Chapter One

INTRODUCTION AND PREVIEW

1.1. INTRODUCTION

Geometric Reflection Groups

Finite groups generated by orthogonal linear reflections on $\mathbb{R}^n$ play a decisive role in

- the classification of Lie groups and Lie algebras;
- the theory of algebraic groups, as well as, the theories of spherical buildings and finite groups of Lie type;
- the classification of regular polytopes (see [69, 74, 201] or Appendix B).

Finite reflection groups also play important roles in many other areas of mathematics, e.g., in the theory of quadratic forms and in singularity theory. We note that a finite reflection group acts isometrically on the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$.

There is a similar theory of discrete groups of isometries generated by affine reflections on Euclidean space $\mathbb{E}^n$. When the action of such a Euclidean reflection group has compact orbit space it is called *cocompact*. The classification of cocompact Euclidean reflection groups is important in Lie theory [29], in the theory of lattices in $\mathbb{R}^n$ and in E. Cartan's theory of symmetric spaces. The classification of these groups and of the finite (spherical) reflection groups can be found in Coxeter's 1934 paper [67]. We give this classification in Table 6.1 of Section 6.9 and its proof in Appendix C.

There are also examples of discrete groups generated by reflections on the other simply connected space of constant curvature, hyperbolic n-space, $\mathbb{H}^n$. (See [257, 291] as well as Chapter 6 for the theory of hyperbolic reflection groups.)

The other symmetric spaces do not admit such isometry groups. The reason is that the fixed set of a reflection should be a submanifold of codimension one (because it must separate the space) and the other (irreducible) symmetric spaces do not have codimension-one, totally geodesic subspaces. Hence, they do not admit isometric reflections. Thus, any truly "geometric" reflection group must split as a product of spherical, Euclidean, and hyperbolic ones.

The theory of these geometric reflection groups is the topic of Chapter 6. Suppose W is a reflection group acting on $\mathbb{X}^n = \mathbb{S}^n, \mathbb{E}^n$, or $\mathbb{H}^n$. Let K be the closure of a connected component of the complement of the union of "hyperplanes" which are fixed by some reflection in W. There are several common features to all three cases:

- K is geodesically convex polytope in $\mathbb{X}^n$.
- K is a "strict" fundamental domain in the sense that it intersects each orbit in exactly one point (so, $\mathbb{X}^n/W \cong K$).
- If S is the set of reflections across the codimension-one faces of K, then each reflection in W is conjugate to an element of S (and hence, S generates W).

Abstract Reflection Groups

The theory of abstract reflection groups is due to Tits [281]. What is the appropriate notion of an "abstract reflection group"? At first approximation, one might consider pairs (W, S), where W is a group and S is any set of involutions which generates W. This is obviously too broad a notion. Nevertheless, it is a step in the right direction. In Chapter 3, we shall call such a pair a "pre-Coxeter system." There are essentially two completely different definitions for a pre-Coxeter system to be an abstract reflection group.

The first focuses on the crucial feature that the fixed point set of a reflection should separate the ambient space. One version is that the fixed point set of each element of S separates the Cayley graph of (W, S) (defined in Section 2.1). In 3.2 we call (W, S) a *reflection system* if it satisfies this condition. Essentially, this is equivalent to any one of several well-known combinatorial conditions, e.g., the Deletion Condition or the Exchange Condition. The second definition is that (W, S) has a presentation of a certain form. Following Tits [281], a pre-Coxeter system with such a presentation is a "Coxeter system" and W a "Coxeter group." Remarkably, these two definitions are equivalent. This was basically proved in [281]. Another proof can be extracted from the first part of Bourbaki [29]. It is also proved as the main result (Theorem 3.3.4) of Chapter 3. The equivalence of these two definitions is the principal mechanism driving the combinatorial theory of Coxeter groups.

The details of the second definition go as follows. For each pair $(s, t) \in S \times S$, let m_{st} denote the order of st. The matrix (m_{st}) is the *Coxeter matrix* of (W, S); it is a symmetric $S \times S$ matrix with entries in $\mathbb{N} \cup \{\infty\}$, 1's on the diagonal, and each off-diagonal entry > 1. Let

$$\mathcal{R} := \{(st)^{m_{st}}\}_{(s,t)\in S\times S}.$$

(W, S) is a *Coxeter system* if $\langle S|\mathcal{R}\rangle$ is a presentation for W. It turns out that, given any $S \times S$ matrix (m_{st}) as above, the group W defined by the presentation $\langle S|\mathcal{R}\rangle$ gives a Coxeter system (W, S). (This is Corollary 6.12.6 of Chapter 6.)

