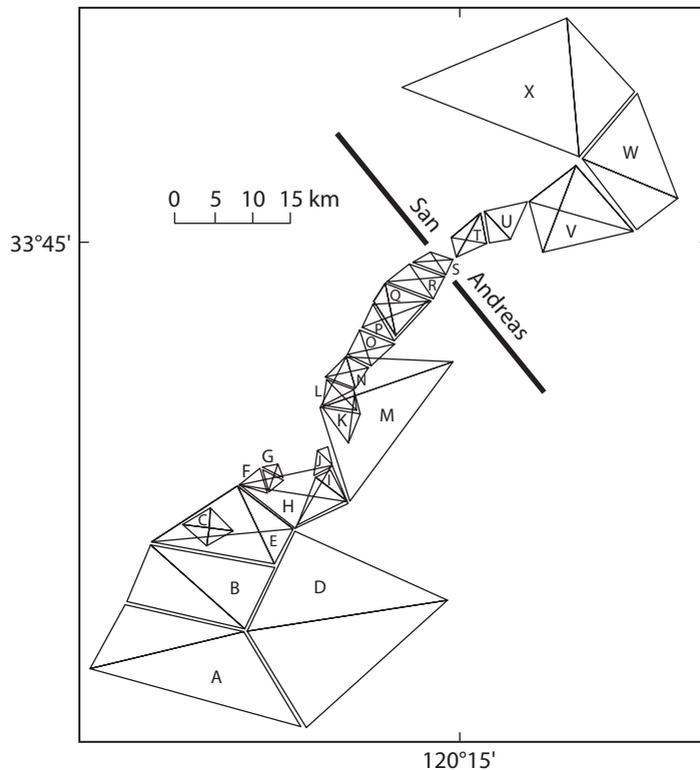


Figure 1.11. Cholame triangulation network.

the mechanics of continuous deformation. We will distinguish between body forces that act at all points in the earth, such as gravity, from surface forces. Surface forces are those that act either on actual surfaces within the earth, such as a fault or an igneous dike, or forces that one part of the earth exerts on an adjoining part. The *traction vector* \mathbf{T} is defined as the limit of the surface force per unit area dF acting on a surface element dA , with unit normal \mathbf{n} as the size of the area element tends to zero (Figure 1.13). The traction thus depends not only on the forces acting within the body, but also on the orientation of the surface elements. The traction components acting on three mutually orthogonal surfaces (nine components in all) populate a second-rank *stress tensor*, σ_{ij} :

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

We will adopt here the convention that the first subscript refers to the direction of the force, whereas the second subscript refers to the direction of the surface normal (Figure 1.14). Also, we will adopt the sign convention that a positive stress is one in which the traction vector acts in a positive direction when the outward-pointing normal to the surface points in a positive direction, and the traction vector points in a negative direction when the unit normal points in a negative direction. Thus, unless otherwise specified, tensile stresses are positive.

Note that this definition of stress makes no mention of the undeformed coordinate system, so this stress tensor, known as the *Cauchy stress*, is a Eulerian description. Lagrangian stress

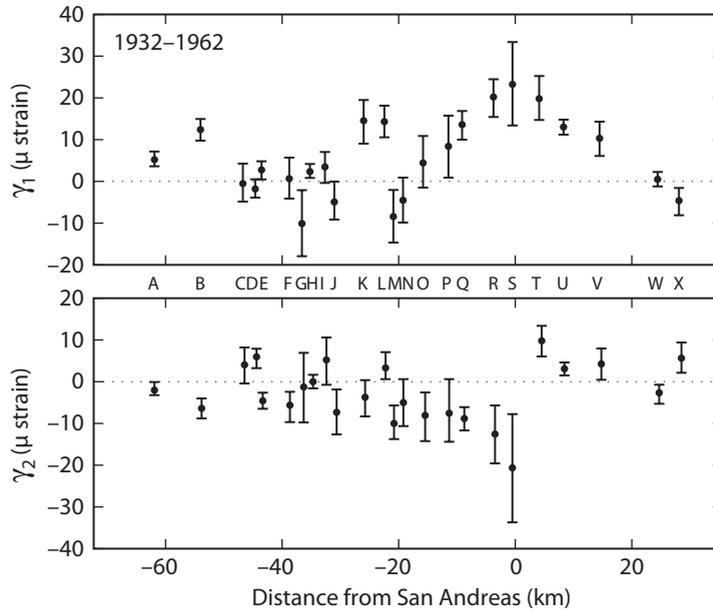


Figure 1.12. Shear strain changes in the Cholame network between 1932 and 1962. After Savage and Burford (1970). The positive values of γ_1 centered on the fault, at quadrilateral S, result from the 1934 Parkfield earthquake northwest of the triangulation network. Slip in this earthquake also caused fault-parallel compression on the east side of the network (positive γ_2) and fault-parallel extension on the west side (negative γ_2).

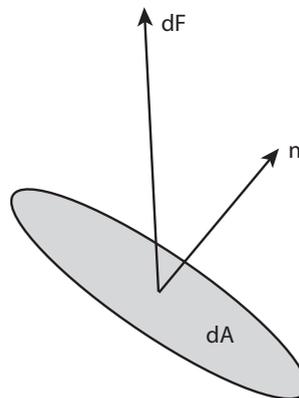


Figure 1.13. Definition of the traction vector. Force dF acts on surface dA with unit normal n . The traction vector is defined as the limit as dA tends to zero of the ratio dF/dA .

tensors, known as *Piola-Kirchhoff stresses*, can be defined where the surface element and unit normal are measured relative to the undeformed state (e.g., Malvern 1969). It is shown in all continuum mechanics texts, and thus will not be repeated here, that in the absence of distributed couples acting on surface elements or within the body of the material, the balance of moments requires that the Cauchy stress tensor be symmetric, $\sigma_{ij} = \sigma_{ji}$. This is not necessarily true for other stress tensors— in particular, the first Piola-Kirchhoff stress is not symmetric.

