CHAPTER 1

Resolution for Curves

Resolution of curve singularities is one of the oldest and prettiest topics of algebraic geometry. In all likelihood, it is also completely explored.

In this chapter I have tried to collect all the different ways of resolving singularities of curves. Each of the thirteen sections contains a method, and some of them contain more than one. These come in different forms: solving algebraic equations by power series, normalizing complex manifolds, projecting space curves, blowing up curves contained in smooth surfaces, birationally transforming plane curves, describing field extensions of Laurent series fields and blowing up or normalizing 1-dimensional rings.

By the end of the chapter we see that the methods are all interrelated, and there is only one method to resolve curve singularities. I found, however, that these approaches all present a different viewpoint or technical twist that is worth exploring.

1.1. Newton's method of rotating rulers

Let $F(x, y)$ be a complex polynomial in two variables. We are interested in finding solutions of $F = 0$ in the form $y = \phi(x)$, where ϕ is some type of function that we are right now unsure about.

Following the classical path of solving algebraic equations, one might start with the case where $\phi(x)$ is a composition of polynomials, rational functions and various mth roots of these. As in the classical case, this will not work if the degree of F is 5 or more in y.

One can also try to look for power series solutions, but simple examples show that we have to work with power series with fractional exponents. The equation $y^m = x + x^2$ has no power series solutions for $m \geq 2$, but it has fractional power series solutions for any $\epsilon^m = 1$ given by

$$y = \epsilon x^{1/m}\Big(1 + \sum_{j\geq 1}\binom{1/m}{j}x^j\Big) \quad \text{for } i = 1, \dots, m.$$

As a more interesting example, $y^m - y^n + x = 0$ for $m > n$ also has a fractional power series solution

$$y = \textstyle\sum_{i\geq 1} a_i x^{i/n},$$

where $a_1 = 1$ and the other a_i are defined recursively by

$$n \cdot a_s = \text{coefficient of } x^{(s+n-1)/n} \text{ in } \quad \left(\sum_{i=1}^{s-1} a_i x^{i/n}\right)^m - \left(\sum_{i=1}^{s-1} a_i x^{i/n}\right)^n.$$

After many more examples, we are led to look for solutions of the form

$$y = \sum_{i=0}^{\infty} c_i x^{i/M},$$

where M is a natural number whose dependence on $\deg F$ we leave open for now. These series, though introduced by Newton, are called *Puiseux series*. We encounter them later several times.

THEOREM 1.1 (Newton, 1676). *Let $F(x, y)$ be a complex polynomial or power series in two variables. Assume that $F(0, 0) = 0$ and that y^n appears in $F(x, y)$ with a nonzero coefficient for some n. Then $F(x, y) = 0$ has a Puiseux series solution of the form*

$$y = \sum_{i=1}^{\infty} c_i x^{i/N}$$

for some integer N.

REMARK 1.2. (1) The original proof is in a letter of Newton to Oldenburg dated October 24, 1676. Two accessible sources are [New60, pp.126–127] and [BK81, pp.372–375].

(2) Our construction gives only a formal Puiseux series; that is, we do not prove that it converges for $|x|$ sufficiently small. Nonetheless, if F is a polynomial or a power series that converges in some neighborhood of the origin, then any Puiseux series solution converges in some (possibly smaller) neighborhood of the origin. This is easiest to establish using the method of Riemann, to be discussed in Section 1.2.

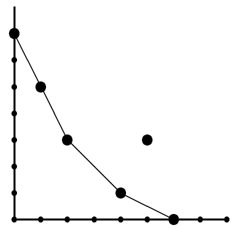
(3) By looking at the proof we see that we get n different solutions (when counted with multiplicity).

The proof of Newton starts with a graphical representation of the "lowest order" monomials occurring in F. This is now called the Newton polygon.

DEFINITION 1.3 (Newton polygon). Let $F = \sum a_{ij} x^i y^j$ be a polynomial or power series in two variables. The *Newton polygon* of f (in the chosen coordinates x and y) is obtained as follows.

In a coordinate plane, we mark the point (i, j) with a big dot if $a_{ij} \neq 0$. Any other monomial $x^{i'} y^{j'}$ with $i' \geq i, j' \geq j$ will not be of "lowest order" in any sense, so we also mark these. (In the figures these markings are invisible, since I do not want to spend time marking infinitely many uninteresting points.)

The Newton polygon is the boundary of the convex hull of the resulting infinite set of marked points.



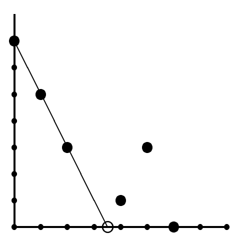
The Newton polygon of
$y^7 + y^5x + y^3x^2 + y^3x^5 + yx^4 + x^6$

Assume now that F contains a nonzero term $a_{0n}y^n$ and n is the smallest possible. This means that the Newton polygon has a corner on the y-axis at the point $(0, n)$. Look at the nonvertical edge of the Newton polygon starting at $(0, n)$. Let us call this the *leading edge* of the Newton polygon. (As Newton explains it, we put a vertical ruler through $(0, n)$ and rotate it till it hits another marked point—hence, the name of the method.)

1.4 (Proof of (1.1)). We construct the Newton polygon of F and concentrate on its leading edge.

If the leading edge is horizontal, then there are no marked points below the $j = n$ line, and hence, y^n divides F and $y = 0$ is a solution.

Otherwise, the extension of the leading edge hits the x-axis at a point which we write as nu/v where u, v are relatively prime. The leading edge is a segment on the line $(v/u)i + j = n$. In the diagram below the leading edge hits the x-axis at $7/2$, so $u = 1$ and $v = 2$.



The leading edge of the
Newton polygon of
$y^7 + y^5x + y^3x^2 + y^3x^5 + yx^4 + x^6$

We use induction on the leading edge, more precisely, on its starting point $(0, n)$ and on its steepness v/u.

Our aim is to make a coordinate change and to obtain another polynomial or power series $F_1(x_1, y_1)$ with leading edge starting at $(0, n_1)$ and steepness v_1/u_1 such that

- either $n_1 < n$,

- or $n_1 = n$, and $v_1/u_1 < v/u$.

Moreover, we can write down a Puiseux series solution of $F(x, y) = 0$ from a Puiseux series solution of $F_1(x_1, y_1)$.

Then we repeat the procedure. The first case can occur at most n-times, so eventually the second case happens all the time. We then construct a Puiseux series solution from this infinite sequence of coordinate transformations.

In order to distinguish the two cases, we consider the terms in F that lie on the leading edge

$$f(x, y) := \sum_{(v/u)i+j=n} a_{ij} x^i y^j,$$

and we think of this as the "lowest terms" of F. If $a_{ij}x^i y^j$ is a nonzero term in $F(x, y)$, then $(v/u)i + j \geq n$, so $f(x, y)$ indeed consists of the lowest degree terms in F if we declare that $\deg x = v/u$ and $\deg y = 1$.

In the above example, $f(x, y) = y^7 + y^5 x + y^3 x^2$.

Note that $(v/u)i + j = n$ has an integer solution only if $v|n - j$; thus we obtain the following.

Claim 1.4.1. We can write

$$f(1, y) = \sum_{0 \leq k \leq n/v} a_k y^{n-kv}.$$

In particular, if $v \neq 1$, then $f(1, y)$ does not contain the term y^{n-1} and so $f(1, y)$ is not an nth power of a linear form. □

We distinguish the two cases based on how $f(1, y)$ factors.

Case 1. $f(1, y)$ is not an nth power.

Let α be a root of $f(1, y)$ with multiplicity $n_1 < n$. Then we make the substitutions

$$x := x_1^v, \ y = y_1 x_1^u + \alpha x_1^u.$$

Note that if $a_{ij}x^i y^j$ is a nonzero term in $F(x, y)$, then $(v/u)i + j \geq n$; thus

$$a_{ij} x^i y^j = a_{ij} x_1^{vi+uj} (y_1 + \alpha)^j$$

and $vi + uj \geq nu$ with equality only if $(v/u)i + j = n$. Thus $F(x_1^v, y_1 x_1^u + \alpha x_1^u)$ is divisible by x_1^{nu}, and we set

$$F_1(x_1, y_1) := x_1^{-nu} F(x_1^v, y_1 x_1^u + \alpha x_1^u).$$

Note furthermore that

$$F_1(0, y_1) = f(1, y_1 + \alpha),$$

and so $y_1^{n_1}$ appears in F_1 with nonzero coefficient.

Furthermore, any Puiseux series solution $y_1 = \phi(x_1)$ of $F_1 = 0$ gives a Puiseux series solution

$$y = \phi(x^{1/v}) x^{u/v} + \alpha x^{u/v}$$

of $F = 0$.

Case 2. $f(1,y)$ is an nth power.

By (1.4.1) this can happen only for $v=1$. Write $f(1,y)=c(y-\alpha)^n$, and make a coordinate change

$$x = x_1,\ y = y_1 + \alpha x_1^u.$$

Under this transformation, $x^i y^j$ becomes a sum of monomials $x_1^{i'} y_1^{j'}$, where $(1/u)i' + j' = (1/u)i + j$. Thus we do not get any new terms below the leading edge of f, and we kill every monomial on the leading edge save y^n, which is now y_1^n.

Hence $n_1 = n$, but the leading edge of the Newton polygon of

$$F_1(x_1, y_1) := F(x_1, y_1 + \alpha x_1^u)$$

is less steep than the leading edge of the Newton polygon of $F(x,y)$.

Next we repeat the procedure with $F_1(x_1,y_1)$ to get $F_2(x_2,y_2)$ and so on.

As we noted, the only remaining question is, what happens when the second case happens infinitely often. This means that we have an infinite sequence of coordinate changes

$$y_s = y_{s+1} + \alpha_{s+1} x_s^{u_{s+1}}, \quad y_{s+1} = y_{s+2} + \alpha_{s+2} x_s^{u_{s+2}}, \dots\ .$$

Here $u_{s+1} < u_{s+2} < \cdots$; thus we can view this sequence as converging to a single power series substitution

$$y_s = y_\infty + \alpha_{s+1} x_s^{u_{s+1}} + \alpha_{s+2} x_s^{u_{s+2}} + \cdots,$$

and then

$$F_s(x_s, y_\infty + \alpha_{s+1} x_s^{u_{s+1}} + \alpha_{s+2} x_s^{u_{s+2}} + \cdots) = y_\infty^n (\text{invertible power series}),$$

giving the power series solution $y_s = -(\alpha_{s+1} x_s^{u_{s+1}} + \alpha_{s+2} x_s^{u_{s+2}} + \cdots)$. □

1.2. The Riemann surface of an algebraic function

The resolution of singularities of analytic curves is due to Riemann. When he constructs the Riemann surface of a function, he goes directly to the smooth Riemann surface, bypassing the singular model; see [Rie90, pp.39–41]. His method is essentially the one given below.

In more contemporary terminology, here is the result.

THEOREM 1.5 (Riemann, 1851). *Let $F(x,y)$ be an irreducible complex polynomial and $C := (F(x,y)=0) \subset \mathbb{C}^2$ the corresponding complex curve. Then there is a 1-dimensional complex manifold $\bar{C}$ and a proper holomorphic map*

$$\sigma : \bar{C} \to C,$$

which is a biholomorphism except at finitely many points.

Proof. Since F is irreducible, F and $\partial F/\partial y$ have only finitely many points $\Sigma \subset C$ in common. By the implicit function theorem, the first coordinate projection $\pi : C \to \mathbb{C}$ is a local analytic biholomorphism on $C \setminus \Sigma$.

We start by constructing a resolution for a small neighborhood of a point $p \in \Sigma$. For notational convenience assume that $p = 0$, the origin.