Geometrization of Abstract Reflection Groups

Can every Coxeter system (W, S) be realized as a group of automorphisms of an appropriate geometric object? One answer was provided by Tits [281]: for any (W, S), there is a faithful linear representation $W \hookrightarrow GL(N, \mathbb{R})$, with $N = \mathrm{Card}(S)$, so that

- Each element of S is represented by a linear reflection across a codimension-one face of a simplicial cone C. (N.B. A "linear reflection" means a linear involution with fixed subspace of codimension one; however, no inner product is assumed and the involution is not required to be orthogonal.)
- If $w \in W$ and $w \neq 1$, then $w(\mathrm{int}(C)) \cap \mathrm{int}(C) = \emptyset$ (here $\mathrm{int}(C)$ denotes the interior of C).
- WC, the union of W-translates of C, is a convex cone.
- W acts properly on the interior $\mathcal{I}$ of WC.
- Let $C^f := \mathcal{I} \cap C$. Then C^f is the union of all (open) faces of C which have finite stabilizers (including the face $\mathrm{int}(C)$). Moreover, C^f is a strict fundamental domain for W on $\mathcal{I}$.

Proofs of the above facts can be found in Appendix D. Tits' result was extended by Vinberg [290], who showed that for many Coxeter systems there are representations of W on $\mathbb{R}^N$, with $N < \mathrm{Card}(S)$ and C a polyhedral cone which is not simplicial. However, the poset of faces with finite stabilizers is exactly the same in both cases: it is the opposite poset to the poset of subsets of S which generate finite subgroups of W. (These are the "spherical subsets" of Definition 7.1.1 in Chapter 7.) The existence of Tits' geometric representation has several important consequences. Here are two:

- Any Coxeter group W is virtually torsion-free.
- $\mathcal{I}$ (the interior of the Tits cone) is a model for $\underline{E}W$, the "universal space for proper W-actions" (defined in 2.3).

Tits gave a second geometrization of (W, S): its "Coxeter complex" Ξ. This is a certain simplicial complex with W-action. There is a simplex $\sigma \subset \Xi$ with $\dim \sigma = \mathrm{Card}(S) - 1$ such that (a) σ is a strict fundamental domain and (b) the elements of S act as "reflections" across the codimension-one faces of σ. When W is finite, Ξ is homeomorphic to unit sphere $\mathbb{S}^{n-1}$ in the canonical representation, triangulated by translates of a fundamental simplex. When (W, S) arises from an irreducible cocompact reflection group on $\mathbb{E}^n$, $\Xi \cong \mathbb{E}^n$. It turns out that Ξ is contractible whenever W is infinite.

The realization of (W, S) as a reflection group on the interior $\mathcal{I}$ of the Tits cone is satisfactory for several reasons; however, it lacks two advantages enjoyed by the geometric examples on spaces of constant curvature:

- The W-action on $\mathcal{I}$ is not cocompact (i.e., the strict fundamental domain C^f is not compact).
- There is no natural metric on $\mathcal{I}$ that is preserved by W. (However, in [200] McMullen makes effective use of a “Hilbert metric” on $\mathcal{I}$.)

In general, the Coxeter complex also has a serious defect—the isotropy subgroups of the W-action need not be finite (so the W-action need not be proper). One of the major purposes of this book is to present an alternative geometrization for (W, S) which remedies these difficulties. This alternative is the cell complex Σ, discusssed below and in greater detail in Chapters 7 and 12 (and many other places throughout the book).

The Cell Complex Σ

Given a Coxeter system (W, S), in Chapter 7 we construct a cell complex Σ with the following properties:

- The 0-skeleton of Σ is W.
- The 1-skeleton of Σ is $\text{Cay}(W, S)$, the Cayley graph of 2.1.
- The 2-skeleton of Σ is a Cayley 2-complex (defined in 2.2) associated to the presentation $\langle S|\mathcal{R}\rangle$.
- Σ has one W-orbit of cells for each spherical subset $T \subset S$. The dimension of a cell in this orbit is $\text{Card}(T)$. In particular, if W is finite, Σ is a convex polytope.
- W acts properly on Σ.
- W acts cocompactly on Σ and there is a strict fundamental domain K.
- Σ is a model for $\underline{E}W$. In particular, it is contractible.
- If (W, S) is the Coxeter system underlying a cocompact geometric reflection group on $\mathbb{X}^n = \mathbb{E}^n$ or $\mathbb{H}^n$, then Σ is W-equivariantly homeomorphic to $\mathbb{X}^n$ and K is isomorphic to the fundamental polytope.