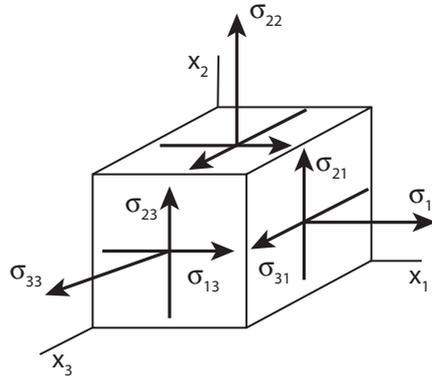


Figure 1.14. Stresses acting on the faces of a cubical element. All of the stress components shown are positive—that is, the traction vector and outward-pointing surface normal both point in positive coordinate directions.

It is also shown in standard texts, by considering a tetrahedron with three surfaces normal to the coordinate axes and one normal to the unit vector \mathbf{n} , that equilibrium requires that

$$T_i = \sigma_{ij}n_j \quad \mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}. \quad (1.46)$$

This is known as *Cauchy's formula*; it states that the traction vector \mathbf{T} acting across a surface is found by the dot product of the stress and the unit normal.

The mean normal stress is equal to minus the pressure $\sigma_{kk}/3 = -p$, the negative sign arising because of our sign convention. The mean stress is a *stress invariant* that is, $\sigma_{kk}/3$ is independent of the coordinate system. It is often useful to subtract the mean stress to isolate the so-called *deviatoric stress*:

$$\tau_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}. \quad (1.47)$$

There are, in fact, three stress invariants. The second invariant is given by $\mathbf{II} = \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj})$. The second invariant of the deviatoric stress, often written as $J_2 = \frac{1}{2}(\tau_{ij}\tau_{ij})$ measures the squared maximum shear stress. It can be shown that

$$J_2 = \frac{1}{6} [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{13}^2. \quad (1.48)$$

The third invariant is given by the determinant of the stress tensor, although it is not widely used.

1.4 Coordinate Transformations

It is often necessary to transform stress or strain tensors from one coordinate system to another—for example, between a local North, East, Up system and a coordinate system that is oriented parallel and perpendicular to a major fault. To begin, we will consider the familiar rotation of a vector to a new coordinate system. For example, a rotation about the x_3 axis from

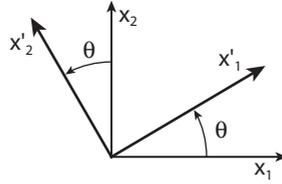


Figure 1.15. Coordinate rotation about the x_3 axis.

the original (x_1, x_2) system to the new (x'_1, x'_2) system (Figure 1.15) can be written in matrix form as

$$\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad (1.49)$$

or compactly as $[\mathbf{v}]' = A[\mathbf{v}]$, where $[\mathbf{v}]'$ and $[\mathbf{v}]$ represent the vector components with respect to the primed and unprimed coordinates, respectively. In indicial notation,

$$v'_i = a_{ij}v_j. \quad (1.50)$$

The elements a_{ij} are referred to as *direction cosines*. It is easy to verify that the inverse transformation is given by $[\mathbf{v}] = A^T[\mathbf{v}]'$:

$$v_i = a_{ji}v'_j. \quad (1.51)$$

The matrix rotating the components of the vector from the primed to the unprimed coordinates is the inverse of the matrix in equation (1.49). Clearly, $a_{ij} = a_{ji}^T$, but also comparing equations (1.50) and (1.51), $a_{ij} = a_{ji}^{-1}$. Therefore, $a_{ji}^T = a_{ji}^{-1}$, so that the transformation is *orthogonal*.

To determine the a_{ij} , let \mathbf{e}_i represent basis vectors in the original coordinate system and \mathbf{e}'_i represent basis vectors in the new coordinates. One can express a vector in either system:

$$\mathbf{x} = x_j\mathbf{e}_j = x'_j\mathbf{e}'_j. \quad (1.52)$$

Taking the dot product of both sides with respect to \mathbf{e}_i , we have

$$\begin{aligned} x_j(\mathbf{e}_j \cdot \mathbf{e}_i) &= x'_j(\mathbf{e}'_j \cdot \mathbf{e}_i), \\ x_j\delta_{ij} &= x'_j(\mathbf{e}'_j \cdot \mathbf{e}_i), \\ x_i &= x'_j(\mathbf{e}'_j \cdot \mathbf{e}_i), \end{aligned} \quad (1.53)$$

so that $a_{ji} = (\mathbf{e}'_j \cdot \mathbf{e}_i)$. A similar derivation gives $a_{ij} = (\mathbf{e}_j \cdot \mathbf{e}'_i)$. For example, for a rotation about the x_3 axis, we have

$$\begin{aligned} a_{11} &= \mathbf{e}_1 \cdot \mathbf{e}'_1 = \cos(\theta), \\ a_{12} &= \mathbf{e}_2 \cdot \mathbf{e}'_1 = \sin(\theta), \\ a_{21} &= \mathbf{e}_1 \cdot \mathbf{e}'_2 = -\sin(\theta), \\ a_{22} &= \mathbf{e}_2 \cdot \mathbf{e}'_2 = \cos(\theta). \end{aligned} \quad (1.54)$$