Let $B_\epsilon \subset \mathbb{C}^2$ denote the ball of radius ϵ around the origin. By choosing ϵ small enough, we may assume that $C \cap (y = 0) \cap B_\epsilon = \{0\}$.

Next, by by choosing η small enough, we can assume that

$$\pi : C \cap B_\epsilon \cap \pi^{-1}(\Delta_\eta) \to \Delta_\eta$$

is proper and a local analytic biholomorphism except at the origin, where $\Delta_\eta \subset \mathbb{C}$ is the disc of radius η. Set

$$C_\eta := C \cap B_\epsilon \cap \pi^{-1}(\Delta_\eta) \quad \text{and} \quad C_\eta^* := C_\eta \setminus \{0\}.$$

We thus conclude that

$$\pi : C_\eta^* \to \Delta_\eta^* \quad \text{is a covering map.}$$

The fundamental group of Δ_η^* is $\mathbb{Z}$; thus for every m, the punctured disc Δ_η^* has a unique connected covering of degree m, namely,

$$\rho_m : \Delta_1^* \to \Delta_\eta^* \quad \text{given by } z \mapsto \eta z^m.$$

Let $C_{\eta,i}^* \subset C_\eta^*$ be any connected component and m_i the degree of the covering $\pi : C_{\eta,i}^* \to \Delta_\eta^*$. We thus have an isomorphism of coverings

$$\begin{tikzcd} \Delta_1^* \arrow[rr, "\sigma_i^*"] \arrow[dr, "\rho_{m_i}"'] & & C_{\eta,i}^* \arrow[dl, "\pi"] \\ & \Delta_\eta^* & \end{tikzcd}$$

More precisely, topology tells us only that σ_i^* is a homeomorphism. However, the maps ρ_{m_i} and π are local analytic biholomorphisms; thus we can assert that σ_i^* is a homeomorphism that is also a local analytic biholomorphism, and hence a global analytic biholomorphism.

The image of σ_i^* lands in the ball B_ϵ, and hence the coordinate functions of σ_i^* are analytic and bounded on Δ_1^*. Thus by the Riemann extension theorem, σ_i^* extends to a proper analytic map

$$\sigma_i : \Delta_1 \to C_\eta.$$

Doing this for every connected component $C_{\eta,i}^* : i \in I$, we obtain a proper analytic map

$$\sigma : \textstyle\coprod_{i\in I} \Delta_1 \to C_\eta \quad \text{such that} \quad \sigma^* : \textstyle\coprod_{i\in I} \Delta_1^* \to C_\eta^*$$

is an isomorphism, where $\coprod_{i\in I}$ denotes disjoint union.

This proves the local resolution for complex algebraic plane curves.

To move to the global case, observe that $\Sigma \subset C$ is a discrete subset. Thus for each $p_i \in \Sigma$ we can choose disjoint open neighborhoods $p_i \in C_i \subset C$. By further shrinking C_i, we have proper analytic maps $\sigma_i : \bar{C}_i \to C_i$, where $\bar{C}_i$ is a disjoint union of open discs and σ_i is invertible outside the singular point $p_i \in C_i$.

We can thus patch together the "big" chart $C \setminus \Sigma$ with the local resolutions $\bar{C}_i$ to get a global resolution $\bar{C}$. □

1.6 (Puiseux expansion). The resolution of the local branches

$$\sigma_i : \Delta_1 \to C_{\eta,i}$$

is given by a power series on Δ_1, and the local coordinate on Δ_1 can be interpreted as x^{1/m_i}.

Thus we obtain that each local branch $C_{\eta,i}$ has a parametrization by a convergent Puiseux series

$$y = \sum_{j=0}^{\infty} a_j x^{j/M}, \quad \text{where } M = m_i \le \deg_y F.$$

Remark 1.7. There is lot more to Puiseux expansions than the above existence theorems.

Let $0 \in C \subset \mathbb{C}^2$ be a curve singularity and S^3_ϵ a 3-sphere of radius ϵ around the origin. Then $C \cap S^3_\epsilon$ is a real 1-dimensional manifold for $0 < \epsilon \ll 1$, and up to diffeomorphism, the pair $(C \cap S^3_\epsilon \subset S^3_\epsilon)$ does not depend on ϵ. It is called the *link* of $0 \in C$.

$C \cap S^3_\epsilon$ is connected iff C is analytically irreducible, in which case it is called a *knot*. One can read off the topological type of this knot from the vanishing of certain coefficients of the Puiseux expansions. See [BK81, Sec.8.4] for a lovely treatment of this classical topic.

The resolution problem for 1-dimensional complex spaces can be handled very similarly. The final result is the following.

Theorem 1.8. *Let C be a 1-dimensional reduced complex space with singular set $\Sigma \subset C$. Then there is a 1-dimensional complex manifold $\bar{C}$ and a proper holomorphic map*

$$\sigma : \bar{C} \to C$$

such that $\sigma : \sigma^{-1}(C \setminus \Sigma) \to C \setminus \Sigma$ is a biholomorphism.

Proof. As before, we start by constructing a resolution for a small neighborhood of a singular point.

Let $0 \in C \subset \mathbb{C}^n$ be a 1-dimensional complex analytic singularity. That is, $0 \in C$ is reduced, and there are holomorphic functions $f_1, \dots, f_k$ such that

$$C \cap B_\epsilon = (f_1 = \cdots = f_k = 0) \cap B_\epsilon,$$

where $B_\epsilon \subset \mathbb{C}^n$ denotes the ball of radius ϵ around the origin.

A general hyperplane $0 \in H \subset \mathbb{C}^n$ intersects C in a discrete set of points, and hence by shrinking ϵ we may assume that $C \cap H \cap B_\epsilon = \{0\}$.

Let $\pi : \mathbb{C}^n \to \mathbb{C}$ denote the projection with kernel H, and choose coordinates $x_1, \ldots, x_n$ such that π is the nth coordinate projection.

By the implicit function theorem, the set of points where $\pi : C \to \mathbb{C}$ is not a local analytic biholomorphism is given by the condition

$$\operatorname{rank}\left(\frac{\partial f_i}{\partial x_j} : 1 \le i \le k, 1 \le j \le n-1\right) \le n-2.$$

It is thus a complex analytic subset of C. By Sard's theorem it has measure zero and, hence, is a discrete set.

Thus by choosing η small enough, we can assume that

$$\pi : C \cap B_\epsilon \cap \pi^{-1}(\Delta_\eta) \to \Delta_\eta$$

is proper and a local analytic biholomorphism except at the origin, where $\Delta_\eta \subset \mathbb{C}$ is the disc of radius η. Set

$$C_\eta := C \cap B_\epsilon \cap \pi^{-1}(\Delta_\eta) \quad \text{and} \quad C_\eta^* := C_\eta \setminus \{0\}.$$

We thus conclude that

$$\pi : C_\eta^* \to \Delta_\eta^* \quad \text{is a covering map.}$$

The rest of the proof now goes the same as before. □

1.3. The Albanese method using projections

In algebraic geometry, the simplest method to resolve singularities of curves was discovered by Albanese [Alb24a]. This relies on comparing the singularities of a curve with the singularities of its projection from a singular point.

In order to get a feeling for this, let us consider some examples.

EXAMPLE 1.9. (1) Let $p \in \mathbb{P}^n$ be a point and $\mathbb{P}^{n-1} \cong H \subset \mathbb{P}^n$ a hyperplane not containing p. The *projection* $\pi_{p,H} : \mathbb{P}^n \dashrightarrow H$ of $\mathbb{P}^n$ from p to H is defined as follows. Pick any point $q \ne p$. Then the line through p, q intersects H in the image point $\pi_{p,H}(q)$.

We can choose coordinates on $\mathbb{P}^n$ such that $p = (0 : 0 : \cdots : 0 : 1)$ and $H = (x_n = 0)$. Then

$$\pi_{p,H}(x_0 : \cdots : x_n) = (x_0 : \cdots : x_{n-1}).$$

(2) Assume that $p \in C \subset \mathbb{P}^n$ is a singular point, where m smooth branches of the curve pass through with different tangent directions. Projecting C from p separates these tangent directions, and the singular point p is replaced by m points with only one local branch through each of them.

(3) Let $C \subset \mathbb{A}^n$ be given by the monomials $t \mapsto (t^{m_1}, t^{m_2}, \dots, t^{m_n})$. We assume that $m_1 < m_i$ for $i \geq 2$.

Projecting from the origin to $\mathbb{P}^{n-1}$, the origin is replaced by a single point $(1 : 0 : \cdots : 0)$, and in the natural affine coordinates we get the parametric curve $t \mapsto (t^{m_2-m_1}, \dots, t^{m_n-m_1})$.

This is quite curious. The new monomial curve looks simpler than the one we started with, but it is hard to pin down in what way. For instance, its multiplicity $\min\{m_i - m_1\}$ may be bigger than the multiplicity of the original curve, which is m_1.

(4) Projection may also create singular points. If a line through p intersects C in two or more points or is tangent to C in one point, we get new singular points after projecting.

For $n \geq 4$ these do not occur when p is in general position, but the singular points of C are not in general position, so it is hard to determine what exactly happens. The best one could hope for is that such projections do not create new singular points for general embeddings $C \hookrightarrow \mathbb{P}^n$.

It is quite surprising that for curves of low degree in $\mathbb{P}^n$ we do not have to worry about general position or about ways of measuring the improvement of singularities step-by-step. The intermediate stages may get worse, but the process takes care of itself in the end.

ALGORITHM 1.10 (Albanese). Let $C_0 \subset \mathbb{P}^n$ be a projective curve. If $C_i \subset \mathbb{P}^{n-i}$ is already defined, then pick any singular point $p_i \in C_i \subset \mathbb{P}^{n-i}$ and set

$$C_{i+1} := \overline{\pi_i(C_i)},$$

where $\pi_i : \mathbb{P}^{n-i} \dashrightarrow \mathbb{P}^{n-i-1}$ is the projection from the point p_i.

THEOREM 1.11 (Albanese, 1924). *Let $C_0 \subset \mathbb{P}^n$ be an irreducible, reduced projective curve spanning $\mathbb{P}^n$ over an algebraically closed field.*

If $\deg C_0 < 2n$, then the Albanese algorithm eventually stops with a smooth projective curve $C_m \subset \mathbb{P}^{n-m}$, which is birational to C_0.

COROLLARY 1.12. *Every irreducible, reduced projective curve C over an algebraically closed field can be embedded into some $\mathbb{P}^n$ such that $\deg C < 2n$ and C spans $\mathbb{P}^n$.*

Thus the Albanese algorithm eventually stops with a smooth projective curve $C_m \subset \mathbb{P}^{n-m}$, which is birational to C. The inverse map $C_m \dashrightarrow C$ is a morphism, and thus $C_m \to C$ is a resolution of C.

Proof. All we need is to find a very ample line bundle L on C such that

$$\deg L < 2(h^0(C, L) - 1) \quad \text{or, equivalently,} \quad h^0(C, L) > \tfrac{1}{2} \deg L + 1.$$

Let us see first how to achieve this using the Riemann-Roch theorem for singular curves (which is way too advanced for such an elementary consequence).

The Riemann-Roch theorem says that if L is any line bundle on an irreducible, reduced projective curve, then

$$h^0(C, L) - h^1(C, L) = \deg L + 1 - p_a(C),$$

where the arithmetic genus $p_a(C)$ is easiest to define as $p_a(C) := h^1(C, \mathcal{O}_C)$.

Thus any very ample line bundle L of degree $\geq 2p_a(C) + 1$ works. $\square$

For those who prefer a truly elementary proof of resolution for curves, here is a method to find the required line bundles.

