Chapter 1

HARD LINES

> I have been obliged to confide the greater portion of the theoretical part of the present work to some mathematical assistants, whose algebra has, I fear, sometimes risen to a needless luxuriance, and in whose superfine speculations the engineer may perhaps discern the hand of a tyro.
>
> Bourne (1846)

There are many convincing ways to justify a result. A scientist gathers *evidence* by undertaking a systematic experiment. One can undertake mathematical experiments, such as a sequence of calculations. Another kind of experiment is to draw a picture, be it on paper or sketched in the sand with a stick. Few, if any, mathematicians would now accept a picture as a valid *proof* but sketches do provide us with the simplest and most direct form of *mathematical experiment*. When undertaking such an experiment we ask you to think of it as representing a whole class of similar ones. What can you change without removing the essence of what you are doing? What must stay the same? And then, of course, decide how you can justify this.

So that we might be definite in the difference between a mathematical proof and an illustration, let us begin with an example. This is a theorem from Euclid's *Elements*, book III, part of proposition 31 (Euclid 1956, volume 2, p. 61), which is encountered early in school geometry connected with a circle.

Theorem 1.1. *Take any circle, and any diameter (from* A *to* B*, say), and any other point* P *on the circle. Then the triangle* APB *is a right-angled triangle, with right angle at* P*. (This is illustrated in figure 1.1.)*

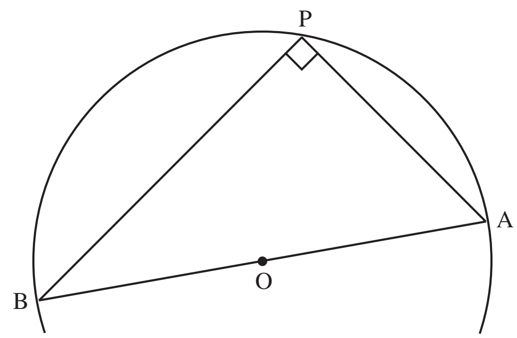


Figure 1.1. Illustrating theorem 1.1.

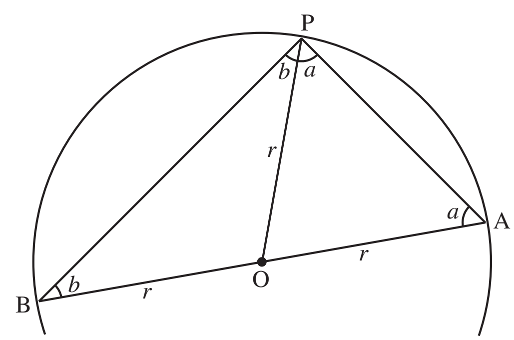


Figure 1.2. Illustrating the proof of theorem 1.1.

Proof. To the diagram add the radius OP. Then OP = OA and OP = OB, so that we have two isosceles triangles AOP and BOP. The base angles of isosceles triangles are equal. Call them a and b, respectively (see figure 1.2). Since the interior angles of any triangle sum to 180°, we have for our triangle APB that $2a+2b = 180°$. That is to say $a + b = 90°$. Hence the angle APB is a right angle. □

This is a surprising result with a mathematical proof which is beautiful and elegant. It removes any doubt as to the truth of the theorem, but to illustrate and motivate what follows we would like to encourage you to make a paper illustration of this theorem, to confirm the result physically. That is, draw a circle, a diameter and chose another point. Cut out the resulting triangle. How does one confirm that the angle is a right angle? One reliable

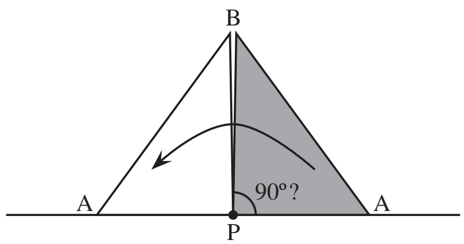


Figure 1.3. Checking that an angle is indeed a right angle.