We can now show that second-rank tensors transform by

$$T'_{ij} = a_{ik}a_{jl}T_{kl}, \quad (1.55)$$

$$T_{ij} = a_{ki}a_{lj}T'_{kl}. \quad (1.56)$$

To do so, consider the second-rank tensor T_{kl} that relates two vectors u and v :

$$v_k = T_{kl}u_l. \quad (1.57)$$

For example, u could be pressure gradient, v could be fluid flux, and T would be the permeability tensor. The components of the two vectors transform according to

$$v'_i = a_{ik}v_k, \quad (1.58)$$

and

$$u_l = a_{jl}u'_j. \quad (1.59)$$

Substituting equations (1.58) and (1.59), into (1.57), we have

$$\begin{aligned} v'_i &= a_{ik}v_k = a_{ik}T_{kl}u_l, \\ v'_i &= a_{ik}T_{kl}a_{jl}u'_j, \\ v'_i &= a_{ik}a_{jl}T_{kl}u'_j. \end{aligned} \quad (1.60)$$

This can be written as $v'_i = T'_{ij}u'_j$, if we define T' in the rotated coordinate system as

$$T'_{ij} \equiv a_{ik}a_{jl}T_{kl}, \quad (1.61)$$

or in matrix notation as

$$[T]' = A[T]A^T, \quad (1.62)$$

where $[T]'$ refers to the matrix of the tensor with components taken relative to the primed coordinates, whereas $[T]$ are the components with respect to the unprimed coordinates. Note that all second-rank tensors, including stress, transform according to the same rules. The derivation can be extended to show that higher rank tensors transform according to

$$\Gamma'_{ijkl\dots} = a_{i\alpha}a_{j\beta}a_{k\gamma}\dots\Gamma_{\alpha\beta\gamma\dots}. \quad (1.63)$$

1.5 Principal Strains and Stresses

The strain tensor can be put into diagonal form by rotating into proper coordinates in which the shear strains vanish:

$$\begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}. \quad (1.64)$$

The strain tensor itself has six independent components. In order to determine the strain, we must specify not only the three principal values, $\epsilon^{(k)}$, $k = 1, 2, 3$, but also the three principal strain directions, so that it still takes six parameters to describe the state of strain. The principal directions are mutually orthogonal (see the following). Thus, while it requires two angles to specify the first axis, the second axis must lie in the perpendicular plane, requiring only one additional angle. The third axis is uniquely determined by orthogonality and the right-hand sign convention.

The principal strains are also the extrema in the linear extension $\Delta l/l_0$. For simplicity, we show this for the case of irrotational deformation, $\omega = 0$; however, see problem 5. In the small strain limit, the linear extension is proportional to the dot product of the relative displacement vector and a unit vector parallel to the baseline:

$$\frac{\Delta l}{l_0} = \frac{du_i dx_i}{dx_k dx_k} = \frac{\mathbf{du} \cdot \mathbf{dx}}{\mathbf{dx} \cdot \mathbf{dx}} \quad (1.65)$$

(equation [1.18]). The maximum and minimum extensions occur when \mathbf{du} and \mathbf{dx} are parallel—that is, $\mathbf{du} = \epsilon \mathbf{dx}$, where ϵ is a constant, or

$$du_i = \epsilon dx_i = \epsilon \delta_{ij} dx_j. \quad (1.66)$$

From equation (1.6), the relative displacement of adjacent points in the absence of rotation is $du_i = \epsilon_{ij} dx_j$. Equating with (1.66) yields

$$\epsilon_{ij} dx_j = \epsilon \delta_{ij} dx_j, \quad (1.67)$$

or rearranging,

$$\epsilon_{ij} - \epsilon \delta_{ij} \quad dx_j = 0. \quad (1.68)$$

Thus, the principal strains ϵ in equation (1.68) are the eigenvalues of the matrix ϵ_{ij} , and the dx_j are the eigenvectors, or principal strain directions. Similarly, the principal stresses and the principal stress directions are the eigenvalues and eigenvectors of the matrix σ_{ij} .

To determine the principal strains and strain directions, we must find the eigenvalues and the associated eigenvectors. Note that the equations (1.68) have nontrivial solutions if and only if

$$\det \quad \epsilon_{ij} - \epsilon \delta_{ij} = 0. \quad (1.69)$$

In general, this leads to a cubic equation in ϵ . The strain tensor is symmetric, which means that all three eigenvalues are real (e.g., Strang 1976). They may, however, be either positive, negative, or zero; some eigenvalues may be repeated. For the distinct (nonrepeated) eigenvalues, the corresponding eigenvectors are mutually orthogonal. With repeated eigenvalues, there exists sufficient arbitrariness in choosing corresponding eigenvectors that they can be chosen to be orthogonal.

For simplicity in presentation, we will examine the two-dimensional case, which reduces to

$$\det \begin{vmatrix} \epsilon_{11} - \epsilon & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} - \epsilon \end{vmatrix} = 0, \quad (1.70)$$

which has the following two roots given by solution of the preceding quadratic:

$$\epsilon = \frac{\epsilon_{11} + \epsilon_{22}}{2} \pm \sqrt{\frac{\epsilon_{11} - \epsilon_{22}}{2}^2 + \epsilon_{12}^2}. \quad (1.71)$$

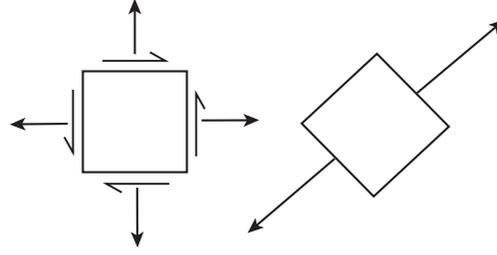


Figure 1.16. Left: Two-dimensional strain state given by equation (1.72), where the arrows denote relative displacement of the two sides of the reference square. Right: Same strain state in the principal coordinate system.