1.13 (Very weak Riemann-Roch on curves). We claim that for any very ample line bundle L,

$$h^0(C, L^m) \geq m \deg L + 1 - \binom{\deg L - 1}{2} \quad \text{for } m \geq \deg L.$$

Indeed, embed C into P^n by L, and then project it generically to a plane curve of degree $\deg L$, $\pi : C \to C' \subset \mathbb{P}^2$. Now, for $m \geq \deg L$,

$$\begin{aligned} h^0(C, L^m) &\geq h^0\big(C', \mathcal{O}_{\mathbb{P}^2}(m)|_{C'}\big) \\ &\geq h^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m)\big) - h^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(m - \deg L)\big) \\ &= m \deg L + 1 - \tbinom{\deg L - 1}{2}. \end{aligned}$$

Taking any $m \geq \deg L$ is sufficient for the Albanese method. $\square$

Before we start the proof of (1.11), we need some elementary lemmas about space curves and their projections.

LEMMA 1.14. *Let $C \subset \mathbb{P}^n$ be an irreducible and reduced curve, not contained in any hyperplane. Then $\deg C \geq n$. Furthermore, if $p_1, \dots, p_n$ are n distinct points of C, then $\deg C \geq \sum_i \operatorname{mult}_{p_i} C$.*

Proof. Pick n points $p_1, \dots, p_n \in C$, and let $L \subset \mathbb{P}^n$ be the linear span of these points. Then $\dim L \leq n - 1$.

By assumption C is not contained in L, and thus there is a hyperplane $H \subset \mathbb{P}^n$ containing L but not containing C. Thus the intersection $H \cap C$ is finite, and it contains at least n points. This implies that $\deg C \geq n$.

If some of the p_i are singular, then we can further improve the estimate to $\deg C \geq \sum_i \operatorname{mult}_{p_i} C$, (cf. (1.20)). $\square$

1.15 (Projections of curves). Let $A \subset \mathbb{P}^n$ be any irreducible, reduced curve and $p \in A$ a point. Let $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ denote the projection from p as in (1.9).

π is not a morphism, but it becomes one after blowing up p. Let $A' \subset B_0\mathbb{P}^n$ be the birational transform of A and let $A'_p \subset A'$ denote the preimage of p. The closure A_1 of the projection of A is the union of $\pi(A \setminus \{p\})$ and of the image of A'_p.

Claim 1.15.1. Let the notation be as above.

(i) If A spans $\mathbb{P}^n$, then A_1 spans $\mathbb{P}^{n-1}$.

(ii) If $A \dashrightarrow A_1$ is birational, then

$$\deg A_1 = \deg A - \operatorname{mult}_p A.$$

(iii) If $A \dashrightarrow A_1$ is not birational, then

$$\deg A_1 \cdot \deg(A/A_1) = \deg A - \operatorname{mult}_p A.$$

Proof. If $A_1 \subset H$ for some hyperplane H, then $A \subset \pi^{-1}(H)$, a contradiction, proving (i).

In order to see (ii), let $H \subset \mathbb{P}^{n-1}$ be a general hyperplane. It intersects A_1 in $\deg A_1$ points. If H avoids all the points over which $\pi : A \setminus \{p\} \to A_1$ is not a local isomorphism, then $\pi^{-1}(H)$ intersects A in the points $H \cap A_1$ and also at p. Here $\pi^{-1}(H)$ is a general hyperplane through p thus the intersection number of A and $\pi^{-1}(H)$ at p is $\operatorname{mult}_p A$ by (1.20).

Finally, the same argument as before shows (iii), once we notice that $\deg(A/A_1)$ points of A lie over a general point of A_1. We have to be a little more careful when $A \to A_1$ is inseparable, when the total number of preimages is the degree divided by the degree of inseparability. However, the local intersection multiplicity goes up by the degree of inseparability (1.20), and so the two changes cancel each other out. □

1.16 (Proof of (1.11)). Starting with $C_0 \subset \mathbb{P}^n$ such that $\deg C_0 < 2n$, we get a sequence of curves $C_i \subset \mathbb{P}^{n-i}$.

If the projection $C_{i-1} \dashrightarrow C_i$ is birational, then $\deg C_i \le \deg C_{i-1} - 2$ by (1.15.1), and thus $\deg C_i < 2(n-i)$.

If $C_i \dashrightarrow C_{i+1}$ is not birational and we project from a point of multiplicity $m_i \ge 2$, then

$$\deg C_{i+1} = \frac{\deg C_i - m_i}{\deg(C_i/C_{i+1})} \le \frac{\deg C_i - m_i}{2} < n - i - 1.$$

On the other hand, $\deg C_{i+1} \ge n - i - 1$ by (1.14), a contradiction.

Thus all the projections are birational. The sequence of projections must stop after at most $n-1$ steps, so at some stage we get $C_m \subset \mathbb{P}^{n-m}$ without any points of multiplicity at least 2. Therefore, C_m is smooth. □

1.17 (The Albanese method over nonclosed fields). Let C be an irreducible and reduced curve over a field k, which is not algebraically closed, and choose an embedding $C \subset \mathbb{P}^n$ such that $\deg C < 2n$.

If $p \in C(k)$ is a singular point defined over k, then we can proceed without any change and project from p as before.

What happens with singular points $p \in C(\bar{k})$ that are defined over an extension field $k' \supset k$?

If k'/k is separable (for instance, if $\operatorname{char} k = 0$), then the point p has several conjugates $p = p_1, \dots, p_d$ and the linear space $L = \langle p_1, \dots, p_d \rangle \subset \mathbb{P}^n$ spanned by them is defined over k. Thus we can project from L to get

$\pi_L : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-d}$, and everything works out as before. (Note that $L \neq \mathbb{P}^n$ by (1.14).)

(As long as the p_i are in general position, over $\bar{k}$ we can also realize π_L as projecting from the points $p_1, \dots, p_d$ successively.)

From this we conclude the following.

Corollary 1.18. *Let k be a perfect field and C an irreducible, reduced projective curve over k.*

Then C can be embedded into some $\mathbb{P}^n$ such that $\deg C < 2n$ and the Albanese algorithm eventually stops with a smooth projective curve, which is a resolution of C. □

1.19 (Curves over nonperfect fields). The typical example of nonperfect fields is a one-variable function field $K = k(t)$, where k is any field of characteristic p.

Over K consider the hyperelliptic curve $C := (y^2 = x^p - t)$. After adjoining $t^{1/p}$ this can be rewritten as $(y^2 = (x - t^{1/p})^p)$, which is singular at the point $(t^{1/p}, 0)$.

Nonetheless, the original curve C is nonsingular; that is, its local rings are regular. Indeed, the only point in question is $(t^{1/p}, 0)$, and over k it is defined by the equations $y = x^p - t = 0$. The maximal ideal of this point in $K[x,y]/(y^2 - x^p + t)$ is thus $(y, x^p - t)/(y^2 - x^p + t)$, which is generated by y alone.

Even worse examples appear over a two-variable function field $K = k(s,t)$. I leave it to the reader to check that $(sx^p + ty^p + z^p = 0) \subset \mathbb{P}^2$ is nonsingular over K, but over the algebraic closure $\bar{K}$ it becomes the p-fold line since

$$sx^p + ty^p + z^p = (s^{1/p}x + t^{1/p}y + z)^p.$$

Curves like this certainly cannot be made smooth by projections since they are not even birational to any smooth projective curve. Of course here the relevant question is, does the Albanese algorithm produce the nonsingular model of C over nonperfect fields?

The basic inductive structure of the proof breaks down in some examples, and I am not sure what happens in general.

1.20 (Review of multiplicities, I). Almost every introduction to algebraic geometry discusses the order of zero or pole of a rational function on a smooth curve, but very few consider these notions for singular curves. Here we give a short sketch of the general case.

Let k be a field, C an irreducible and reduced curve defined over k and $P = \{p_1, \dots, p_n\} \subset C$ a finite number of closed points. Let $R = \mathcal{O}_{P,C}$ be the semi-local ring of P; that is, we invert every function that is nonzero at all the p_i. Then R is a 1-dimensional integral domain with finitely many maximal ideals $m_i = m_{p_i}$ and $\dim_k R/m_i < \infty$. For us the best definition of 1-dimensional is that $\dim_k R/(r) < \infty$ for every $r \in R^* := R \setminus \{0\}$. Let $K \supset R$ denote the quotient field with multiplicative group $K^* := K \setminus \{0\}$.

More generally, everything works if R is a 1-dimensional semi-local ring without nilpotent elements, $R^* \subset R$ is the group of non–zero divisors, $K \supset R$ is obtained by inverting R^* and $K^* \subset K$ denotes the subgroup of invertible elements.

If $R = k[x]_{(x)}$ and $f = x^m(\text{unit})$, then $m = \dim_k R/fR$ is the usual order of vanishing or multiplicity of f. Based on this, for any R and any $f \in R^*$ we call $e_R(f) := \dim_k R/fR$ the multiplicity of f.

If R/m is bigger than k, then one may think that this is the "usual" order of vanishing times $\dim_k R/m$. A typical example is $C = (y^2 = x^2 - 1) \subset \mathbb{A}^2_{\mathbb{R}}$ and P corresponds to the maximal ideal (x). We compute that $\dim_{\mathbb{R}} \mathcal{O}_{P,C}/(x) = 2$. Looking at it over $\mathbb{C}$, we see that x vanishes at two points $(0, \pm i)$, with multiplicity 1 at each. Thus the total order of vanishing is 2, and our definition works well after all.

It is convenient to think of a function with a pole as having a negative order of vanishing. With this in mind, we define the multiplicity of any $f \in K^*$ as follows. Write $f = r_1/r_2$, where $r_i \in R^*$. Thus R/r_1R and R/r_2R are both finite dimensional. Set

$$e_P(f) := e_R(f) := \dim_k R/r_1R - \dim_k R/r_2R, \tag{1.20.1}$$

and call it the *multiplicity* of f at p. If we write $f = (r_1r_3)/(r_2r_3)$, then

$$\begin{aligned}
&\dim_k R/r_1r_3R - \dim_k R/r_2r_3R \\
&\quad = \dim_k R/r_1R + \dim_k r_1R/r_1r_3R - \dim_k R/r_2R - \dim_k r_2R/r_2r_3R \\
&\quad = \dim_k R/r_1R - \dim_k R/r_2R,
\end{aligned}$$

where the last equality holds since multiplication by r_2/r_1 gives an isomorphism $r_1R/r_1r_3R \to r_2R/r_2r_3R$. So the notion is well defined. As we change field from k to $k' \supset k$, the k-vector space R/r_iR is replaced by a k'-vector space of the same dimension (namely, $k' \otimes_k R/r_iR$), so the multiplicity does not depend on k.

Note however that $e_P(f) = 0$ does not imply that f is a unit at P, not even if P consists of a single point. For instance, $f = x/y$ has multiplicity 0 at the origin of the irreducible cubic $(xy + x^3 + y^3 = 0)$.

The multiplicity is additive; that is,

$$e_P(fg) = e_P(f) + e_P(g). \tag{1.20.2}$$

Indeed, write $f = r_1/r_2$ and $g = s_1/s_2$. Then

$$\begin{aligned}
e_P(fg) &= \dim_k R/r_1s_1R - \dim_k R/r_2s_2R \\
&= \dim_k R/r_1R + \dim_k r_1R/r_1s_1R - \dim_k R/r_2R - \dim_k r_2R/r_2s_2R \\
&= \dim_k R/r_1R + \dim_k R/s_1R - \dim_k R/r_2R - \dim_k R/s_2R \\
&= e_P(f) + e_P(g).
\end{aligned}$$

Another important property of multiplicity is semi-continuity.

Lemma 1.20.3. For $f_1, \dots, f_n \in K^*$, the (partially defined) function $k^n \to \mathbb{Z}$ given by

$$\sum t_i f_i \to e_R(\sum t_i f_i)$$

takes its minimum on an open subset of k^n.