method is to take a fresh sheet of paper and draw a straight line. Mark a point on this line and match it up with the point P, with the side PA on the line itself. Now mark the position of the point B, and hence the line PB. Flip the triangle out of the plane of the paper and place the side PA onto the line again, but this time on the other side. Now look at the position of B. This will remain unchanged if and only if the angle APB is a right angle. This procedure is reliable in practice and is illustrated in figure 1.3. We can also cut along the radius OP and confirm that the two resulting triangles are isosceles by folding. This also gives us the opportunity to confirm the steps in the proof which assert that the base angles in an isosceles triangle (a and b in our case) are equal. For our purposes in proving theorem 1.1, it does not matter how many degrees are in a triangle, as long as every triangle has the same quantity of internal angle. This can be physically demonstrated in a final destructive act, by ripping off the corners of a triangle and arranging them in a straight line.

Actually, we need to backtrack a little. While the above procedure confirms that the angle is indeed a right angle, we need to think carefully about some assumptions we have taken for granted.

There is no doubt that a pair of compasses will draw a circle. Since a circle is *defined* to be the set of points a given distance from the centre, the shape you draw with a good pair of compasses is a circle. However, we need to be a little more careful when we draw the diameter and radius. To do this we need to draw a *straight line*. How does one do this? If you were tempted to reach for a ruler, then consider the problem of making a ruler in the first place. We shall explain how to check the straightness of a ruler at the end of this chapter, but the problem of drawing a *straight line* will have to wait until chapter 2.

To illustrate further the spirit of trial and experiment in mathematics, a very simple model of this geometry can be made on a wooden board with two pins and a standard set square from a geometry set. Draw a semicircle with AB as diameter, and stick the pins firmly and vertically at A and B. Slide the 90° corner against these pins and observe that it follows the semicircle exactly, just as was expected. Now turn the square and repeat the exercise with the 30°, 60° and 45° corners, all of which can be found on various set squares. What is the curve along which the corner moves?

All the curves look like arcs of circles of different radii, with AB no longer a diameter: they are all larger. Next, make a 'square' from a piece of card or plywood with a corner angle greater than 90°. The resulting arc still looks circular and of much larger radius than any of the others.

Two questions now arise. Are the arcs truly circular and how might we complete them to form the circles, if in fact they are all circular arcs? Figure 1.4 shows the arc produced by a corner of angle 45°. By symmetry, if it is circular the centre must lie on the perpendicular bisector of the line AB, which we might as well take to be the y-axis. If we do indeed have such a circle, then using theorem 1.1, we draw AQ to be perpendicular to AP. Where it crosses the y-axis we mark Q. We push in pins at P and at Q, and sliding a 90° square against these should produce the required circular arc.

Without detailed working, if angle APB is t° and AB $= 2l$, the equation of the circle is

$$x^2 + \left(y - \frac{l}{\tan(t)}\right)^2 = \frac{l^2}{\sin^2(t)}.$$

Having drawn this circle we can now justify why the point P, constrained to move by pins at A and B, really does move around it using a generalization of theorem 1.1. This generalization says that if we take a chord of a circle, such as AB, and any point on the circle, such as P of figure 1.1, then the angle APB is constant.

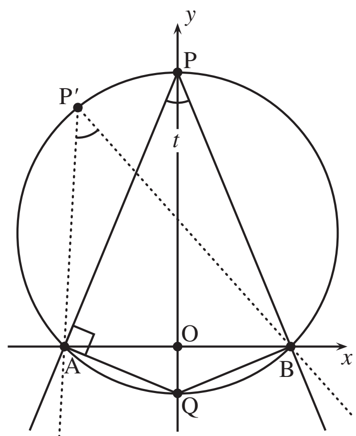


Figure 1.4. Extending theorem 1.1.

In fact, the theorem tells us what that angle is, but that is another story.