The principal strain directions are found by substituting the principal values back into equation (1.68) and solving for the $dx^{(k)}$. As an example, consider the simple two-dimensional strain tensor illustrated in Figure 1.16:

$$\epsilon_{ij} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \quad (1.72)$$

From equation (1.71), we find that $\epsilon^{(1)} = 2$ and $\epsilon^{(2)} = 0$. Now to find the principal strain direction corresponding to the maximum principal strain, we substitute $\epsilon^{(1)} = 2$ into equation (1.68), yielding

$$\begin{pmatrix} 1-2 & 1 \\ 1 & 1-2 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = 0. \quad (1.73)$$

These are two redundant equations that relate dx_1 and dx_2 :

$$-dx_1 + dx_2 = 0, \quad (1.74)$$

$$dx_1 - dx_2 = 0. \quad (1.75)$$

The eigenvectors are usually normalized so that

$$\mathbf{dx}^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}. \quad (1.76)$$

For this particular example, the principal direction corresponding to the maximum extension bisects the x_1 and x_2 axes in the first and third quadrants (Figure 1.16). We know that the second principal direction has to be orthogonal to the first, but we could find it also by the same procedure. Substituting $\epsilon^{(2)} = 0$ into equation (1.68) yields

$$\begin{pmatrix} 1-0 & 1 \\ 1 & 1-0 \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = 0, \quad (1.77)$$

or $dx_1 = -dx_2$. Again, we normalize the eigenvector so that

$$\mathbf{dx}^{(2)} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \text{ or } \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad (1.78)$$

showing that the second eigenvector, corresponding to the minimum extension, is also 45 degrees from the coordinate axes, but in the second and fourth quadrants.

1.6 Compatibility Equations

Recall the kinematic equations relating strain to displacement (1.4). Given the strains, these represent six equations in only three unknowns, the three displacement components. This implies that there must be relationships between the strains. These relations, known as *compatibility equations*, guarantee that the displacement field be single valued. To be more concrete, this means that the displacement field does not admit gaps opening or material interpenetrating.

In two dimensions, we have

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad (1.79)$$

$$\epsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad (1.80)$$

$$\epsilon_{xy} = \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right). \quad (1.81)$$

Notice that

$$2 \frac{\partial^2 \epsilon_{xy}}{\partial x \partial y} = \frac{\partial^3 u_x}{\partial y^2 \partial x} + \frac{\partial^3 u_y}{\partial x^2 \partial y} = \frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2}. \quad (1.82)$$

An example of a strain field that is not compatible is as follows. Assume that the only nonzero strain is $\epsilon_{xx} = Ay^2$, where A is a constant. Note that such a strain could not occur without a shear strain, nor in fact without gradients in shear strain. This follows from equation (1.82).

In three dimensions, the compatibility equations can be written compactly as:

$$\epsilon_{ij,kl} + \epsilon_{kl,ij} - \epsilon_{ik,jl} - \epsilon_{jl,ik} = 0, \quad (1.83)$$

where we have used the notation $_{,i} \equiv \partial/\partial x_i$. The symmetry of the strain tensor can be used to reduce the 81 equations (1.83) to 6 independent equations. Furthermore, there are three relations among the *incompatibility elements*, so there are only three truly independent compatibility relations. The interested reader is referred to Malvern (1969).

When do you have to worry about compatibility? If you solve the governing equations in terms of stress, then you must guarantee that the implied strain fields are compatible. If, on the other hand, the governing equations are written in terms of displacements, then compatibility is automatically satisfied assuming that the displacements are continuous (true except perhaps along a fault or crack), one can always derive the strains by differentiation.

1.7 Conservation Laws

The behavior of a deforming body is governed by conservation laws that generalize the concepts of conservation of mass and momentum. Before detailing these, we will examine some important relations that will be used in subsequent derivations.

The *divergence theorem* relates the integral of the outward component of a vector \mathbf{f} over a closed surface S to the volume integral of the divergence of the vector over the volume V

enclosed by S :

$$\int_S f_i n_i \, dS = \int_V \frac{\partial f_i}{\partial x_i} \, dV, \quad (1.84)$$

$$\int_S \mathbf{f} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{f} \, dV. \quad (1.85)$$

The time rate of change of a quantity, f , is different in a fixed Eulerian frame from a moving Lagrangian frame. Any physical quantity must be the same, regardless of the system. This equivalence can be expressed as

$$f^L(\mathbf{x}, t) = f^E[\boldsymbol{\xi}(\mathbf{x}, t), t], \quad (1.86)$$

where the superscripts L and E indicate Lagrangian and Eulerian descriptions, respectively. Here f can be a scalar, vector, or tensor. Recall also that \mathbf{x} refers to initial coordinates, whereas $\boldsymbol{\xi}$ refers to current coordinates. Differentiating with respect to time and using the chain rule yields

$$\frac{df^L}{dt} = \frac{\partial f^E}{\partial t} + \frac{\partial f^E}{\partial \xi_i} \frac{\partial \xi_i}{\partial t}. \quad (1.87)$$

The first term on the right-hand side refers to the time rate of change of f at a fixed point in space, whereas the second term refers to the convective rate of change as particles move to a place having different properties. The Lagrangian derivative is often written, where using an Eulerian description, as Df/Dt , and is also known as the *material time derivative*:

$$\frac{Df}{Dt} = \frac{\partial f^E}{\partial t} + \mathbf{v} \cdot \nabla f^E. \quad (1.88)$$

A linearized, time-integrated form of equation (1.88) valid for small displacements:

$$\delta f^L = \delta f^E + \mathbf{u} \cdot \nabla f^E, \quad (1.89)$$

relates the Lagrangian description of a perturbation in f to the Eulerian description (Dahlen and Tromp 1998, equation 3.16). The spatial derivative in equation (1.89) is with respect to the current coordinates, although this distinction can be dropped in a linearized analysis.