Proof. Multiply through the common denominator to assume that $f_i \in R$. Fix $r \in m_R \cap R^*$. For every natural number s,

$$\begin{aligned} e_R(\sum t_i f_i) &= \dim_k R/(\sum t_i r_i)R \ \geq \ \dim_k R/(r^s, \sum t_i r_i)R \\ &= \dim_k R/r^s R - \operatorname{rank}_k \big[\sum t_i r_i : R/r^s R \to R/r^s R\big]. \end{aligned}$$

The rank is a lower-semi-continuous function for families of k-linear maps of finite-dimensional k-vector spaces. We are done if we can prove that there is a fixed s such that $r^s \in (\sum t_i r_i)R$ for every $(t_1, \dots, t_n)$ in a suitable dense open subset of k^n.

To show this, introduce new independent variables T_i, and work over the field $k' = k(T_1, \dots, T_n)$. The new ring $R_{k'}$ is still 1-dimensional and local, and so $r^s \in (\sum T_i r_i)R_{k'}$ for some s. This means that

$$r^s = \big(\sum T_i r_i\big) \cdot \sum_j \frac{\phi_j(T_1, \dots, T_n)}{\psi_j(T_1, \dots, T_n)} \cdot b_j$$

for some $b_j \in R$ and $\phi_j, \psi_j \in k[T_1, \dots, T_n]$. For $t_1, \dots, t_n$ in the open set $\prod_j \psi_j \neq 0$ we can substitute $T_i = t_i$ to obtain $r^s \in (\sum t_i r_i)R$. □

When $P = p$ is a single point and k is an infinite field, we define the *multiplicity* of $p \in C$ by

$$\operatorname{mult}_p C := \frac{1}{\dim_k \mathcal{O}_{p,C}/m_p} \min\{e_p(f) : f \in m_p\}. \tag{1.20.4}$$

Thus $\operatorname{mult}_p C = 1$ iff $m_p \subset \mathcal{O}_{p,C}$ is a principal ideal.

Assume for example that $C \subset \mathbb{A}^2$ is defined by an equation $(g = 0)$ and $p \in C$ is the origin. We show that $\operatorname{mult}_p C$ coincides with the multiplicity of g at the origin, that is, the degree of the lowest monomial in g.

If $\operatorname{mult}_p g = m$, then $g \in (x, y)^m$, and hence we get a surjection $\mathcal{O}_C \twoheadrightarrow k[x, y]/(x, y)^m$. We can assume that $f = y + (\text{other terms})$ and then

$$e_p(f) \geq \dim_k k[x, y]/\big(f, (x, y)^m\big) = \dim_k k[x]/(x^m) = m.$$

Conversely, choose a general linear function for f. After a coordinate change we can assume that $f = y$, and the generic choice assures that x^m appears in g with nonzero coefficient. Thus we can write $g = yu + x^m v$, where $v(p) \neq 0$ and set $L := (y = 0)$. Then

$$\mathcal{O}_{p,C}/(f) = \mathcal{O}_{p,\mathbb{A}^2}/(g, y) = \mathcal{O}_{p,L}/(g) = \mathcal{O}_{p,L}/(x^m v) \cong k[x]/(x^m).$$

This shows that $e_p(f) \leq m$; thus in fact $e_p(f) = m$.

If k is finite, a dense open subset of $\mathbb{A}^n_k$ need not have any k-points, so the above definition does not work. See Section 2.9 for a general definition.

Lemma 1.20.5. Let C be a reduced, irreducible projective curve. Then $\sum_{p\in C} e_P(f) = 0$ for every nonzero rational function $f \in k(C)$.

Proof. If C is smooth, then $f : C \to \mathbb{P}^1$ is everywhere defined. If C is singular, then we only get a rational map $f : C \dashrightarrow \mathbb{P}^1$ and this causes problems. Our first step is to reduce to the case where $f : C \to \mathbb{P}^1$ is a morphism.

To do this, let h be a rational function on C, which is contained in the local ring of every singular point. Then fh^s is also in the local ring of every singular point for $s \gg 1$. Writing $f = (fh^s)/h^s$ and using additivity (1.20.2), it is enough to prove the assertion for those functions f that are contained in the local ring of every singular point. Such an f can be viewed as a finite morphism $f : C \to \mathbb{P}^1$. Let $C_0 := C \setminus (\text{polar set of } f)$ and $C_\infty := C \setminus (\text{zero set of } f)$. Then $k[C_0]$ is a finite and torsion-free $k[f]$-module, and hence free. Similarly, $k[C_\infty]$ is a free $k[f^{-1}]$-module. Thus

$$\begin{aligned} \textstyle\sum_{p\in C_0} e_p(f) &= \dim_k k[C_0]/f k[C_0] = \operatorname{rank}_{k[f]} k[C_0], \quad \text{and} \\ -\textstyle\sum_{p\in C_\infty} e_p(f) &= \dim_k k[C_\infty]/f^{-1} k[C_\infty] = \operatorname{rank}_{k[f^{-1}]} k[C_\infty]. \end{aligned}$$

Since $\operatorname{rank}_{k[f]} k[C_0] = \operatorname{rank}_{k[f,f^{-1}]} k[C_0 \cap C_\infty] = \operatorname{rank}_{k[f^{-1}]} k[C_\infty]$, we are done. □

Let L be a line bundle on C and s_1 any nonzero rational section of L. At each point $p \in C$ we can identify the $\mathcal{O}_{p,C}$-module L with $\mathcal{O}_{p,C}$ and define the multiplicity $e_p(s_1)$. The *degree* of L on C is defined by the formula

$$\deg_C L := \sum_p e_p(s_1). \tag{1.20.6}$$

If s_2 is another section, then by (1.20.2) and (1.20.3) we get that

$$\sum_p e_p(s_1) - \sum_p e_p(s_2) = \sum_p e_p(s_1/s_2) = 0,$$

so $\deg_C L$ is well defined and does not depend on k.

From the definition we see that if L has a nonzero section, then $\deg_C L \geq 0$. If s is a nonzero section of L and $p \in C$ a k-point such that $s(p) \neq 0$ then in passing from L to $L(-p)$ we lose one section and lower the degree by 1. If k is algebraically closed, then we have plenty of points such that $s(p) \neq 0$. Repeating this if necessary we get the basic inequality between degrees and the space of global sections:

$$\dim_k H^0(C, L) \leq \deg_C L + 1 \quad \text{if } \deg_C L \geq 0. \tag{1.20.7}$$

This holds for arbitrary k as well, since both sides remain unchanged under field extensions. (For the left-hand side, this is explained in [Sha94, III.3.5].)

Remark 1.20.8. Lemma (1.20.5) also holds for invertible rational functions if C is 1-dimensional and projective, with no embedded points. This can be used to define the degree of any line bundle. The inequality (1.20.7), however, no longer holds in general, since even $\mathcal{O}_C$ can have many sections.

1.4. Normalization using commutative algebra

These days, commutative algebra is part of the familiar foundations of algebraic geometry, and it is hard to imagine that the relationship between resolutions and normalization was not always obvious. So we start this section by trying to explain how one may be led from complex analytic resolutions to normalization. Then we prove that normalization does give the resolution for 1-dimensional integral domains that are finitely generated over a field.

1.21 (Why normalization?). Let C be a complex analytic curve with resolution $n : \bar{C} \to C$. Let $\Sigma \subset C$ denote the set of singular points and $C^0 \subset C$ the open subset of smooth points. Let $\bar{\Sigma} = n^{-1}(\Sigma)$ and $\bar{C}^0 = n^{-1}(C^0)$ denote their preimages. Then $n : \bar{C}^0 \to C^0$ is an isomorphism.

By restriction we obtain the maps below, where the first one is an isomorphism by the Riemann extension theorem and the second is an isomorphism since $\bar{C}^0 \cong C^0$:

$$\left\{ \begin{array}{l} \text{holomorphic} \\ \text{functions on } \bar{C} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{holomorphic} \\ \text{functions on } \bar{C}^0 \\ \text{bounded near } \bar{\Sigma} \end{array} \right\} \cong \left\{ \begin{array}{l} \text{holomorphic} \\ \text{functions on } C^0 \\ \text{bounded near } \Sigma \end{array} \right\}.$$

Looking at the right-hand side, we obtain an intrinsic way of defining the ring (or sheaf) of holomorphic functions on $\bar{C}$ as the ring (or sheaf) of holomorphic functions on C^0 that are bounded near the singular points.

It is quite natural to expect that a similar description would hold in the algebraic setting as well.

Thus let C be a complex, affine algebraic curve with singular set Σ and let $C^0 := C \setminus \Sigma$, be the set of smooth points. In analogy with the holomorphic case, one arrives at the conjecture

$$\left\{ \begin{array}{l} \text{regular} \\ \text{functions on } \bar{C} \end{array} \right\} \stackrel{?}{\cong} \left\{ \begin{array}{l} \text{regular} \\ \text{functions on } C^0 \\ \text{bounded near } \Sigma \end{array} \right\}.$$

This looks pretty good, but boundedness is not really a concept of algebraic geometry. To understand it, let us look again on $\bar{C}$.

If $n : \bar{C} \to C$ is a resolution, then n is birational; thus every (regular or rational) function on $\bar{C}$ is also a rational function on C.

Let f be a regular function on $\bar{C}^0$ that is not bounded along $\bar{\Sigma}$. Since f is a rational function on $\bar{C}$, it must have a pole at one of the points $p \in \bar{\Sigma}$. Thus by looking at Laurent series around p, we get a homomorphism

$$\Phi_\Sigma : \left\{ \begin{array}{l} \text{rational} \\ \text{functions on } C \end{array} \right\} \to \sum_{p \in \bar{\Sigma}} \left\{ \begin{array}{l} \text{Laurent series} \\ \text{around } p \end{array} \right\},$$

which has the property that a rational function f is bounded around Σ iff its Laurent series expansion has no pole for every $p \in \bar{\Sigma}$.

For algebraic varieties one could keep working with Laurent series, but in general it is more convenient to change to arbitrary DVRs.

Definition 1.22. A *discrete valuation ring* or *DVR* is a Noetherian integral domain R with a single maximal ideal that is also principal; that is, $m = (t)$ for some $t \in R$.

This easily implies (cf. [AM69, 9.2]) that every element of the quotient field $Q(R)$ can be written uniquely as $t^n u$, where $n \in \mathbb{Z}$ and $u \in R \setminus m$ is a unit. Hence, just as in Laurent series fields, we can talk about an element having a pole of order $-n$ (if $n < 0$) or a zero of order n (if $n > 0$).

We can now state the first definition of normality, which is inspired by the above considerations.

Definition 1.23. Let S be an integral domain with quotient field $Q(S)$. The *normalization* of S in $Q(S)$, denoted by $\bar{S}$, is the unique largest subring $\bar{S} \subset Q(S)$ such that every homomorphism $\phi : S \to R$ to a DVR extends to a homomorphism $\bar{\phi} : \bar{S} \to R$.

Note that ϕ has a unique extension to a partially defined homomorphism $\Phi : Q(S) \dashrightarrow Q(R)$ between the quotient fields given by

$$\Phi(s_1/s_2) := \phi(s_1)/\phi(s_2), \quad \text{whenever } \phi(s_2) \neq 0.$$

Thus $\bar{S} = \cap_{\phi : S \to R} \Phi^{-1}(R)$.

In general, it is quite difficult to use this definition to construct the normalization, but here is a useful case that is easy.

Lemma 1.24. *A unique factorization domain is normal. In particular, any polynomial ring $k[x_1, \dots, x_n]$ over a field is normal.*

Proof. If $p_i \in S$ are the irreducible elements, then every element of $Q(S)$ can be uniquely written as a finite product $u \prod_i p_i^{m_i}$, where $u \in S$ is a unit and $m_i \in \mathbb{Z}$.