Inspection of the figure shows that angle AQB must be 135°, and a 'square' with this corner could be used to draw the part of the circle below the x-axis. In general, if the angle APB is $t°$ we can see that the angle of the other square should be $(180 - t)°$. This is hardly a practical way of drawing a *circle*, but before dismissing it completely two pins and a 'square' of an angle approaching 180°, say 165° for example, could be used to draw short *arcs* of circles with large radii, which is itself an important application. As we shall see later, there is another way of achieving this but at the expense of much more complicated linkages.

1.1 Cutting Lines

It would be a strange book about mathematics that began by telling its readers that all its drawings, such as those of figure 1.1, were at best only approximations to the truth. However this necessarily follows from Euclid's definition of points and lines. Heath translated the opening of Euclid's *Elements*, book I, with the immortal sentences:

> A *point* is that which has no part. A *line* is a breadthless length.

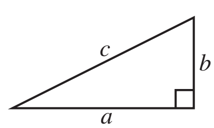
The idea of such a breadthless length is, of course, a mathematical fantasy, for in practice every line must have some width, otherwise it would be invisible! So this makes drawing a line actually a little more tricky than one might suppose.

Had this book been concerned only with pure geometry, these remarks would not have been necessary since the mathematical definitions integral to geometry are understood and accepted. While Hobbes (1588–1679) tried in vain to develop a *theory* in which lines have width, our concern in this chapter is more *practical*. Lines are used for all sorts of other purposes—art, dressmaking patterns and electrical circuit diagrams are just some examples. In the rest of this chapter we discuss the physical and practical consequences of having to work with broad lines.

1.2 The Pythagorean Theorem

A problem which anyone who has done some home improvement is bound to have stumbled across is that of cutting out a shape. For example, when sawing plywood some material is lost in the cut. We shall illustrate this problem and explain one solution with models, the first of which encapsulates the elegant demonstration of the Pythagorean theorem given by Kelland in 1864.

This theorem, also known as Euclid, book I, proposition 47, is perhaps the most famous mathematical result. As everyone learns at school, this relates the lengths of sides in a right-angled planar triangle. In particular, it states that for any right-angled planar triangle with sides of length a, b and c, as shown below,



we have the following relationship between the lengths:

$$a^2 + b^2 = c^2.$$

This is an algebraic statement of the theorem. A geometrical statement is that 'the area of the square on the hypotenuse is

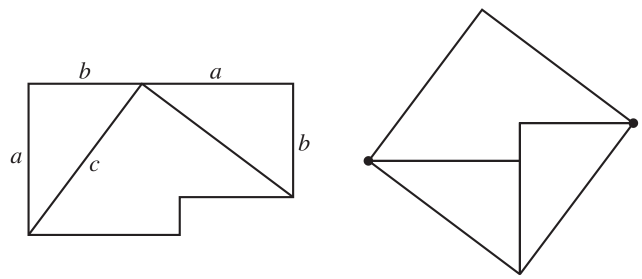


Figure 1.5. The Pythagorean theorem according to Kelland.

equal to the sum of the areas of the squares on the other two sides'. Both amount to the same thing, but it is through this geometrical statement that we obtain a physical demonstration by constructing, dissecting and comparing physical areas.

Kelland's physical model, or jigsaw, consisting of only three pieces, is shown in figure 1.5. On the left we have the two squares, b^2 and a^2, joined. These are made from three pieces, two of which are copies of the original triangle. These can then be rearranged on the right to give a square of area c^2.

Actually, what is particularly satisfying about this dissection is that the pieces may be hinged. One arrangement of the hinges is shown by the dots on the right-hand figure. When hinged in this way the pieces may be moved continuously from one configuration to the other and this makes a most excellent model. Not only is this much more satisfying than a jigsaw, but the pieces are much less likely to become lost. In practice, card with thread attached carefully by sticky tape makes an acceptable, if not a robust, model. Wood or plastic is much more durable, of course, and the classical solution is to use polished hardwood with inlaid brass hinges. However, when cutting wood or plastic, we have to be careful about the material lost by the width of the saw blade and it is to the solution of this problem we now direct our attention.