Moreover, the cell structure on Σ is dual to the cellulation of $\mathbb{X}^n$ by translates of the fundamental polytope.

- The elements of S act as "reflections" across the "mirrors" of K. (In the geometric case where K is a polytope, a mirror is a codimension-one face.)
- Σ embeds in $\mathcal{I}$ and there is a W-equivariant deformation retraction from $\mathcal{I}$ onto Σ. So Σ is the "cocompact core" of $\mathcal{I}$.
- There is a piecewise Euclidean metric on Σ (in which each cell is identified with a convex Euclidean polytope) so that W acts via isometries. This metric is CAT(0) in the sense of Gromov [147]. (This gives an alternative proof that Σ is a model for $\underline{E}W$.)

The last property is the topic of Chapter 12 and Appendix I. In the case of "right-angled" Coxeter groups, this CAT(0) property was established by Gromov [147]. ("Right angled" means that $m_{st} = 2$ or ∞ whenever $s \neq t$.) Shortly after the appearance of [147], Moussong proved in his Ph.D. thesis [221] that Σ is CAT(0) for any Coxeter system. The complexes Σ gave one of the first large class of examples of "CAT(0)-polyhedra" and showed that Coxeter groups are examples of "CAT(0)-groups." This is the reason why Coxeter groups are important in geometric group theory. Moussong's result also allowed him to find a simple characterization of when Coxeter groups are word hyperbolic in the sense of [147] (Theorem 12.6.1).

Since W acts simply transitively on the vertex set of Σ, any two vertices have isomorphic neighborhoods. We can take such a neighborhood to be the cone on a certain simplicial complex L, called the "link" of the vertex. (See Appendix A.6.) We also call L the "nerve" of (W, S). It has one simplex for each nonempty spherical subset $T \subset S$. (The dimension of the simplex is $\text{Card}(T) - 1$.) If L is homeomorphic to S^{n-1}, then Σ is an n-manifold (Proposition 7.3.7).

There is great freedom of choice for the simplicial complex L. As we shall see in Lemma 7.2.2, if L is the barycentric subdivision of any finite polyhedral cell complex, we can find a Coxeter system with nerve L. So, the topological type of L is completely arbitrary. This arbitrariness is the source of power for the using Coxeter groups to construct interesting examples in geometric and combinatorial group theory.

Coxeter Groups as a Source of Examples in Geometric and Combinatorial Group Theory

Here are some of the examples.

- The Eilenberg-Ganea Problem asks if every group π of cohomological dimension 2 has a two-dimensional model for its classifying space $B\pi$

(defined in 2.3). It is known that the minimum dimension of a model for $B\pi$ is either 2 or 3. Suppose L is a two-dimensional acyclic complex with $\pi_1(L) \neq 1$. Conjecturally, any torsion-free subgroup of finite index in W should be a counterexample to the Eilenberg-Ganea Problem (see Remark 8.5.7). Although the Eilenberg-Ganea Problem is still open, it is proved in [34] that W is a counterexample to the appropriate version of it for groups with torsion. More precisely, the lowest possible dimension for any $\underline{E}W$ is 3 ($= \dim \Sigma$) while the algebraic version of this dimension is 2.

- Suppose L is a triangulation of the real projective plane. If $\Gamma \subset W$ is a torsion-free subgroup of finite index, then its cohomological dimension over $\mathbb{Z}$ is 3 but over $\mathbb{Q}$ is 2 (see Section 8.5).
- Suppose L is a triangulation of a homology $(n-1)$-sphere, $n \geqslant 4$, with $\pi_1(L) \neq 1$. It is shown in [71] that a slight modification of Σ gives a contractible n-manifold not homeomorphic to $\mathbb{R}^n$. This gave the first examples of closed apherical manifolds not covered by Euclidean space. Later, it was proved in [83] that by choosing L to be an appropriate "generalized homology sphere," it is not necessary to modify Σ; it is already a CAT(0)-manifold not homeomorphic to Euclidean space. (Such examples are discussed in Chapter 10.)