For every p_j we define

$$S_{p_j} := \{u \prod_i p_i^{m_i} : m_j \geq 0\} \subset Q(S).$$

Then S_{p_j} is a DVR whose maximal ideal is generated by p_j and $S = \cap_{p_j} S_{p_j}$. □

The following observation leads to a quite different definition of normality, which comes more from the study of algebraic number fields.

LEMMA 1.25. *Let the notation be as in (1.23). Assume that $t \in Q(S)$ satisfies a monic equation*

$$t^m + s_{m-1}t^{m-1} + \cdots + s_0 = 0, \quad \textit{where } s_i \in S.$$

Then $t \in \bar{S}$.

Proof. Pick any $\phi : S \to R$, and consider its extension Φ. We get that

$$\Phi(t)^m + \phi(s_{m-1})\Phi(t)^{m-1} + \cdots + \phi(s_0) = 0.$$

If $\Phi(t)$ has a pole of order $r > 0$, then $\Phi(t)^m$ has a pole of order mr, while all the other terms of the equation have a pole of order at most $(m-1)r$. This is impossible, and hence $\Phi(t) \in R$. □

DEFINITION 1.26. Let $S \subset S'$ be a ring extension. We say that $s \in S'$ is *integral* over S if one of the following equivalent conditions holds.

(1) s satisfies a monic equation

$$s^m + r_{m-1}s^{m-1} + \cdots + r_0 = 0, \quad \text{where } r_i \in S.$$

(2) The subring $S[s] \subset S'$ is a finitely generated S-module.

It is easy to see that all elements integral over S form a subring, called the *integral closure* or *normalization* of S in S'.

DEFINITION 1.27. Let S be an integral domain. The *normalization* of S is its integral closure in its quotient field $Q(S)$.

This is now the standard definition in commutative algebra books. See, for instance, [AM69, Chap.5.Sec.1] for the basic properties that we use.

REMARK 1.28. The fact that these definitions (1.23) and (1.27) are the same is not obvious. One implication is given by (1.25).

The easy argument that every normal integral domain S is the intersection of all the valuation rings V sitting between it and its quotient field $S \subset V \subset Q(S)$ is in [AM69, 5.22]. Working only with discrete valuation rings is a bit harder. The strongest theorem in this direction is Serre's criterion for normality; see [Mat70, 17.I] or [Mat89, 23.8]. We do not need it for now.

1.29 (Is normalization useful?). With the concept of normalization established, we have to see how useful it is. In connection with resolutions, two questions come to mind.

(1) If C is an affine curve whose coordinate ring $k[C]$ is normal, does that imply that C is smooth?

(2) Does the normalization give the resolution?

The answer to the first question is yes, as we see it in (1.30). This of course begs another question, how do we see smoothness in terms of commutative algebra?

Just look at a plane curve C with coordinate ring $k[x,y]/(f(x,y))$ and let $m \subset k[x,y]/(f(x,y))$ be the ideal of the origin. The following are easily seen to be equivalent.

(i) C is smooth at the origin.
(ii) $f(0,0) = 0$ and f contains a linear term.
(iii) $\dim_k m/m^2 = 1$.

In general, we say that a 1-dimensional ring is *regular* if for any maximal ideal $m \subset R$ the quotient m/m^2 is 1-dimensional as an R/m-vector space. We end up proving that every 1-dimensional normal ring is regular. (The converse is easy.)

The second question is more troublesome. When constructing resolutions, we need $f : \bar{C} \to C$ to be surjective. By the going-up and -down theorems (cf. [AM69, 5.11, 5.16]) we are in good shape if $k[\bar{C}]$ is *finite* over $k[C]$. (That is, $k[\bar{C}]$ is finitely generated as a module over $k[C]$.) This is nowadays adopted as a necessary condition of resolution. For coordinate rings of algebraic varieties, the normalization is finite (1.33).

It should be noted though that finiteness of normalization fails in some examples (cf. (1.103) or (1.105)), but one could reasonably claim that the normalization is nonetheless the resolution.

The following is the nicest result relating normalization to resolutions.

THEOREM 1.30. *Let R be a 1-dimensional, normal, Noetherian integral domain. Then R is regular. That is, for every maximal ideal $m \subset R$, the quotient m/m^2 is 1-dimensional (over R/m).*

1.31 (Nuts-and-bolts proof). For C a singular, reduced and affine curve over an algebraically closed field k, we construct an explicit rational function in $k(C) \setminus k[C]$ that is integral over $k[C]$. Besides proving (1.30), this also gives an algorithm to construct the normalization.

Pick $p \in C$, and assume that m_p/m_p^2 is at least 2-dimensional. Pick any $x, y \in m_p$ that are independent in m_p/m_p^2.

Claim 1.31.1. There are $a \in k$ and $u \in k[C] \setminus m_p$ such that

$$\frac{y}{x+ay}u \notin k[C] \quad \text{is integral over } k[C].$$

Proof. An element of this form is definitely not in $k[C]$. Indeed, otherwise we would have that

$$uy = (x+ay)\frac{y}{x+ay}u \in (x+ay)k[C],$$

so uy would be a multiple of $x + ay$ in m_p/m_p^2, a contradiction.

x, y can be viewed as elements of the function field $k(C)$, which has transcendence degree 1 over k. Thus x, y satisfy an algebraic relation $f(x, y) = 0$. Let $f_m(x, y)$ be the lowest degree homogeneous part.

Make a substitution $x \mapsto x+ay$. The coefficient of y^m in $f_m(x, y)$ is now $f_m(a, 1)$. We can choose a different from zero such that after multiplying by a suitable constant we achieve that

$$f(x, y) = y^m + \big(\text{other terms } (x + ay)^i y^j \text{ of degree at least } m\big).$$

We can rewrite this as

$$y^m\big(1 + g_m(x, y)\big) + y^{m-1}(x + ay)g_{m-1}(x, y) + \cdots + (x + ay)^m g_0(x, y) = 0,$$

where $g_m(x, y) \in (x, y) \subset m$. Hence $u := 1 + g_m(x, y)$ is not in m_p and $g_i \in k[C]$ for $i < m$. Thus over $k[C]$ we get the integral dependence relation

$$\left(\frac{y}{x+ay}u\right)^m + \left(\frac{y}{x+ay}u\right)^{m-1}\big(g_{m-1}(x, y)\big) + \cdots + \big(u^{m-1}g_0(x, y)\big) = 0. \quad \square$$

1.32 (Slick proof). By localizing at m_p we are reduced to the case where (R, m) is local. Pick an element $x \in m \setminus m^2$. If $m = (x)$, then we are done. Otherwise, $m/(x)$ is nonzero. Since R has dimension 1, $R/(x)$ is 0-dimensional, so $m/(x)$ is killed by a power of m. Thus there is a $y \in m \setminus (x)$ such that $my \in (x)$. Equivalently,

$$\frac{y}{x}m \subset R.$$

If $\frac{y}{x}m$ contains a unit, then $\frac{y}{x}z = u$ for some $z \in m$ and a unit u; thus $x = yzu^{-1} \in m^2$, which is impossible.

Thus $\frac{y}{x}m \subset m$. Now we can run the beautiful proof of the Nakayama lemma (cf. [AM69, 2.4]), which is worth repeating.

Write $m = (x_1, \ldots, x_n)$. Then there are $r_{ij} \in R$ such that

$$\frac{y}{x}x_j = \sum_i r_{ij}x_j.$$

Thus the vector $(x_1, \ldots, x_n)$ is a null vector of the matrix

$$\frac{y}{x}\mathbf{1}_n - (r_{ij}),$$

and hence its determinant is zero. This determinant is a monic polynomial in $\frac{y}{x}$ with coefficients in R, and hence $\frac{y}{x} \in R$ since R is normal.

This, however, means that $y \in (x)$, contrary to our choice of y. $\square$

THEOREM 1.33. *Let S be an integral domain that is finitely generated over a field k, and let $F \supset Q(S)$ be a finite field extension of its quotient field. Then the normalization of S in F is finite over S.*

Proof. One's first idea might be to write down any finite extension S'/S whose quotient field is F and reduce everything to the case where $F = Q(S)$. We will see how this works for curves, but in general it seems better to go the other way around, even when $F = Q(S)$ to start with.

By the Noether normalization theorem (1.35), S is finite over a polynomial ring R. So it is enough to prove (1.33) for a polynomial ring $R = k[x_1, \dots, x_n]$. The key advantage of this reduction is that R is normal.

A short argument proving (1.33) in the case where $F/Q(S)$ is separable and S is normal is in [AM69, 5.17]. Thus we are done in characteristic zero, but in positive characteristic we still have to deal with inseparable extensions.

F is a finite extension of $k(x_1, \dots, x_n)$, so there is a finite purely inseparable extension $E/k(x_1, \dots, x_n)$ such that EF/E is separable. Every finite purely inseparable extension of $k(x_1, \dots, x_n)$ is contained in a field $k'(x_1^{p^{-m}}, \dots, x_n^{p^{-m}})$, where k'/k is finite and purely inseparable. The normalization of $k[x_1, \dots, x_n]$ in $k'(x_1^{p^{-m}}, \dots, x_n^{p^{-m}})$ is $k'[x_1^{p^{-m}}, \dots, x_n^{p^{-m}}]$ since the latter is a unique factorization domain, hence normal by (1.24).

Thus every finite extension of $k[x_1, \dots, x_n]$ is contained in a finite and separable extension of $k'[x_1^{p^{-m}}, \dots, x_n^{p^{-m}}]$, and hence it is finite by the first argument. □

1.34. Here is another approach to (1.33) for projective curves.

Let C be an irreducible, reduced projective curve over an algebraically closed field k. If $C_0 := C$ is not smooth, then as in (1.31) we can write down another curve C_1 and a proper birational map $\pi_1 : C_1 \to C_0$, which is not an isomorphism. We prove that after finitely many iterations we get a smooth curve.

In (1.13) we constructed a line bundle L on C such that

$$h^0(C_0, L^m) \geq m \deg L + 1 - p(C_0, L) \quad \text{for some } p(C_0, L) \geq 0 \text{ and } m \gg 1.$$

Choose that smallest possible value for $p(C_0, L)$. (This is the arithmetic genus, but we do not need to know this for the present argument.) The line bundle $\pi_1^* L^m$ is very ample for $m \gg 1$, so not all of its sections pull back from C_0. Thus

$$h^0(C_1, \pi_1^* L^m) > h^0(C_0, L^m) \quad \text{for } m \gg 1.$$

Hence there is a $p(C_1, L) < p(C_0, L)$ such that

$$h^0(C_1, \pi_1^* L^m) \geq m \deg L + 1 - p(C_1, L) \quad \text{for } m \gg 1.$$

Iterating this procedure, we get curves $\pi_i : C_i \to C_0$ such that

$$h^0(C_i, \pi_i^* L^m) \geq m \deg L + 1 - p(C_0, L) + i \quad \text{for } m \gg 1.$$

Since $h^0(D, M) \leq \deg M + 1$ for any irreducible and reduced curve D and for any line bundle M, we conclude that we get a smooth curve C_j for some $j \leq p(C_0, L)$. □

In the proofs above we have used the following theorem, due to E. Noether. It is traditionally called the normalization theorem, though it has nothing to do with normalization as defined in this section. Rather, it creates some sort of a "normal form" for all rings that are finitely generated over a field k. This form is not at all unique, but it is very useful in reducing many questions involving general affine varieties to affine spaces.