We start by drawing the left-hand diagram of figure 1.5 on a piece of plywood thick enough to allow hinges to be fixed. Although it is tempting to start with a piece of wood with two straight edges at right angles it is better to work the whole construction on a piece that is larger all round.

Before we cut any lines the centres of the two hinges must be marked and drilled so that their centres will fall exactly at the centres of the intersections of the lines. The next step is to cut out the joined squares. There are two possible approaches we could use. The first is to saw out a slightly oversized piece and sand down to the outer edge of the lines. The second is to saw down the centres of the lines. Since we can only saw down the two diagonals along their centres it is this second method that is needed for all cuts, otherwise there will be noticeable changes in sizes between the parts where they meet.

A saw whose *kerf*, i.e. width of cut, is twice the thickness of veneer, available in handicraft shops, is ideal. Once all the sawn edges are covered with veneer the original shape will be restored, and they will consequently fit together well in both configurations.

It may be thought, with some justification, that the procedures we have outlined above to make the model of this mathematical dissection are over elaborate and too demanding. This is true enough for a simple demonstration model, but to exhibit your craftsmanship to its fullest extent they are worth following. With a little care the lines where the pieces meet will be better approximations to Euclidean lines than those on the original drawing.

This form of construction was followed exactly in the hinged model of Dudeney's dissection, which is a dissection of an equilateral triangle into pieces which can be rearranged into a square. The solution to this problem is often attributed to Henry Ernest Dudeney (1857–1930), one of the greatest nineteenth-century puzzlers. This dissection appears in his book (Dudeney 1907) as the haberdasher's problem, although it had previously been posed in his puzzle column in *Weekly Dispatch* on 6 April 1902. Two weeks later he reported that many people had spotted that this was possible using five pieces, beginning by cutting the triangle into two right-angled triangles. The only person to correctly send in a solution using *four* pieces was a Mr C. W. McElroy of Manchester. Intriguingly, Frederickson (2002) leaves us in doubt as to whether Dudeney knew how to solve the puzzle using only four pieces before he posed it.

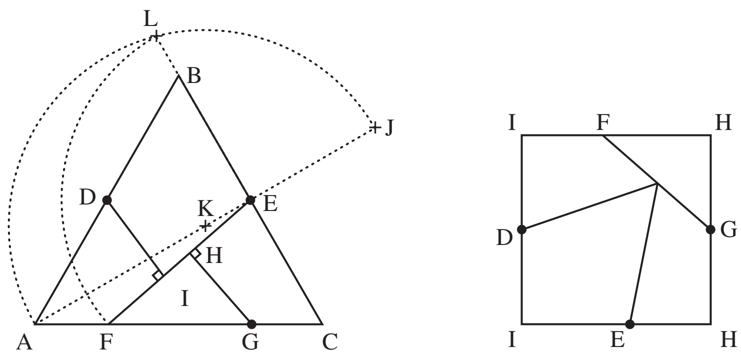


Figure 1.6. Dudeney's dissection.

The dissection of an equilateral triangle into just four pieces, which can be rearranged into a square, is achieved as follows. Begin with an equilateral triangle ABC with sides of length l. A calculation shows that the area of this shape is

$$\frac{\sqrt{3}}{4} l^2. \tag{1.1}$$

Hence, the square will have sides of length $\frac{\sqrt[4]{3}}{2} l$. For our purposes we do not wish to measure these lengths, which would be inaccurate. Rather than taking a metrical approach, we would like to use *geometrical constructions* with only a straight edge and a pair of compasses. If you are unsure how to actually carry out these constructions, then skip ahead to chapter 4 where we examine the topic in detail.