We will also state here, without proof, the *Reynolds transport theorem*, which gives the material time derivative of a volume integral—that is, following a set of points in volume V —as the region moves and distorts. The theorem states (e.g., Malvern 1969) that for a property \mathcal{A} ,

$$\frac{D}{Dt} \int_V \rho \mathcal{A} \, dV = \int_V \rho \frac{D\mathcal{A}}{Dt} \, dV, \quad (1.90)$$

where ρ is the Eulerian mass density.

We are now ready to state the important conservation laws of continuum mechanics. Mass must be conserved during deformation. The time rate of change of mass in a current element with volume V , $\int_V (\partial \rho / \partial t) \, dV$, must be balanced by the flow of material out of the element

$\int_V \rho \mathbf{v} \cdot \mathbf{n} dS$, where S is the surface bounding V . Making use of the divergence theorem (1.85):

$$\begin{aligned} \int_V \frac{\partial \rho}{\partial t} dV + \int_S \rho \mathbf{v} \cdot \mathbf{n} dS &= 0, \\ \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} \right] dV &= 0. \end{aligned} \quad (1.91)$$

This must hold for every choice of volume element, which yields the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0. \quad (1.92)$$

Note that for an incompressible material with uniform density, the continuity equation reduces to $\nabla \cdot \mathbf{v} = 0$.

Conservation of linear momentum for the current volume element V surrounded by surface S requires that the rate of change of momentum be balanced by the sum of surface forces and body forces:

$$\frac{D}{Dt} \int_V \rho \mathbf{v} dV = \int_S \mathbf{T} dS + \int_V \rho \mathbf{f} dV. \quad (1.93)$$

Here, \mathbf{T} is the traction acting on the surface S , and \mathbf{f} are body forces per unit mass, such as gravity or electromagnetic forces. The traction is related to the stress via $\mathbf{T} = \boldsymbol{\sigma} \cdot \mathbf{n}$ (equation [1.46]). Application of the divergence theorem (1.85) yields

$$\frac{D}{Dt} \int_V \rho \mathbf{v} dV = \int_V [\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f}] dV. \quad (1.94)$$

Application of the Reynolds transport theorem (1.90) yields

$$\int_V \left[\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} - \rho \frac{D\mathbf{v}}{Dt} \right] dV = 0. \quad (1.95)$$

Again, this must hold for all volume elements so that

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} = \rho \frac{D\mathbf{v}}{Dt}. \quad (1.96)$$

For small deformations, $D\mathbf{v}/Dt = \mathbf{0}$. Furthermore, if accelerations can be neglected, the equations of motion (1.96) reduce to the *quasi-static* equilibrium equations:

$$\begin{aligned} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i &= 0, \\ \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} &= 0. \end{aligned} \quad (1.97)$$

Note that this is an Eulerian description. In chapter 9, we will write Lagrangian equations of motion for perturbations in stress and displacement relative to a gravitationally equilibrated reference state. In this formulation, an advective stress term appears, which from equation (1.89) is of the form $\mathbf{u} \cdot \nabla \sigma^0$, where σ^0 is the stress in the reference state.

1.7.1 Equilibrium Equations in Curvilinear Coordinates

In cylindrical polar coordinates,

$$\begin{aligned} \frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rr}) + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} &= 0, \\ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \sigma_{r\theta}) + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} &= 0, \\ \frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rz}) + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} &= 0. \end{aligned} \quad (1.98)$$

In spherical coordinates,

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r} (2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta} \cot \phi) &= 0, \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r} [3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\phi\phi}) \cot \theta] &= 0, \\ \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r} (3\sigma_{r\phi} + 2\sigma_{\theta\phi} \cot \theta) &= 0. \end{aligned} \quad (1.99)$$

All equations are shown without body forces (Malvern 1969).

1.8 Constitutive Laws

It is worth summarizing the variables and governing equations discussed to this point. There are three displacement components, u_i ; six strains, ϵ_{ij} ; six stresses, σ_{ij} ; and density, ρ , for a total of 16 variables (all of which are generally functions of position and time). In terms of the governing equations, we have the six kinematic equations that relate strain and displacement (1.4), the three equilibrium equations (1.97), and the single continuity equation (1.92), for a total of 10. At this point, we have six more unknowns than equations, which makes sense because we have not yet defined the material behavior. The so-called *constitutive equations* provide the remaining six equations relating stress and strain (possibly including time derivatives) so that we have a closed system of equations. It should be noted that this discussion is limited to a purely mechanical theory. A more complete, thermomechanical description, including temperature, entropy, and conservation of energy, is discussed briefly in chapter 10.

For a linear elastic material, the strain is proportional to stress:

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl}, \quad (1.100)$$

where C_{ijkl} is the symmetric fourth-rank stiffness tensor. Equation (1.100) is referred to as generalized *Hooke's law*. The symmetry of the strain tensor makes it irrelevant to consider cases other than for which the C_{ijkl} are symmetric in k and l . The stress is also symmetric, so

the C_{ijkl} must be symmetric in i and j . Last, the existence of a strain energy function requires $C_{ijkl} = C_{klij}$, as follows. The strain energy per unit volume in the undeformed state is defined as

$$dW = \sigma_{ij} d\epsilon_{ij}, \quad (1.101)$$

so that $\sigma_{ij} = \partial W / \partial \epsilon_{ij}$. From the mixed partial derivatives $\partial^2 W / \partial \epsilon_{ij} \partial \epsilon_{kl}$, it follows that $\partial \sigma_{ij} / \partial \epsilon_{kl} = \partial \sigma_{kl} / \partial \epsilon_{ij}$. Making use of equation (1.100) leads directly to $C_{ijkl} = C_{klij}$.