The Reflection Group Trick

This a technique for converting finite aspherical CW complexes into closed aspherical manifolds. The main consequence of the trick is the following.

THEOREM. (Theorem 11.1). *Suppose π is a group so that $B\pi$ is homotopy equivalent to a finite CW complex. Then there is a closed aspherical manifold M which retracts onto $B\pi$.*

This trick yields a much larger class of groups than Coxeter groups. The group that acts on the universal cover of M is a semidirect product $\widetilde{W} \rtimes \pi$, where $\widetilde{W}$ is an (infinitely generated) Coxeter group. In Chapter 11 this trick is used to produce a variety examples. These examples answer in the negative many of questions about aspherical manifolds raised in Wall's list of problems in [293]. By using the above theorem, one can construct examples of closed aspherical manifolds M where $\pi_1(M)$ (a) is not residually finite, (b) contains infinitely divisible abelian subgroups, or (c) has unsolvable word problems. In 11.3, following [81], we use the reflection group trick to produce examples of closed aspherical topological manifolds not homotopy equivalent to closed smooth manifolds. In 11.4 we use the trick to show that if the Borel Conjecture (from surgery theory) holds for all groups π which are fundamental groups of closed aspherical manifolds, then it must also hold for any π with a finite classifying space. In 11.5 we combine a version of the reflection group trick with the examples of Bestvina and Brady in [24] to show that there are Poincaré duality groups which are not finitely presented. (Hence, there are Poincaré duality groups which do not arise as fundamental groups of closed aspherical manifolds.)

Buildings

Tits defined the general notion of a Coxeter system in order to develop the general theory of buildings. Buildings were originally designed to generalize certain incidence geometries associated to classical algebraic groups over finite fields. A building is a combinatorial object. Part of the data needed for its definition is a Coxeter system (W, S). A building of type (W, S) consists of a set Φ of "chambers" and a collection of equivalence relations indexed by the set S. (The equivalence relation corresponding to an element $s \in S$ is called "s-adjacency.") Several other conditions (which we will not discuss until 18.1) also must be satisfied. The Coxeter group W is itself a building; a subbuilding of Φ isomorphic to W is an "apartment." Traditionally (e.g., in [43]), the geometric realization of the building is defined to be a simplicial complex with one top-dimensional simplex for each element of Φ. In this incarnation, the realization of each apartment is a copy of the Coxeter complex Ξ. In view of our previous discussion, one might suspect that there is a better definition of the geometric realization of a building where the realization of each chamber is isomorphic to K and the realization of each apartment is isomorphic to Σ. This is in fact the case: such a definition can be found in [76], as well as in Chapter 18. A corollary to Moussong's result that Σ is CAT(0) is that the geometric realization of any building is CAT(0). (See [76] or Section 18.3.)

A basic picture to keep in mind is this: in an apartment exactly two chambers are adjacent along any mirror while in a building there can be more than two. For example, suppose W is the infinite dihedral group. The geometric realization of a building of type W is a tree (without endpoints); the chambers are the edges; an apartment is an embedded copy of the real line.

(Co)homology

A recurrent theme in this book will be the calculation of various homology and cohomology groups of Σ (and other spaces on which W acts as a reflection group). This theme first occurs in Chapter 8 and later in Chapters 15 and 20 and Appendix J. Usually, we will be concerned only with cellular chains and cochains. Four different types of (co)homology will be considered.

(a) Ordinary homology $H_*(\Sigma)$ and cohomology $H^*(\Sigma)$.

(b) Cohomology with compact supports $H_c^*(\Sigma)$ and homology with infinite chains $H_*^{lf}(\Sigma)$.

(c) Reduced L^2-(co)homology $L^2\mathcal{H}^*(\Sigma)$.

(d) Weighted L^2-(co)homology $L^2_{\mathbf{q}}\mathcal{H}^*(\Sigma)$.

The main reason for considering ordinary homology groups in (a) is to prove Σ is acyclic. Since Σ is simply connected, this implies that it is contractible (Theorem 8.2.13).