THEOREM 1.35 (Noether normalization theorem). *Let S be a ring that is finitely generated over a field k. Then there is a polynomial ring $k[x_1, \dots, x_n] \cong R \subset S$ such that S is finite over R (that is, finitely generated as an R-module).*

Proof. By assumption S can be written as $k[y_1, \dots, y_m]/(f_1, \dots, f_s)$.

Look at the first equation f_1, say, of degree d_1. If $y_m^{d_1}$ appears in f_1 with nonzero coefficient, then f_1 shows that y_m is integral over $k[y_1, \dots, y_{m-1}]$ and we finish by induction.

Otherwise we try to make a change of variables $y_i = y_i' + g_i(y_m)$ for $i < m$ to get a new polynomial equation

$$f_1' := f_1\bigl(y_1' + g_1(y_m), \dots, y_{m-1}' + g_{m-1}(y_m), y_m\bigr),$$

which shows that y_m is integral over $k[y_1', \dots, y_{m-1}']$.

If k is infinite, then we can get by with a linear change $y_i = y_i' + a_i y_m$. Indeed, after this change the coefficient of $y_m^{d_1}$ in f_1' is $f_{1,d_1}(a_1, \dots, a_{m-1}, 1)$, where f_{1,d_1} denotes the degree d_1 homogeneous part of f_1. This is nonzero for general $a_1, \dots, a_{m-1}$.

For finite fields such coordinate changes may not work, and here we use $y_i = y_i' + y_m^{n_i}$.

Assume that f_1 contains a term $x_1^{a_1} \cdots x_m^{a_m}$ with nonzero coefficient, and this is lexicographically the largest one. (That is, a_1 is the largest possible, among those with maximal a_1, then a_2 is the largest, etc.)

As long as the sequence $n_1, \dots, n_m$ satisfies $n_i > d_1 n_{i+1}$, the highest y_m-power in f_1' is $y_m^{\sum n_i a_i}$, and it occurs with nonzero constant coefficient. So y_m is integral over $k[y_1', \dots, y_{m-1}']$. □

1.5. Infinitely near singularities

Let $C \subset \mathbb{C}^2$ be a plane curve given by an equation $f(x, y) = 0$. In order to study the singularity of C at the origin, we write f as a sum of homogeneous terms

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots .$$

The degree of the lowest nonzero term f_m is the *multiplicity* of C at $0 \in \mathbb{C}^2$. It is denoted by $\operatorname{mult}_0 f$ or by $\operatorname{mult}_0 C$.

The multiplicity is pretty much the only invariant of the singularity that is easily computable from the equation, but it is not a really good measure of the complexity of the singularity. M. Noether realized that one should view a curve singularity together with all other singularities that can be obtained from it by blowing up. These are the so-called *infinitely near singularities*. The basic observation is that the true complexity of the singularity can be computed from the multiplicities of all infinitely near singularities.

This idea also leads to several nice methods to resolve singularities.

1.36 (Blowing up a smooth surface). Let $0 \in S$ be a point on a smooth surface, and let x, y be local coordinates. Assume for simplicity that x, y do not have any other common zero outside 0. (This can always be achieved by replacing S by a smaller affine neighborhood of $0 \in S$.)

The blow-up of $0 \in S$, denoted by $B_0 S$, is the surface

$$B_0 S := (xu - yv = 0) \subset S \times \mathbb{P}^1_{(u:v)}.$$

Let $\pi : B_0 S \to S$ denote the first coordinate projection. The curve $E = (x = y = 0) \subset B_0 S$, called the *exceptional curve*, is contracted by π to the point 0, and $\pi : B_0 S \setminus E \cong S \setminus \{0\}$ is an isomorphism.

It is usually convenient to work with an affine covering of $B_0 S$. On the $v \neq 0$ chart, we can use $y_1 := u/v$ as a coordinate, and then we have

$$B_0 S_{v \neq 0} = (x y_1 - y = 0) \subset S \times \mathbb{A}^1_{y_1}.$$

Thus if $S = \operatorname{Spec} R$, then $B_0 S_{v \neq 0} = \operatorname{Spec} R[\frac{y}{x}]$. Similarly, the $u \neq 0$ chart can be represented as $\operatorname{Spec} R[\frac{y}{x}]$. From this we see that $B_0 S$ is smooth.

See [Sha94, Sec.II.4] for more details.

1.37 (Digression on fields of representatives). If X is an algebraic variety over a field k and $x \in X$ is a point, then we have a corresponding local ring $\mathcal{O}_{x,X}$ with maximal ideal m_x. $\mathcal{O}_{x,X}$ is a k-algebra, but its residue field $\mathcal{O}_{x,X}/m_x$ is usually bigger than k if k is not algebraically closed or if x is not a closed point.

In general $\mathcal{O}_{x,X}$ does not contain any field K that maps isomorphically onto $\mathcal{O}_{x,X}/m_x$. For instance, if $k = \mathbb{C}$ and $x = Y \subset X$ is a subvariety, then the existence of such a field K is equivalent to a rational "retraction" map $X \dashrightarrow Y$, which is the identity on Y.

There is no such map if $X = \mathbb{C}^2$ and if $Y \subset \mathbb{C}^2$ is a nonrational plane curve.

This is somewhat awkward since in any local ring (R, m) it is very convenient to imagine that the elements of m^s/m^{s+1} are degree s homogeneous polynomials in a basis of m/m^2.

Although we do not have any natural section of the quotient map $R \to R/m$, we just pick any set-theoretic lifting $R/m \dashrightarrow R$ (which is neither additive nor multiplicative in general), and this allows us to write things like $\sigma(a)x^i y^j$ for any $a \in R/m$ and $x, y \in m$. This will be unique modulo m^{i+j+1}.

One has to be careful not to get carried away, but occasionally this makes it easier to see the analogy between the classical case and more general rings.

1.38 (Infinitely near singularities). Let the notation be as in (1.36). Consider a curve $0 \in C \subset S$ with equation $(f = 0)$. There is a largest power $(x, y)^m$ of the maximal ideal that contains f; this m is the *multiplicity* of C at $0 \in S$, cf. (1.20.4).

Thus f modulo $(x, y)^{m+1}$ can be identified with a degree m homogeneous polynomial $f_m(x, y)$, called the *leading term* of f.

What happens to C under the blow-up?

On the chart $B_0S_{v\neq 0} = \operatorname{Spec} R[\frac{y}{x}]$, the pullback of $f \in R$ is again $f \in R[\frac{y}{x}] = R[y_1]$. The change is that $y = y_1 x$, and thus $(x, y)^m \subset x^m(y_1, 1)^m$. This means that we can write f as $f = x^m f^1$ for some $f^1 \in R[y_1]$.

Thus the pullback of C contains the exceptional curve E with multiplicity m (defined by $(x^m = 0)$ on the $v \neq 0$ chart), and the *birational transform* of C, denoted by C_1, is defined by $(f^1 = 0)$ on the $v \neq 0$ chart:

$$\pi^* C = (\operatorname{mult}_0 C) \cdot E + C_1. \tag{1.38.1}$$

Definition 1.39. The singularities of C_1 lying above $0 \in C$ are called the *infinitely near singularities* in the first infinitesimal neighborhood of $0 \in C$.

Similarly, the singularities in the first infinitesimal neighborhood of $\pi^{-1}(0) \subset C_1$ are called the infinitely near singularities in the second infinitesimal neighborhood of $0 \in C$, and so on.

Thus starting with a singular curve $0 \in C \subset S$, we get a towering system of infinitely near singularities on various blow-ups of S.

It is very easy to convince yourself through examples that each blow-up improves the singularities—that is, the infinitely near singularities are "better" than the original one—but it is nonetheless hard to come up with a good general statement about what actually improves. We will see several ways of doing it, but for now let us just see some simple results and examples.

Lemma 1.40. *Let the notation be as above. The intersection points $C_1 \cap E$ are the roots of $(f_m = 0) \subset \mathbb{P}^1 \cong E$. More precisely, when counted with multiplicities, $C_1 \cap E = (f_m = 0)$. Thus,*

(1) *the intersection number $(C_1 \cdot E)$ equals* $\operatorname{mult}_0 C$, *and*

(2) $\sum_{p \in C_1 \cap E} \operatorname{mult}_p C_1 \le \operatorname{mult}_0 C$.

Proof. Write $f = f_m + r$, where $r \in (x,y)^{m+1}$ (cf. (1.37)).

In the $v \neq 0$ chart $f^1 = f_m(y_1, 1) + xr_1$; hence the intersection points of C_1 and E, that is, the solutions of $f_1 = x = 0$, are the points $f_m(y_1, 1) = 0$ on E, proving (1). Furthermore, by (1.20.4), the multiplicity of C_1 at $(y_1 = a, x = 0)$ is $\le$ the multiplicity of $y_1 = a$ as a root of $f_m(y_1, 1) = 0$, which gives (2). □

EXAMPLE 1.41. Consider the singularity $C := (x^a = y^b)$. If $a < b$, then on the first infinitesimal neighborhood we find $C_1 = (x_1^a = y_1^{b-a})$.

If $a < b - a$, then on the second infinitesimal neighborhood we get $C_2 = (x_1^a = y_1^{b-2a})$ and so on.

Eventually we reach a stage where $b - ka \le a$. Here we reverse the role of x and y and continue as before. Thus we see the usual Euclidean algorithm for the pair a, b carried out on exponents.

At some point we get an equation $x_s^c = y_s^c$. After one more blow-up we get c different smooth points. The singularity is thus resolved and we stop.

Thus far the resolutions constructed were abstract curves. In some cases the existence was only local, and in some other cases we ended up with a smooth curve or Riemann surface without any specific embedding into $\mathbb{C}^n$ or $\mathbb{P}^n$. In the Riemann surface case it is not even obvious that $\bar{C}$ has any embeddings into some $\mathbb{C}^n$.

It is frequently very useful to have a resolution that takes into account the ambient affine or projective space containing the curve C.

We summarize the above examples into an algorithm. Later we prove several cases when the algorithm does work.

ALGORITHM 1.42 (Weak embedded resolution algorithm). Let S_0 be a smooth surface over a perfect field and $C_0 \subset S_0$ a curve.

If $C_i \subset S_i$ is already constructed then pick any singular point $p_i \in C_i$, let $S_{i+1} \to S_i$ be the blow-up of p_i, and let $C_{i+1} \subset S_{i+1}$ be the birational transform of C_i.

For now we only state the weak embedded resolution theorem. The next six sections contain seven versions of its proof.

THEOREM 1.43 (Weak embedded resolution). *Let S be a smooth surface over a perfect field k and $C \subset S$ a reduced curve. After finitely many steps the weak embedded resolution algorithm (1.42) stops with $S_m \to S$ such that $C_m \subset S_m$ is smooth.*

It is often convenient to have an embedded resolution where not only is the birational transform of C smooth, but all the exceptional curves of the blow-ups behave as nicely as possible. We cannot make everything disjoint, but we can achieve the next best situation, simple normal crossing.

DEFINITION 1.44. Let X be a smooth variety and $D \subset X$ a divisor. We say that D is a *simple normal crossing* divisor (abbreviated as *snc* divisor) if every irreducible component of D is smooth and all intersections are transverse.

That is, for every $p \in X$ we can choose local coordinates $x_1, \dots, x_n$ and natural numbers $m_1, \dots, m_n$ such that $D = (\prod_i x_i^{m_i} = 0)$ in a neighborhood of x.

REMARK 1.45. A frequently occurring variant of this concept is that of a *normal crossing* divisor. Here we assume that for every $p \in X$ there are local analytic or formal coordinates $x_1, \dots, x_n$ and natural numbers $m_1, \dots, m_n$ such that $D = (\prod_i x_i^{m_i} = 0)$ in a local analytic or formal neighborhood of p.