We begin by marking the points D and E midway along the sides AB and BC. Extend the line AE to J so that EJ equals EB. Bisect the line AJ at the point K, and draw an arc centred at K through A and J. Extend EB to l on this arc. Now, with E as centre, draw an arc of radius El. Where this crosses the line AC mark a point F. In fact EF is the length of the side of the square. Make FG equal to EB. Drop a perpendicular to EF through G, and denote by H where these two lines intersect. Lastly, drop a perpendicular to EF through D, and denote by I where these two lines intersect. Note that the distances FH and EI are equal. This is shown in figure 1.6.

Figure 1.7. Extending lines to complete the corner.

Just as with Kelland's demonstration of the Pythagorean theorem, it is particularly satisfying that the pieces may be *hinged*. By placing hinges at, for example, the points D, E and G, it is possible to transform one shape continuously into the other. This makes it possible to produce a table, a teapot stand or other objects which are useful for two, three, four or six people. An example of a teapot stand is shown in plate 1. What is even more surprising about this particular dissection is that the grain of the wood lines up correctly in *both configurations*.

1.3 Broad Lines

The marking out of sports pitches and tennis courts are two familiar examples of using broad lines. Lines of the width suitable for geometrical drawings would be useless as they would be invisible to player, referee and spectator. In these two examples lines meet at right angles, and so we must consider what happens at their junction. The probable result is illustrated in exaggerated form on the left of figure 1.7. Notice the annoying missing part, caused by the intersection of the rectangles which represent the lines. What we really would prefer here, but by no means always, is for the lines to extend a little beyond where they should, to give a corner as shown on the right of the figure.

A consequence of having broad lines is the need for very careful interpretation of what is meant by 'over the line'. Traffic police and soccer referees clearly have different views about its meaning when considering whether or not to prosecute a motorist or to allow a goal. Depending on the interpretation, the broad lines might be redrawn to make explicit a choice from the two possibilities shown in figure 1.8, which illustrate the fact that when two lines meet, the coordinate should really label the corner of the shape and not the middle of the line.

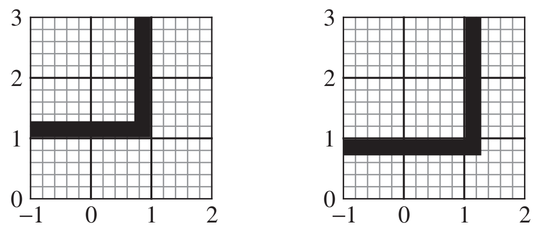


Figure 1.8. Emphasizing the inside and outside of a corner.

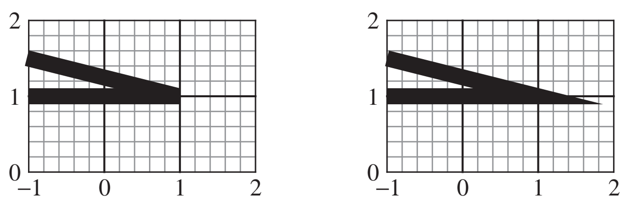


Figure 1.9. A more acute problem.

Of course, not all lines are perpendicular, as is the case with some standard road markings, and in particular the isosceles triangle found at road junctions. Here the problem of the lines meeting at a corner is even more acute (excuse the pun). Two options for drawing this are shown in figure 1.9. On the left are two lines drawn independent of each other and on the right is the shape obtained if we extend the lines to fill in the gap as we did in figure 1.7. While extending the lines appeared to be a sensible policy for the right angle, a small angle between the lines leads to a huge overshoot here.

The problem of cutting a 'vee' notch is exactly the same as that already noted with road markings. If the marked angle is 90° or greater, two straight saw cuts with the blade perpendicular to the wood are adequate. If the angle is less than 90° it is impossible to saw right into the notch of the 'vee', and a different tool, such as a chisel, will be necessary to cut out the final piece.

A completely different approach is demonstrated by two standard pieces of mechanical workshop equipment: a vee-block and a centre drill. Blocks are used to support work in milling machines and on drill tables, and usually come in matching pairs with ground faces. One block is shown in figure 10.18.