The reason for considering cohomology with compact supports in (b) is that $H_c^*(\Sigma) \cong H^*(W; \mathbb{Z}W)$. We give a formula for these cohomology groups in Theorem 8.5.1. This has several applications: (1) knowledge of $H_c^1(\Sigma)$ gives the number of ends of W (Theorem 8.7.1), (2) the virtual cohomological dimension of W is $\max\{n | H_c^n(\Sigma) \neq 0\}$ (Corollary 8.5.5), and (3) W is a virtual Poincaré duality group of dimension n if and only if the compactly supported cohomology of Σ is the same as that of $\mathbb{R}^n$ (Lemma 10.9.1). (In Chapter 15 we give a different proof of this formula which allows us to describe the W-module structure on $H^*(W; \mathbb{Z}W)$.)

When nonzero, reduced L^2-cohomology spaces are usually infinite-dimensional Hilbert spaces. A key feature of the L^2-theory is that in the presence of a group action it is possible to attach "von Neumann dimensions" to these Hilbert spaces; they are nonnegative real numbers called the "L^2-Betti numbers." The reasons for considering L^2-cohomology in (c) involve two conjectures about closed aspherical manifolds: the Hopf Conjecture on their Euler characteristics and the Singer Conjecture on their L^2-Betti numbers. The Hopf Conjecture (called the "Euler Characteristic Conjecture" in 16.2) asserts that the sign of the Euler characteristic of a closed, aspherical $2k$-manifold M^{2k} is given by $(-1)^k \chi(M^{2k}) \geqslant 0$. This conjecture is implied by the Singer Conjecture (Appendix J.7) which asserts that for an aspherical M^n, all the L^2-Betti numbers of its universal cover vanish except possibly in the middle dimension. For Coxeter groups, in the case where Σ is a $2k$-manifold, the Hopf Conjecture means that the rational Euler characteristic of W satisfies $(-1)^k \chi(W) \geqslant 0$. In the right-angled case this can be interpreted as a conjecture about a certain number associated to any triangulation of a $(2k-1)$-sphere as a "flag complex" (defined in 1.2 as well as Appendix A.3). In this form, the conjecture is known as the Charney-Davis Conjecture (or as the Flag Complex Conjecture). In [91] Okun and I proved the Singer Conjecture in the case where W is right-angled and Σ is a manifold of dimension ≤ 4 (see 20.5). This implies the Flag Complex Conjecture for triangulations of S^3 (Corollary 20.5.3).

The fascinating topic (d) of weighted L^2-cohomology is the subject of Chapter 20. The weight $\mathbf{q}$ is a certain tuple of positive real numbers. For simplicity, let us assume it is a single real number q. One assigns each cell c in Σ a weight $\|c\|_q = q^{l(w(c))}$, where $w(c)$ is the shortest $w \in W$ so that $w^{-1}c$ belongs to the fundamental chamber and $l(w(c))$ is its word length. $L^2_qC^*(\Sigma)$ is the Hilbert space of square summable cochains with respect to this new inner product. When $q = 1$, we get the ordinary L^2-cochains. The group W no longer acts orthogonally; however, the associated Hecke algebra of weight q is a $*$-algebra of operators. It can be completed to a von Neumann algebra $\mathcal{N}_q$ (see Chapter 19). As before, the "dimensions" of the associated reduced cohomology groups give us L^2_q-Betti numbers (usually not rational numbers). It turns out that the "L^2_q-Euler characteristic" of Σ is $1/W(q)$ where $W(q)$ is the growth series of W. $W(q)$ is a rational function of q. (These growth series are the subject of Chapter 17.) In 20.7 we give a complete calculation of these L^2_q-Betti numbers for $q < \rho$ and $q > \rho^{-1}$, where ρ is the radius of convergence of $W(q)$. When q is the "thickness" (an integer) of a building Φ of type (W, S) with a chamber transitive automorphism group G, the L^2_q-Betti numbers are the ordinary L^2-Betti numbers (with respect to G) of the geometric realization of Φ (Theorem 20.8.6).

What Has Been Left Out

A great many topics related to Coxeter groups do not appear in this book, such as the Bruhat order, root systems, Kazhdan–Lusztig polynomials, and the relationship of Coxeter groups to Lie theory. The principal reason for their omission is my ignorance about them.

1.2. A PREVIEW OF THE RIGHT-ANGLED CASE

In the right-angled case the construction of Σ simplifies considerably. We describe it here. In fact, this case is sufficient for the construction of most examples of interest in geometric group theory.