Thus the nodal curve $y^2 = x^3 + x^2$ is a normal crossing divisor in $\mathbb{C}^2$ but not a simple normal crossing divisor. Indeed, we can write

$$y^2 - x^3 - x^2 = (y - x\sqrt{1+x})(y + x\sqrt{1+x})$$

as a power series, but $y^2 - x^3 - x^2$ is irreducible as a polynomial.

Be warned that in the literature the distinction between normal crossing and simple normal crossing is not systematic. In most cases the difference between them is a small technical matter, but occasionally it can cause difficulties.

ALGORITHM 1.46 (Strong embedded resolution algorithm). Let S_0 be a smooth surface over a perfect field k and $C_0 \subset S_0$ a curve.

If $\pi_i : S_i \to S$ is already constructed, then pick any point $p_i \in \pi_i^{-1}(C_0)$ where $\pi_i^{-1}(C_0)$ is not a simple normal crossing divisor, let $\sigma_{i+1} : S_{i+1} \to S_i$ be the blow-up of p_i and let $\pi_{i+1} := \pi_i \circ \sigma_{i+1} : S_{i+1} \to S_0$ be the composite.

THEOREM 1.47 (Strong embedded resolution). *Let S be a smooth surface over a perfect field k and $C \subset S$ a curve. After finitely many steps the strong embedded resolution algorithm (1.46) stops with $\pi_m : S_m \to S$ such that $\pi_m^{-1}(C)$ is a simple normal crossing divisor.*

1.48 (Proof of (1.43) $\Rightarrow$ (1.47)). The algorithm (1.46) does not specify the order of the blow-ups, but let us be a little more systematic first.

Starting with $C \subset S$, use (1.43) to get $C_m \subset S_m$ such that C_m is smooth. Let $E_m \subset S_m$ be the exceptional divisor of $S_m \to S$. Then C_m is smooth and E_m is a simple normal crossing divisor, but $C_m + E_m$ need not be a simple normal crossing divisor.

Let us now apply (1.43) again to $C_m + E_m \subset S_m$. We get $S_m^* \to S_m$ with exceptional divisor F^* such that the birational transform $C_m^* + E_m^* \subset S_m^*$ is smooth.

Since every irreducible component of $C_m + E_m$ is smooth, we see that every irreducible component of $C_m^* + E_m^*$ has only simple normal crossings with F^*. As $C_m^* + E_m^*$ is smooth, this implies that $C_m^* + E_m^* + F^*$ is a simple normal crossing divisor, and thus we have achieved strong embedded resolution.

We still need to show that blowing up in any other order also gets to strong embedded resolution. Even more, we claim that we always end up with a surface $S_n \to S$ that is dominated by S_m^*. That is, there is factorization $S_m^* \to S_n \to S$.

Assume that we already know that S_m^* dominates some S_i. If we blow up $p_i \in S_i$ as the next step of the algorithm (1.46), then $\pi_i^{-1}(C)$ does not have simple normal crossing at p, and hence the point p_i is also blown up when we construct S_m^* but maybe at a later stage. For blowing up points on a surface it does not matter in which order we blow them up, so S_m^* also dominates S_{i+1}, and we are done by induction. □

Note that in getting $S_m^* \to S$ we may have performed some unnecessary blow-ups as well. Indeed, if $C \subset S$ has some simple normal crossing points, then these were blown up in constructing $S_m \to S$ but there is no need to blow these up in the algorithm (1.46).

1.49 (Digression on blow-ups over imperfect fields). I heartily recommend avoiding blowing up over imperfect fields unless it is absolutely necessary. Here are two examples to show what can happen.

Let u and v be indeterminates over a field k of prime characteristic p. Consider the affine plane $\mathbb{A}^2$ over the field $k(u,v)$.

Let $P \in \mathbb{A}^2$ denote the closed point corresponding to $(0, \sqrt[p]{v})$. The ideal of P is $(x, y^p - v)$. Thus the blow-up is given in $\mathbb{A}^2_{x,y} \times \mathbb{P}^1_{s,t}$ by the equation $xs - (y^p - v)t = 0$. Over the algebraic closure we can introduce a new coordinate $y_1 = y - \sqrt[p]{v}$, and the equation becomes $xs - y_1^p t = 0$. The resulting hypersurface is singular at $x = s = y_1 = 0$.

Let $Q \in \mathbb{A}^2$ denote the closed point corresponding to $(\sqrt[p]{u}, \sqrt[p]{v})$. The ideal of Q is $(x^p - u, y^p - v)$, and the blow-up is given by the equation $(x^p - u)s - (y^p - v)t = 0$. Over the algebraic closure we can introduce new coordinates $x_1 = x - \sqrt[p]{u}, y_1 = y - \sqrt[p]{v}$ and the equation becomes $x_1^p s - y_1^p t = 0$. The corresponding hypersurface is singular along the line $x_1 = y_1 = 0$.

On the other hand, both of the blow-ups are nonsingular hypersurfaces; that is, their local rings are regular.

In the first case, the only question is at the point with maximal ideal $(x, y^p - v, \frac{s}{t})$. The equation of the blown-up surface is $x\frac{s}{t} = y^p - v$, so x and $\frac{s}{t}$ are local coordinates on $B_P\mathbb{A}^2$.

In the second case, look at any point along the exceptional curve, say with maximal ideal $(x^p - u, y^p - v, \frac{s}{t} - a)$. The equation of the blown-up surface is

$$(x^p - u)(\tfrac{s}{t} - a) + a(x^p - u) = y^p - v,$$

and thus $x^p - u$ and $\frac{s}{t} - a$ are local coordinates on $B_Q\mathbb{A}^2$.

1.6. Embedded resolution, I: Global methods

Another observation of M. Noether is that it is easier to measure the singularities of a global curve. In effect, he considered the arithmetic genus of the curve. (Working on $\mathbb{P}^2$, he used the closely related intersection number between the affine curve $f(x,y) = 0$ and its polar curve $\partial f/\partial y = 0$.)

In hindsight it is not difficult to localize the proof, as we see in Section 1.8, but we start with the global version. For anyone familiar with basic algebraic geometry, the global version is faster, and the proof is a very nice application of intersection theory on surfaces. The local proofs are, however, more elementary.

We use the basic properties of intersection numbers of curves on smooth projective surfaces and of the canonical class of a surface. We start by recalling the relevant facts.

1.50 (Intersection numbers on smooth surfaces). Here is a summary of the properties that we use. For proofs see [Sha94, IV.1 and IV.3.2].

Let S be a smooth projective surface over a field k and C, D divisors on S. Then one can define their *intersection number*, $(C \cdot D)$ which has the following properties. (In fact, it is defined by the properties (1.50.1–3).)

(1) $(C \cdot D) \in \mathbb{Z}$ is bilinear.
(2) If $C_1 \sim C_2$ are linearly equivalent, then $(C_1 \cdot D) = (C_2 \cdot D)$.
(3) If C, D are effective and $C \cap D$ is finite, then

$$(C \cdot D) = \sum_{P \in C \cap D} \dim_k \mathcal{O}_{P,S}/(f_P, g_P),$$

where f_P (resp., g_P) is a local equation for C (resp., D) at P.
(4) Let $h : S' \to S$ be a birational morphism. Then $(h^*C \cdot h^*D) = (C \cdot D)$.
(5) Let $h : S' \to S$ be a birational morphism and $E \subset S'$ an h-exceptional divisor. Then $(f^*C \cdot E) = 0$.
(6) Let $h : S' \to S$ be the blow-up of a smooth k-point and $E \subset S'$ the exceptional curve. Then $(E \cdot E) = -1$.

The number $(C \cdot D)_P = \dim_k \mathcal{O}_{P,S}/(f_P, g_P)$ is called the *local intersection number* of C and D at P. (The local intersection number is defined only if $P \in C \cap D$ is an isolated point.)

1.51 (Canonical divisor). Let X be a smooth variety of dimension n. Divisors of rational differential n-forms on X are linearly equivalent, and they form the *canonical class* denoted by K_X; see, for instance, [Sha94, Sec.III.6.3]. It is a long-standing tradition to pretend sometimes that K_X is a divisor, but it is only a linear equivalence class.

One checks easily that $K_{\mathbb{P}^n} = -(n+1) \cdot$ (hyperplane class).

Let $h : S' \to S$ be the blow-up of a k-point P of a smooth surface defined over k and $E \subset S'$ the exceptional curve. Choose local coordinates (x, y) at P. An affine chart of S' is given by $y_1 = y/x, x_1 = x$ and

$$h^*(dx \wedge dy) = dx_1 \wedge d(x_1 y_1) = x_1 \cdot dx_1 \wedge dy_1.$$

Thus we conclude that

$$K_{S'} = h^* K_S + E. \tag{1.51.1}$$

THEOREM 1.52 (Weak embedded resolution, I). *Let S_0 be a smooth projective surface over a perfect field k and $C_0 \subset S_0$ a reduced projective curve. After finitely many steps, the weak embedded resolution algorithm (1.42) stops with a smooth curve $C_m \subset S_m$.*

Proof. We look at the intersection number $C \cdot (C + K_S)$ and prove two properties:

(1.52.1) $C_{i+1} \cdot (C_{i+1} + K_{S_{i+1}}) < C_i \cdot (C_i + K_{S_i})$, and
(1.52.2) $C_i \cdot (C_i + K_{S_i})$ is bounded from below.

These two imply the termination of the blow-up process, and they even give a bound on the number of necessary steps.

Another variant of this method uses $h^1(C, \mathcal{O}_C)$ instead of $C \cdot (C + K_S)$; cf. [Har77, V.3.8].

The proof of (1.52.1) is a straightforward local computation using (1.53).

The proof of (1.52.2) is again not difficult. We discuss two methods that do not need the knowledge of resolution of curves.

Method 1 using duality theory. If you know enough duality theory (say, as in [Har77, Sec.III.8]) then you know that $\mathcal{O}_S(C + K_S)|_C$ is isomorphic to the dualizing sheaf ω_C. Then we claim that

$$\deg \omega_C \geq (-2) \cdot \#\{\text{irreducible geometric components of } C\}.$$

(With a little more care one could sharpen this to the number of connected geometric components.) This is easy if we know that C has a resolution, but here is an argument that does not rely on this.

If C is irreducible over an algebraically closed field k, pick two smooth points $p, q \in C$. From

$$0 \to \mathcal{O}_C(-p-q) \to \mathcal{O}_C \to k_p + k_q \to 0$$

we conclude that $h^1(C, \mathcal{O}_C(-p-q)) > 0$. Since $H^1(C, \mathcal{O}_C(-p-q))$ is dual to $\operatorname{Hom}(\mathcal{O}_C(-p-q), \omega_C)$, we conclude that $\deg \omega_C \geq \deg \mathcal{O}_C(-p-q) = -2$. (It is here that we use that C is reduced, and hence ω_C has no nilpotents. Otherwise, the homomorphism $\mathcal{O}_C(-p-q) \to \omega_C$ could not be used to bound the degree.)

If $C = \cup C_i$ are the irreducible geometric components, then

$$\big(C \cdot (C + K_S)\big) = \sum_i \big(C_i \cdot (C_i + K_S)\big) + \sum_{i \neq j} (C_i \cdot C_j) \geq \sum_i \big(C_i \cdot (C_i + K_S)\big)$$

and we are done. □

The next method is entirely elementary but gives a weaker bound.

Method 2 using differential forms. We prove two assertions.

(1.52.3) There is an injection $\Omega_C/(\text{torsion}) \hookrightarrow \mathcal{O}_S(C + K_S)|_C$.

(1.52.4) Let $f : C \to \mathbb{P}^1$ be a separable morphism of degree n. Then there is an injection $f^*\mathcal{O}_{\mathbb{P}^1}(-2) \cong f^*\Omega_{\mathbb{P}^1} \hookrightarrow \Omega_C$.