To overcome the problem of machining right into the hollow of the vee it is usual practice to cut a slot, as is shown below. In this way square work, for example, can be held firmly:

The same problem arises with the conical centre used to support the end of a long slender piece of work at the end of a lathe. It is easy enough to turn the conical centre itself, but it is extremely difficult to drill the matching hole in the workpiece. As with vee-blocks the solution is not to attempt to drill to a point, but to remove the point altogether with a special centre drill so that the cone can sit properly, without shaking.

A true cone can only be turned on the end of a round bar if the tool is at centre height. If it is low then the result is not a cone, not even a truncated cone, but a section of a hyperbola of revolution: a hyperboloid. We leave this as an exercise for you to prove.

1.4 Cutting Lines

Although it is very unlikely to be found in a home workshop, we discuss some of the geometrical problems that arise when using a computerized numerical control (CNC) milling machine. The machine tool is essentially a vertical milling machine with the cutting tool rotating about a vertical axis whose height, the z-coordinate, is controlled together with the x- and y-coordinates of the work table.

For simplicity we assume that the height of the tool is fixed and we wish to cut to some defined curve on a piece of metal plate to make a template. Examples of this can be seen in figure 10.5. The milling cutter is similar to a drill but much more rigid and uses both its end and side faces to cut away material. When moved sideways it is effectively drawing a broad line.

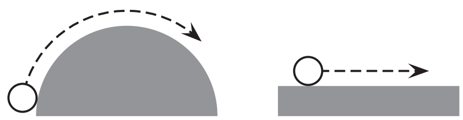


Figure 1.10. Cutting a circle and a straight line.

It turns out that there are only two curves for which the tool's path has the same *form* as the desired curve: they are the circle and the (closely related) straight line. The reason for this is that the evolute, which is the locus of the centres of curvature, of a circle is a point at its centre. For a straight line the equivalent point lies an infinite distance away. For a circle, then, the tool's path is another circle with the same centre and whose radius is increased by the radius of the cutting tool. A straight line obviously has a straight-line tool path, displaced again by the radius of the cutting tool, as illustrated in figure 1.10. No other curves have a point evolute, and so tool paths are necessarily of a different shape from the desired form.

Returning briefly to the drawing board, you may well be familiar with sets of radius curves used for drawing arcs of circles of small radius. Careful measurement of the sizes show that each is smaller by $\frac{1}{32}$ in than its stated size to allow for broad pencils or pens.

After a circle and a straight line one would argue that the mathematically simplest shapes are the conic sections. These include the ellipse and the parabola, and we shall meet all these shapes again—for example, in chapter 2 we show how two identical parabolic templates can be used to draw a straight line. Here we concentrate on cutting out a parabolic shape. As one example, let us see what curve a tool of radius R should take to produce an artefact where the boundary has the equation

$$x^2 = 4ay \text{ mm}^2.$$

Because we are about to make a real template the equation of the parabola above must have its dimensions clearly defined, otherwise it is meaningless.

At some point $\mathrm{P} = \mathrm{P}(x, y)$ on the parabola the centre of the tool is at $\mathrm{T} = \mathrm{T}(X, Y)$. The problem is to relate the two sets

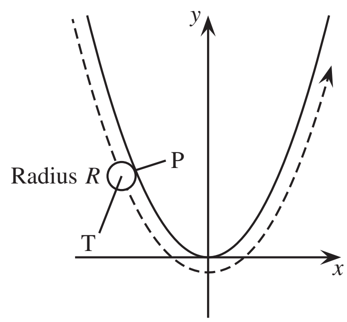


Figure 1.11. Cutting out a parabola on a CNC machine.