It can be shown that for an isotropic medium—that is, one in which the mechanical properties do not vary with direction—the most general fourth-rank tensor exhibiting the preceding symmetries is

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \quad (1.102)$$

(e.g., Malvern 1969). Substituting equation (1.102) into (1.100) yields the isotropic form of Hooke's law:

$$\sigma_{ij} = 2\mu \epsilon_{ij} + \lambda \epsilon_{kk} \delta_{ij}, \quad (1.103)$$

where μ and λ are the Lamé constants, μ being the shear modulus that relates shear stress to shear strain.

Other elastic constants, which are convenient for particular applications, can be defined. For example, the bulk modulus, K , relates mean normal stress $\sigma_{kk}/3$ to volumetric strain ϵ_{kk} :

$$\sigma_{kk} = 3K \epsilon_{kk}. \quad (1.104)$$

Alternatively, equation (1.103) can be written in terms of Young's modulus E and Poisson's ratio ν as

$$E \epsilon_{ij} = (1 + \nu) \sigma_{ij} - \nu \sigma_{kk} \delta_{ij}. \quad (1.105)$$

Young's modulus relates stress and strain for the special case of uniaxial stress. Applying stress in one direction generally leads to strain in an orthogonal direction. Poisson's ratio measures the ratio of strain in the orthogonal direction to that in the direction the stress is applied. Notice that while there are five widely used elastic constants μ , λ , K , ν , and E , only two are independent. Sometimes the symbol G is used for shear modulus, but we reserve this for the universal gravitational constant. There are a host of relationships between the five elastic constants—some prove particularly useful:

$$\begin{aligned} K &= \frac{2\mu(1+\nu)}{3(1-2\nu)} = \lambda + \frac{2}{3}\mu, \\ \lambda &= \frac{2\mu\nu}{(1-2\nu)}, \\ E &= 2\mu(1+\nu), \\ \frac{\mu}{\lambda + \mu} &= 1 - 2\nu, \\ \frac{\lambda}{\lambda + 2\mu} &= \frac{\nu}{1-\nu}. \end{aligned} \quad (1.106)$$

The existence of a positive definite strain energy requires that both μ and K are nonnegative. This implies, from the first of equations (1.106), that ν is bounded by $-1 \leq \nu \leq 0.5$. For $\nu = 0.5$, $1/K = 0$, and the material is incompressible.

It is possible to write the compatibility equations in terms of stresses using Hooke's law. Inverting equations (1.103) to write strain in terms of stress and substituting into equation (1.83) yields

$$\sigma_{ij,kl} + \sigma_{kl,ij} - \sigma_{ik,jl} - \sigma_{jl,ik} = \frac{\nu}{1+\nu} (\sigma_{mn,kl}\delta_{ij} + \sigma_{mn,ij}\delta_{kl} - \sigma_{mn,jl}\delta_{ik} - \sigma_{mn,ik}\delta_{jl}) . \quad (1.107)$$

As before, only six of these equations are independent. Setting $k = l$ and summing,

$$\sigma_{ij,kk} + \sigma_{kk,ij} - \sigma_{ik,jk} - \sigma_{jk,ik} = \frac{\nu}{1+\nu} (\sigma_{mn,kk}\delta_{ij} + \sigma_{mn,ij}) . \quad (1.108)$$

This result can be further simplified using the equilibrium equations (1.97), yielding the *Beltrami-Michell compatibility equations*:

$$\sigma_{ij,kk} + \frac{1}{1+\nu} \sigma_{kk,ij} - \frac{\nu}{1+\nu} \sigma_{kk,mm}\delta_{ij} = -(\rho f_i)_{,j} - (\rho f_j)_{,i} . \quad (1.109)$$

A useful relationship can be obtained by further contracting equation (1.109), setting $i = j$ and summing, which gives

$$\nabla^2 \sigma_{kk} = - \frac{1+\nu}{1-\nu} \nabla \cdot (\rho \mathbf{f}) . \quad (1.110)$$

There will be occasion to discuss fluidlike behavior of rocks at high temperature within the earth. The constitutive equations for a linear Newtonian fluid are

$$\sigma_{ij} = 2\eta D'_{ij} - p\delta_{ij} , \quad (1.111)$$

where η is the viscosity, p is the fluid pressure, and D'_{ij} are the components of the deviatoric rate of deformation, defined by equation (1.20) and

$$D'_{ij} = D_{ij} - \frac{D_{kk}}{3} \delta_{ij} . \quad (1.112)$$

Equation (1.111) is appropriate for fluids that may undergo arbitrarily large strains. In chapter 6, we will consider viscoelastic materials and implicitly make the approximation that displacement gradients remain small so that the rate of deformation tensor can be approximated by the strain rate, $D_{ij} \approx \dot{\epsilon}_{ij}$. Constitutive equations for viscoelastic and poroelastic materials will be introduced in chapters 6 and 10.

We will now return to the question of the number of equations and the number of unknowns for an elastic medium. If it is desirable to explicitly consider the displacements— for example, if the boundary conditions are given in terms of displacement— then the kinematic equations (1.4) can be used to eliminate the strains from Hooke's law, leaving 10 unknowns (three displacement, six stresses, and density) and 10 equations (three equilibrium equations, six constitutive equations, and continuity). If, on the other hand, the boundary conditions are given in terms of tractions, it may not be necessary to consider the displacements explicitly. In this case, there are 13 variables (six stresses, six strains, and density). The governing equations are: equilibrium (three), the constitutive equations (six), continuity (one), and compatibility (three), leaving 13 equations and 13 unknowns.