Cubes and Cubical Complexes

Let $I := \{1, \dots, n\}$ and $\mathbb{R}^I := \mathbb{R}^n$. The *standard n-dimensional cube* is $[-1, 1]^I := [-1, 1]^n$. It is a convex polytope in $\mathbb{R}^I$. Its vertex set is $\{\pm 1\}^I$. Let $\{e_i\}_{i \in I}$ be the standard basis for $\mathbb{R}^I$. For each subset J of I let $\mathbb{R}^J$ denote the linear subspace spanned by $\{e_i\}_{i \in J}$. (If $J = \emptyset$, then $\mathbb{R}^\emptyset = \{0\}$.) Each face of $[-1, 1]^I$ is a translate of $[-1, 1]^J$ for some $J \subset I$. Such a face is said to be of *type* J.

For each $i \in I$, let $r_i : [-1, 1]^I \to [-1, 1]^I$ denote the orthogonal reflection across the hyperplane $x_i = 0$. The group of symmetries of $[-1, 1]^n$ generated by $\{r_i\}_{i\in I}$ is isomorphic to $(\mathbf{C}_2)^I$, where $\mathbf{C}_2$ denotes the cyclic group of order 2. $(\mathbf{C}_2)^I$ acts simply transitively on the vertex set of $[-1,1]^I$ and transitively on the set of faces of any given type. The stabilizer of a face of type J is the subgroup $(\mathbf{C}_2)^J$ generated by $\{r_i\}_{i\in J}$. Hence, the poset of nonempty faces of $[-1,1]^I$ is isomorphic to the poset of cosets

$$\coprod_{J\subset I}(\mathbf{C}_2)^I/(\mathbf{C}_2)^J.$$

$(\mathbf{C}_2)^I$ acts on $[-1,1]^I$ as a group generated by reflections. A fundamental domain (or "fundamental chamber") is $[0,1]^I$.

A *cubical cell complex* Λ is a regular cell complex in which each cell is combinatorially isomorphic to a standard cube. (A precise definition is given in Appendix A.) The *link* of a vertex v in Λ, denoted $Lk(v,\Lambda)$, is the simplicial complex which realizes the poset of all positive dimensional cells which have v as a vertex. If v is a vertex of $[-1,1]^I$, then $Lk(v,[-1,1]^I)$ is the $(n-1)$-dimensional simplex, Δ^{n-1}.

The Cubical Complex P_L

Given a simplicial complex L with vertex set $I=\{1,\dots,n\}$, we will define a subcomplex P_L of $[-1,1]^I$, with the same vertex set and with the property that the link of each of its vertices is canonically identified with L. The construction is similar to the standard way of realizing L as a subcomplex of Δ^{n-1}. Let $\mathcal{S}(L)$ denote the set of all $J\subset I$ such that $J=\mathrm{Vert}(\sigma)$ for some simplex σ in L (including the empty simplex). $\mathcal{S}(L)$ is partially ordered by inclusion. Define P_L to be the union of all faces of $[-1,1]^I$ of type J for some $J\in\mathcal{S}(L)$. So, the poset of cells of P_L can be identified with the disjoint union

$$\coprod_{J\in\mathcal{S}(L)}(\mathbf{C}_2)^I/(\mathbf{C}_2)^J.$$

(This construction is also described in [37, 90, 91].)

Example 1.2.1. Here are some examples of the construction.

- If $L=\Delta^{n-1}$, then $P_L=[-1,1]^n$.
- If $L=\partial(\Delta^{n-1})$, then P_L is the boundary of an n-cube, i.e., P_L is homeomorphic to S^{n-1}.
- If L is the disjoint union of n points, then P_L is the 1-skeleton of an n-cube.
- If $n=3$ and L is the disjoint union of a 1-simplex and a 0-simplex, then P_L is the subcomplex of the 3-cube consisting of the top and bottom faces and the 4 vertical edges. (See Figure 1.1.)

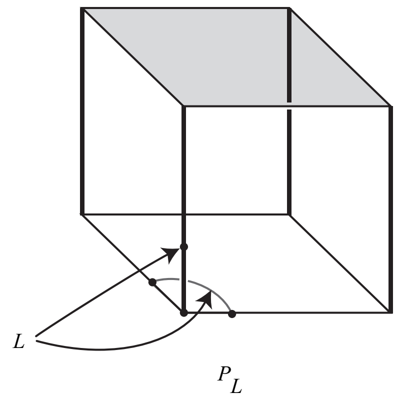

Figure 1.1. L is the union of a 1-simplex and a 0-simplex.