of coordinates. This relation can be described in language by saying that both T and P are moving in the same direction, a fixed distance apart. This is expressed in mathematical terms by equating the tangents of the curves. Second, we know that P and T are a distance R apart, for which we return to our friend the Pythagorean theorem. Lastly, since the tool and the work surface are tangential we have the normal line to the parabola passing through both (x, y) and (X, Y). No explicit attempt has been made here to solve the resulting equations to find the fundamental relation between X and Y, except to note that it is not simple. We have illustrated the solution in figure 1.11. Indeed, although in this case it is possible to solve the resulting system of differential equations exactly this is generally not the case. Hence in practice X and Y would be approximated with straight-line segments for some given, short, step length. Modern and necessarily expensive machines have the inbuilt facility to do this provided that they are given the equations of the curve and the tool radius. Most CNC machine centres have the capacity to cut straight or circular arcs after being given the tool radius.

This discussion of CNC machining has been included not because such tools are likely to be in a home workshop, but to show how apparently simple tasks such as cutting a shape like a parabola can lead to real geometric problems. The same can be said of a CNC lathe: the solids of constant breadth we examine in chapter 10 were turned this way. All the arcs are circular, but had the arcs not been of this form, the programming would have been much more difficult and tedious.

We need to look at this geometry in a reverse sense when considering what shape of cam and its follower should be used to provide some specified linear movement. The follower is effectively a point bearing on the face of the cam. In reality the follower is not quite a point, it is more typically in the form of a circular arc. In effect the tool path is dictated by the motion required of the follower and this in turn defines the shape of the cam.

1.5 Trial by Trials

This chapter has really been about the interpretation of lines and marking out work prior to cutting. It is very appropriate then that it finishes by describing how three of the essential pieces of workshop equipment can be made. Although chapter 2 shows how straight lines can be drawn by link-work they are hardly practicable when marking out in the workshop. What is needed is a straight edge. These can be obtained commercially, and for home use, at least, steel rules are not only straight but also graduated, something we address in chapter 4. In order to make one straight edge from scratch with no reference edges or planes available is effectively impossible. However, if *three* are made together they can be checked against each other and their ultimate straightness ensured.

Consider a first attempt A, shown in a misaligned form in figure 1.12. It is not straight so make another B. A and B fit together perfectly when they slide over each other. The only conclusion is that either they are both straight or they are both arcs of some circle of large radius. Their straightness is only confirmed if a third edge is made, C, and all three pairings of edges in all positions match. Then, and only then, are they confirmed to be straight.

A similar argument is applied to making a truly flat surface, usually called a surface plate if designed for use on a bench or a surface table for larger stand-alone models.

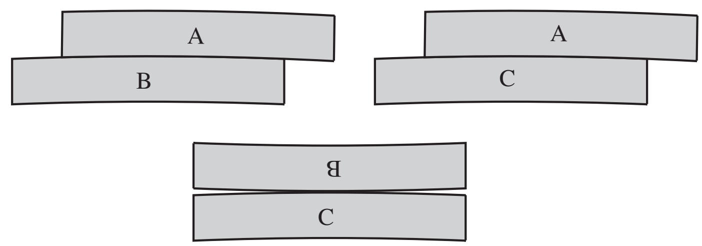


Figure 1.12. Comparing three edges.

Lastly we consider an engineer's 90° square. Again, three are made, A, B and C. When the stocks are supported on a straight edge or surface plate and match over the full length of the blades we therefore have for the corresponding angles

$$\alpha + \beta = 180^\circ,$$
$$\beta + \gamma = 180^\circ,$$
$$\gamma + \alpha = 180^\circ,$$

so that $\alpha = \beta = \gamma = 90^\circ$.

Squares, like the edges and surface plates, are manufactured to very high degrees of precision. British Standard 939 ('Engineers' squares (including cylindrical and block squares)', 1977, British Standards Institute) specifies the tolerances for squares and these are given in the seven tables within the specification. What is perhaps curious at a first reading is that it is only after the first five tolerances have been satisfied that it is possible to define the tolerances of the 90° angle. Even this is not specified as an angle, but as a linear departure from squareness between the stock and the blade.