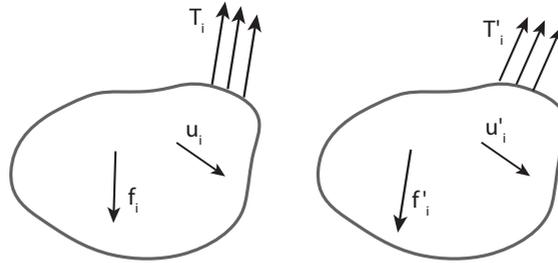


Figure 1.17. Two sets of body forces and surface tractions used to illustrate the reciprocal theorem. Left: Unprimed system. Right: Primed system.

In general, solving 13 coupled partial differential equations can be a formidable task. Fortunately, we can begin in the next chapter with a far simpler two-dimensional problem that does a reasonably good job of describing deformation adjacent to long strike-slip faults.

1.9 Reciprocal Theorem

The *Betti-Rayleigh reciprocal theorem* is an important tool from which many useful results can be derived. As one example, in chapter 3, the reciprocal theorem is employed to derive *Volterra's formula*, which gives the displacements due to slip on a general planar discontinuity in an elastic medium. We will present a derivation of the reciprocal theorem here, following, for example, Sokolnikoff (1983). Imagine two sets of self-equilibrating body forces and surface tractions, f_i , T_i and f'_i , T'_i , acting on the same body (Figure 1.17). The unprimed forces and tractions generate displacements, u_i , whereas the primed generate u'_i , both unique to within rigid body motions. We will consider the work done by forces and tractions in the unprimed state acting through the displacements in the primed state:

$$\int_A T_i u'_i dA + \int_V f_i u'_i dV = \int_A T'_i u_i dA - \int_V \sigma_{ij,j} u'_i dV, \quad (1.113)$$

where we have rewritten the volume integral making use of the quasi-static equilibrium equations (1.97). Use of the divergence theorem on the first integral on the right-hand side leads to

$$\begin{aligned} \int_A T_i u'_i dA + \int_V f_i u'_i dV &= \int_V (\sigma_{ij} u'_i)_{,j} dV - \int_V \sigma_{ij,j} u'_i dV, \\ &= \int_V \sigma_{ij} u'_{i,j} dV, \\ &= \int_V C_{ijkl} u_{k,l} u'_{i,j} dV, \end{aligned} \quad (1.114)$$

the last step resulting from application of Hooke's law (1.100). You have seen that the C_{ijkl} have the symmetry $C_{ijkl} = C_{klij}$, implying that the left-hand side of equation (1.114) is symmetric with respect to primed and unprimed coordinates. In other words, we can interchange the primed and unprimed states:

$$\int_A T_i u'_i dA + \int_V f_i u'_i dV = \int_A T'_i u_i dA + \int_V f'_i u_i dV. \quad (1.115)$$

Equation (1.115) is one form of the reciprocal theorem. It states that the work done by the primed body forces and tractions acting through the unprimed displacements is equal to the work done by the unprimed body forces and tractions acting through the primed displacements.

1.10 Problems

- (a) Derive expression (1.12) for the Almansi strain.
(b) Show that in terms of the displacement u_i , the Almansi strain tensor is given by equation (1.17).
- Consider a line segment $\widehat{\mathbf{dx}}$ that lies in the (x_1, x_2) plane. Show that the elongation is given by

$$\frac{\Delta l}{l} = \epsilon_{11} \cos^2 \theta + 2\epsilon_{12} \cos \theta \sin \theta + \epsilon_{22} \sin^2 \theta,$$

where θ is the orientation of the baseline with respect to the x_1 axis. Given the strain

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},$$

graph $\Delta l/l$ between 0 and π . What is the meaning of the maximum and minimum?

- Prove that a pure rotation involves no extension in any direction.
Hint: Replace ω_{ij} for ϵ_{ij} in equation (1.18), and note the antisymmetry of ω_{ij} .
- Consider the rotation of a line segment due to strain. Derive an expression for Θ_3 in terms of the strain components. For simplicity, take \mathbf{dx} to lie in the (x_1, x_2) plane:

$$\mathbf{dx}^T = (dx_1, dx_2, 0).$$

Draw diagrams showing that the sign of the rotation you compute for each strain component makes sense physically.

- Show formally that the principal strains are the extrema in the linear extension. Recall from equation (1.18) that for small deformations,

$$\frac{\Delta l}{l_0} = \frac{\epsilon_{ij} x_i x_j}{x_k x_k}.$$

To find the extrema, set the gradient of $\Delta l/l_0$ with respect to x_n to zero. First show that the gradient is given by

$$\frac{\partial (\Delta l/l_0)}{\partial x_n} = \frac{2\epsilon_{in} x_i}{x_k x_k} - \frac{\epsilon_{ij} x_i x_j}{x_k x_k} \frac{2x_n}{x_m x_m}. \quad (1.116)$$

Notice that the quantity in parentheses, $\epsilon_{ij} x_i x_j / x_k x_k$, is a scalar. It is the value of $\Delta l/l_0$ at the extremum, which we denote ϵ . Show that the condition that the gradient (1.116) vanish leads to

$$(\epsilon_{in} - \epsilon \delta_{in}) x_i = 0,$$

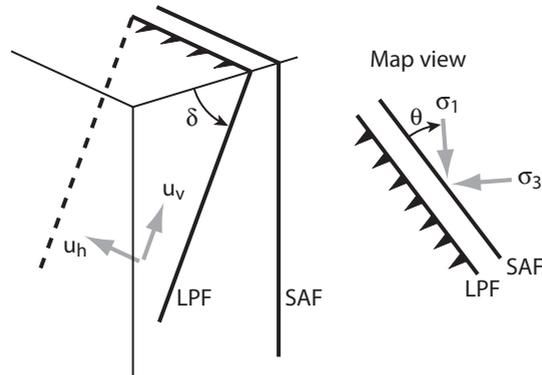


Figure 1.18. Loma Prieta fault geometry. SAF = San Andreas fault; LPF = Loma Prieta fault.

thus demonstrating that the maximum and minimum elongations are the eigenvalues of the strain.