- Suppose L is the join of two simplicial complexes L_1 and L_2. (See Appendix A.4 for the definition of "join.") Then $P_L = P_{L_1} \times P_{L_2}$.
- So, if L is a 4-gon (the join of S^0 with itself), then P_L is the 2-torus $S^1 \times S^1$.
- If L is an n-gon (i.e., the triangulation of S^1 with n vertices), then P_L is an orientable surface of Euler characteristic $2^{n-2}(4-n)$.

P_L is stable under the $(\mathbf{C}_2)^I$-action on $[-1,1]^I$. A fundamental chamber K is given by $K := P_L \cap [0,1]^I$. Note that K is a cone (the cone point being the vertex with all coordinates 1). In fact, K is homeomorphic to the cone on L. Since a neighborhood of any vertex in P_L is also homeomorphic to the cone on L we also get the following.

PROPOSITION 1.2.2. *If L is homeomorphic to S^{n-1}, then P_L is an n-manifold.*

Proof. The cone on S^{n-1} is homeomorphic to an n-disk. □

The Universal Cover of P_L and the Group W_L

Let $\widetilde{P}_L$ be the universal cover of P_L. For example, the universal cover of the complex P_L in Figure 1.1 is shown in Figure 1.2. The cubical cell structure on P_L lifts to a cubical structure on $\widetilde{P}_L$. Let W_L denote the group of all lifts of elements of $(\mathbf{C}_2)^I$ to homeomorphisms of $\widetilde{P}_L$ and let $\varphi : W_L \to (\mathbf{C}_2)^I$ be the homomorphism induced by the projection $\widetilde{P}_L \to P_L$. We have a short exact sequence,

$$1 \longrightarrow \pi_1(P_L) \longrightarrow W_L \xrightarrow{\varphi} (\mathbf{C}_2)^I \longrightarrow 1.$$

Since $(\mathbf{C}_2)^I$ acts simply transitively on $\mathrm{Vert}(P_L)$, W_L is simply transitive on $\mathrm{Vert}(\widetilde{P}_L)$. By Theorem 2.1.1 in the next chapter, the 1-skeleton of $\widetilde{P}_L$ is

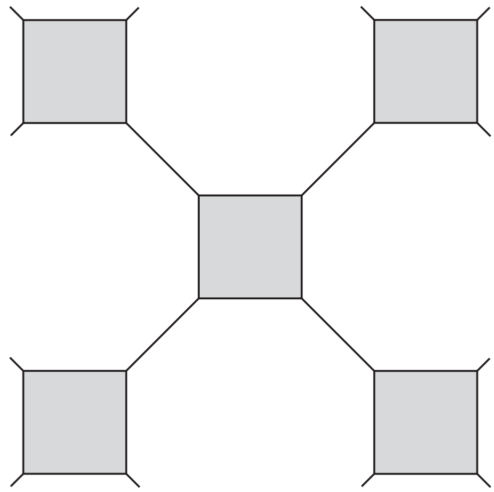

Figure 1.2. The universal cover of P_L.

$\mathrm{Cay}(W_L, S)$ for some set of generators S and by Proposition 2.2.4, the 2-skeleton of $\widetilde{P}_L$ is a "Cayley 2-complex" associated with some presentation of W_L. What is this presentation for W_L?

The vertex set of P_L can be identified with $(\mathbf{C}_2)^I$. Fix a vertex v of P_L (corresponding to the identity element in $(\mathbf{C}_2)^I$). Let $\tilde{v}$ be a lift of v in $\widetilde{P}_L$. The 1-cells at v or at $\tilde{v}$ correspond to vertices of L, i.e., to elements of I. The reflection r_i stabilizes the i^{th} 1-cell at v. Let s_i denote the unique lift of r_i which stabilizes the i^{th} 1-cell at $\tilde{v}$. Then $S := \{s_i\}_{i \in I}$ is a set of generators for W_L. Since s_i^2 fixes $\tilde{v}$ and covers the identity on P_L, we must have $s_i^2 = 1$. Suppose σ is a 1-simplex of L connecting vertices i and j. The corresponding 2-cell at $\tilde{v}$ is a square with edges labeled successively by s_i, s_j, s_i, s_j. So, as explained in Section 2.2, we get a relation $(s_i s_j)^2 = 1$ for each 1-simplex $\{i, j\}$ of L. By Proposition 2.2.4, W_L is the group defined by this presentation, i.e., (W_L, S) is a right-angled Coxeter system, with $S := \{s_1, \dots, s_n\}$. Examining the presentation, we see that the abelianization of W_L is $(\mathbf{C}_2)^I$. Thus, $\pi_1(P_L)$ is the commutator subgroup of W_L.