6. In two dimensions, show that the principal strain directions θ are given by

$$\tan 2\theta = \frac{2\epsilon_{12}}{\epsilon_{11} - \epsilon_{22}},$$

where θ is measured counterclockwise from the x_1 axis.

Hint: if you use an eigenvector approach, the following identity may be useful:

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}.$$

However, other methods also lead to the same result.

7. This problem explores Hooke's law for two-dimensional deformation. For *plane strain* deformation, all strains in the 3-direction are assumed to be zero, $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$.

(a) Show that for this case, $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$.

(b) Show that for plane strain, Hooke's law (1.103) for an isotropic medium reduces to

$$2\mu\epsilon_{\alpha\beta} = \sigma_{\alpha\beta} - \nu(\sigma_{11} + \sigma_{22})\delta_{\alpha\beta} \quad \alpha, \beta = 1, 2. \quad (1.117)$$

(c) Show that for plane stress conditions $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$, Hooke's law reduces to the same form as in (b), but with ν replaced by $\nu/(1 + \nu)$.

8. The M 6.9 1989 Loma Prieta, California, earthquake occurred with oblique slip on a fault that strikes parallel to the San Andreas fault (SAF) and dips to the southwest approximately 70 degrees (Figure 1.18). We believe that the SAF last slipped in a purely right-lateral event, in 1906, on a vertical fault plane. The problem is to find whether a uniform stress state could drive pure, right-lateral, strike slip on a vertical fault *and* the observed oblique slip on the parallel Loma Prieta fault (LPF).

The important points to note are that the SAF is vertical; the LPF dips to the southwest ($\delta \approx 70$ degrees); slip in 1906 was pure, right-lateral, strike slip; and slip in 1989 was $u_h = \beta u_v$, where u_h is horizontal displacement, u_v is vertical displacement, and $\beta \approx 1.4$.

Assume that the fault slips in the direction of the shear stress acting on the fault plane, so that $\tau_h = \beta \tau_v$, where τ_h and τ_v are, respectively, the horizontal and vertical components of the resolved shear stress on the LPF. You can assume that the intermediate principal stress, σ_2 , is vertical.

Hint: Solve for τ_h and τ_v in terms of the principal stresses. The constraint that $\tau_h = \beta \tau_v$ will allow you to find a family of solutions (stress states) parameterized by

$$\phi \equiv \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}.$$

Note that

$$\begin{aligned}\phi &\rightarrow 1 \quad \text{as } \sigma_2 \rightarrow \sigma_1, \\ \phi &\rightarrow 0 \quad \text{as } \sigma_2 \rightarrow \sigma_3,\end{aligned}$$

and thus $0 < \phi < 1$. Express the family of admissible stress states in terms of ϕ as a function of θ .

9. Use the strain-displacement equations (1.4) and the isotropic form of Hooke's law (1.103) to show that the quasi-static equilibrium equations (1.97) can be written as

$$\mu \nabla^2 u_i + (\lambda + \mu) \frac{\partial u_{k,k}}{\partial x_i} + \rho f_i = 0. \quad (1.118)$$

These equations are known as the *Navier form* of the equilibrium equations in elasticity theory.

10. Consider a solid elastic body with volume V and surface S with unit normal n_i . Traction T_i are applied to the S ; in general, they are nonuniform and may be zero in places. We want to find the volume change

$$\Delta V = \int_S u_i n_i ds,$$

where u_i is the displacement. Use the reciprocal theorem to show that the volume change can be computed by

$$\Delta V = \frac{1}{3K} \int_S T_i x_i ds,$$

where K is the bulk modulus.

1.11 References

- Agnew, D. C. 1985. Strainmeters and tiltmeters. *Reviews of Geophysics* **24**(3), 579–624.
- Dahlen, F. A., and J. Tromp. 1998. *Theoretical global seismology*. Princeton, NJ: Princeton University Press.
- Frank, F. C. 1966. Deduction of earth strains from survey data. *Bulletin of the Seismological Society of America* **56**, 35–42.
- Johnson, H. O., D. C. Agnew, and F. K. Wyatt. 1994. Present-day crustal deformation in Southern California. *Journal of Geophysical Research* **B 99**(12), 23,951–23,974.
- Malvern, L. E. 1969. *Introduction to the mechanics of continuous media*. Englewood Cliffs, NJ: Prentice Hall.

- Savage, J. C., and R. O. Burford. 1970. Accumulation of tectonic strain in California. *Bulletin of the Seismological Society of America* **60**, 1877–1896.
- Savage, J. C., W. Gan, and J. L. Svarc. 2001. Strain accumulation and rotation in the Eastern California Shear Zone. *Journal of Geophysical Research* **106**, 21,995–22,007.
- Segall, P., and Y. Du. 1993. How similar were the 1934 and 1966 Parkfield earthquakes? *Journal of Geophysical Research* **98**, 4527–4538.
- Sokolnikoff, I. S. 1983. *Mathematical theory of elasticity*. New York: McGraw-Hill; reprinted Malabar, FL: Robert E. Krieger Publishing, pp. 390–391.
- Strang, G. 1976. *Linear algebra and its applications*. New York: Academic Press, pp. 206–220.