For each subset J of I, W_J denotes the subgroup generated by $\{s_i\}_{i \in J}$. If $J \in \mathcal{S}(L)$, then W_J is the stabilizer of the corresponding cell in $\widetilde{P}_L$ which contains $\tilde{v}$ (and so, for $J \in \mathcal{S}(L)$, $W_J \cong (\mathbf{C}_2)^J$). It follows that the poset of cells of $\widetilde{P}_L$ is isomorphic to the poset of cosets,

$$\coprod_{J \in \mathcal{S}(L)} W_L / W_J.$$

When Is $\widetilde{P}_L$ Contractible?

A simplicial complex L is a *flag complex* if any finite set of vertices, which are pairwise connected by edges, spans a simplex of L. (Flag complexes play an important role throughout this book, e.g., in Sections 7.1 and 16.3 and Appendices A.3 and I.6.)

PROPOSITION 1.2.3. *The following statements are equivalent.*

(i) L is a flag complex.

(ii) $\widetilde{P}_L$ is contractible.

(iii) The natural piecewise Euclidean structure on $\widetilde{P}_L$ is CAT(0).

Sketch of Proof. One shows (ii) $\Longrightarrow$ (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii). If L is not a flag complex, then it contains a subcomplex L' isomorphic to $\partial\Delta^n$, for some $n \geqslant 2$, but which is not actually the boundary complex of any simplex in L. Each component of the subcomplex of $\widetilde{P}_L$ corresponding to L' is homeomorphic to S^n. It is not hard to see that the fundamental class of such a sphere is nontrivial in $H_n(\widetilde{P}_L)$ (cf. Sections 8.1 and 8.2). So, if L is not a flag complex, then $\widetilde{P}_L$ is not contractible, i.e., (ii) $\Longrightarrow$ (i). As we explain in Appendix I.6, a result of Gromov (Lemma I.6.1) states that a simply connected cubical cell complex is CAT(0) if and only if the link of each vertex is a flag complex. So, (i) $\Longrightarrow$ (iii). Since CAT(0) spaces are contractible (Theorem I.2.6 in Appendix I.2), (iii) $\Longrightarrow$ (ii). □

When L is a flag complex, we write Σ_L for $\widetilde{P}_L$. It is the cell complex referred to in the previous section.

Examples 1.2.4. In the following examples we assume L is a triangulation of an $(n-1)$-manifold as a flag complex. Then P_L is a manifold except possibly at its vertices (a neighborhood of the vertex is homeomorphic to the cone on L). If L is the boundary of a manifold X, then we can convert P_L into a manifold $M_{(L,X)}$ by removing the interior of each copy of K and replacing it with a copy of the interior of X. We can convert Σ_L into a manifold $\widehat{\Sigma}_{(L,X)}$ by a similar modification.

A metric sphere in Σ_L is homeomorphic to a connected sum of copies of L, one copy for each vertex enclosed by the sphere. When $n \geqslant 4$, the fundamental group of such a connected sum is the free product of copies of $\pi_1(L)$ and hence, is not simply connected when $\pi_1(L) \neq 1$. It follows that Σ_L is not simply connected at infinity when $\pi_1(L) \neq 1$. (See Example 9.2.7.) As we shall see in 10.3, for each $n \geqslant 4$, there are $(n-1)$-manifolds L with the same homology as S^{n-1} and with $\pi_1(L) \neq 1$ (the so-called "homology spheres"). Any such L bounds a contractible manifold X. For such L and X, we have that $M_{(L,X)}$ is homotopy equivalent to P_L. Its universal cover is $\widehat{\Sigma}_{(L,X)}$, which is contractible. Since $\widehat{\Sigma}_{(L,X)}$ is not simply connected at infinity, it is not homeomorphic to $\mathbb{R}^n$. The $M_{(L,X)}$ were the first examples of closed manifolds with contractible universal cover not homeomorphic to Euclidean space. (See Chapter 10, particularly Section 10.5, for more details.)

Finally, suppose $L = \partial X$, where X is an aspherical manifold with boundary (i.e., the universal cover of X is contractible). It is not hard to see that the closed manifold $M_{(L,X)}$ is also aspherical. This is the "reflection group trick" of Chapter 11.