Assuming these, take a separable morphism $f_0 : C_0 \to \mathbb{P}^1$, say, of degree n. Since $C_i \to C_0$ are birational, we get degree n separable morphisms $f_i : C_i \to \mathbb{P}^1$ for every i. Thus $\deg \Omega_{C_i}/(\text{torsion}) \geq -2n$, and so $C_i \cdot (C_i + K_{S_i}) \geq -2n$, proving (1.52.2).

The proof of (1.52.4) is easy since differential forms can be pulled back and the pullback map is injective for a separable map.

Let us study the map in (1.52.3) in local coordinates x, y. Here Ω_S is generated by dx, dy and Ω_C is generated by the restrictions $dx|_C, dy|_C$.

What about $\mathcal{O}_S(C + K_S)|_C$? The local generator is $f^{-1}dx \wedge dy$, and then we take its residue along C. That is, we write

$$\frac{1}{f} dx \wedge dy = \frac{df}{f} \wedge \sigma,$$

and then $\sigma|_C$ is the local generator of $\mathcal{O}_S(C+K_S)|_C$. Thus we see that along the smooth points of C we can identify $\mathcal{O}_S(C + K_S)|_C$ with Ω_C, and even near singular points, $dx|_C, dy|_C$ give rational sections of $\mathcal{O}_S(C + K_S)|_C$. We only need to prove that they do not have poles. (In fact we see that they have zeros of quite high order.)

Since $df = (\partial f/\partial x)dx + (\partial f/\partial y)dy$, we get that

$$\frac{1}{f} dx \wedge dy = \frac{df}{f} \wedge \frac{dy}{\partial f/\partial x} = -\frac{df}{f} \wedge \frac{dx}{\partial f/\partial y},$$

and hence the local generator of $\mathcal{O}_S(C + K_S)|_C$ is

$$\sigma|_C = \frac{dy}{\partial f/\partial x} = -\frac{dx}{\partial f/\partial y}.$$

Therefore, $dx|_C = -(\partial f/\partial y)\sigma_C$ and $dy|_C = (\partial f/\partial x)\sigma_C$ both have zeros. □

LEMMA 1.53. *Let S be a smooth surface over a perfect field k and $C \subset S$ a projective curve. Let $p \in C$ be a closed point of degree d; that is, p is a conjugation-invariant set of d points in $C(\bar{k})$. Let $m = \operatorname{mult}_p C$ be its multiplicity. Let $\pi : S' \to S$ denote the blow-up of p and $C' \subset S'$ the birational transform of C. Then*

(1) $(C' \cdot C') = (C \cdot C) - dm^2$,
(2) $(C' \cdot K_{S'}) = (C \cdot K_S) + dm$, *and*
(3) $(C' \cdot (C' + K_{S'})) = (C \cdot (C + K_S)) - dm(m-1)$.

Proof. Over $\bar{k}$, the blow-up of p is just the blow-up of d distinct points, so it is enough to compute what happens under one such blow-up. Thus assume that k is algebraically closed.

We have already computed in (1.38.1) that $\pi^*C = C' + mE$, where $E \subset B_pS$ is the exceptional curve. Thus by (1.50.4–6) we get that

$$\begin{array}{rcl}(C' \cdot C') & = & (\pi^*C - mE) \cdot (\pi^*C - mE) \\ & = & (\pi^*C \cdot \pi^*C) - 2m(\pi^*C \cdot E) + m^2(E^2) \\ & = & (C \cdot C) - m^2.\end{array}$$

Similarly, using (1.51.1) we get that

$$\begin{array}{rcl}(C' \cdot K_{S'}) & = & \big((\pi^*C - mE) \cdot (\pi^*K_S + E)\big) \\ & = & (C \cdot K_S) - m(E \cdot E).\end{array}$$

Finally (3) is a combination of (1) and (2). □

1.7. Birational transforms of plane curves

If we start the embedded resolution process of the previous section with a plane curve $C \subset \mathbb{P}^2$, the method produces a smooth curve that sits in a plane blown up in many points. Classical geometers were foremost interested in plane curves, and from their point of view the natural problem was to start with a projective plane curve $f_1(x_0 : x_1 : x_2) = 0$ and to perform birational transformations of $\mathbb{P}^2$ that "improve" the equation step-by-step until one gets another equation $f_m(x_0 : x_1 : x_2) = 0$, which is the resolution of the original curve.

This is, however, too much to hope for, since many curves are not isomorphic to smooth plane curves at all (for instance, because a smooth plane curve of degree d has genus $(d-1)(d-2)/2$). Therefore, we have to settle for a model $f_m(x_0 : x_1 : x_2) = 0$, which has "relatively simple" singularities.

From our modern point of view, the method and its difficulties are the following.

Starting with $C \subset \mathbb{P}^2$, as a first resolution step in (1.52) we get $C_1 \subset B_p\mathbb{P}^2$. Usually C_1 is not even isomorphic to any plane curve, but we want to force it back into the plane $\mathbb{P}^2$. There are two complications.

- In our effort to put C_1 back into $\mathbb{P}^2$, we have to introduce new singular points.
- There is no "canonical" way of creating a plane curve out of C_1, and we have to make some additional choices. This makes the

process somewhat arbitrary, and it can happen that over finite fields there are no suitable choices at all.

With these limitations, there are two very nice classical solutions.

The one by M. Noether uses birational transformations $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to create a curve $C_m \subset \mathbb{P}^2$, which is birational to C and has only ordinary multiple points (1.54), that is, any number of smooth branches intersecting pairwise transversally.

The method of Bertini uses degree 2 maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to create a curve $C_m \subset \mathbb{P}^2$, which is birational to C and has only ordinary double points.

The Bertini method is lovely geometry, but now it survives only as a curiosity. The Noether method, however, still lies at the heart of our understanding of birational transformations between algebraic varieties.

DEFINITION 1.54. A singular point of a plane curve $p \in C$ is an *ordinary multiple point* if C has smooth branches intersecting pairwise transversally at p. Equivalently, in local coordinates (x, y) the equation of C is written as $\prod_i (a_i x + b_i y) + (\text{higher terms}) = 0$, where none of the $a_i x + b_i y$ is a constant multiple of another.

DEFINITION 1.55 (Cremona transformation). Consider $\mathbb{P}^2$ with coordinates $(x : y : z)$. The standard *Cremona transformation* or *quadratic transformation* of $\mathbb{P}^2$ is the birational involution

$$\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \quad \text{given by } (x : y : z) \mapsto (x^{-1} : y^{-1} : z^{-1}) = (yz : xz : xy).$$

Thus σ is not defined at the three coordinate vertices $(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)$, and the three coordinate lines are contracted to points.

Alternatively, σ can be viewed as first blowing up the three coordinate vertices to get $B_3\mathbb{P}^2$, and then contracting the birational transforms of the three coordinate lines.

Consider a curve $C = (f(x : y : z) = 0)$, which has multiplicity a at $(0 : 0 : 1)$, b at $(0 : 1 : 0)$ and c at $(1 : 0 : 0)$. Then the birational transform of $C = (f(x : y : z) = 0)$ is

$$\sigma_* C = \bigl(x^{-c} y^{-b} z^{-a} f(yz : xz : xy) = 0\bigr). \tag{1.55.1}$$

More generally, let $p, q, r \in \mathbb{P}^2$ be three noncollinear points. We can choose a coordinate system such that p, q, r are the three coordinate vertices. Thus there is a quadratic transformation $\sigma_{p,q,r}$ with base points p, q, r. We blow up the points p, q, r, and then contract the birational transforms of the three lines ℓ_p, ℓ_q and ℓ_r through any two of p, q, r.

1.56 (The singularities of $\sigma_* C$). Let $C \subset \mathbb{P}^2$ be a plane curve of degree n and p, q, r noncollinear points with multiplicities a, b, c on C. (We allow a, b or c to be zero.) What are the singularities of the resulting new curve $(\sigma_{p,q,r})_* C$?

First, we perform the blow-up of the three points to get $B_{p,q,r}\mathbb{P}^2$. Here the singularities outside $\{p,q,r\}$ are unchanged, and the singularities at each of these 3 points are replaced by the singularities in their first infinitesimal neighborhood.

Then we contract the birational transforms of the three lines ℓ_p, ℓ_q and ℓ_r. This creates new singularities.

Assume now that the line ℓ_p is not contained in the tangent cones of C at q and r, and has only transverse intersections with C at other points.

Then the birational transforms of C and of ℓ_p intersect transversally, and so $(\sigma_{p,q,r})_*C$ has an ordinary $(n-b-c)$-fold point at the image of ℓ_p.

Algorithm 1.57 (Birational transformation algorithm). Let k be an algebraically closed field and $C_0 \subset \mathbb{P}^2$ an irreducible plane curve. If $C_i \subset \mathbb{P}^2$ is already defined, pick any nonordinary point $p \in C$ and choose $q, r \in \mathbb{P}^2$ in a general position.

Let $\sigma_{p,q,r} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the quadratic transformation with base points p, q, r, and set $C_{i+1} := (\sigma_{p,q,r})_* C_i$.

A slight modification may be needed in positive characteristic. In order to avoid some inseparable projections, before each of the above steps we may have to perform an auxiliary quadratic transformation, where p is either a general point of $\mathbb{P}^2$ or a general point of C and $q, r \in \mathbb{P}^2$ are in general position.

The algorithm first appears in [Noe71] but without any proof. The first substantial proof is in [Nöt75]. (The first paper spells his name as M. Noether, the second as M. Nöther. Most of his papers use the first variant.)

Theorem 1.58 (M. Noether, 1871). *Let k be an algebraically closed field and $C \subset \mathbb{P}^2$ an irreducible plane curve. Then the algorithm (1.57) eventually stops with a curve $C_m \subset \mathbb{P}^2$, which has only ordinary multiple points (1.54).*

Thus the composite of all the quadratic transformations of the algorithm is a birational map $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\Phi_*(C) \subset \mathbb{P}^2$ has only ordinary multiple points.

We give two proofs. The first one assumes that we already know embedded resolution as in (1.52). The second, following Noether's original approach, gives another proof of embedded resolution.

Proof using resolution. Pick any point $p \in C$, and let $\pi : C \to \mathbb{P}^1$ be the projection from p. In characteristic zero or if the characteristic does not divide $\deg C - \operatorname{mult}_p C$, the projection π is separable. Thus we can take two general lines through p such that they are not contained in the tangent cone of C at p and they have only transverse intersections with C at other points. Take also a third line (not through p) that intersects C transversally everywhere.

The genericity conditions of (1.56) on the corresponding quadratic transformation are satisfied. Thus we get a curve C_1 such that

- the singularities of C outside p are unchanged,
- the singularity at p is replaced by the singularities in its first infinitesimal neighborhood, and
- we have created one new ordinary multiple point of multiplicity $\deg C$ and two new ordinary multiple points of multiplicity $\deg C - \operatorname{mult}_p C$.

Thus we can follow along the resolution algorithm (1.42) and end up with a curve $C_m \subset \mathbb{P}^2$ with only ordinary multiple points.

In positive characteristic, we first perform an auxiliary quadratic transformation, where p is either a general point of $\mathbb{P}^2$ or a general point of C and $q, r \in \mathbb{P}^2$ are in general position. In the first case, we get a curve of degree $2\deg C$, in the second case a curve of degree $2\deg C - 1$. Either $2\deg C - \operatorname{mult}_p C$ or $2\deg C - 1 - \operatorname{mult}_p C$ is not divisible by the characteristic, and then we can proceed as above. $\square$

Noether's proof. We start as above, but we prove that eventually we get a curve with ordinary multiple points without assuming the existence of resolution.

We use a first approximation of the genus of the curve, which is called the *apparent genus* or *deficiency* in the classical literature. If $C \subset \mathbb{P}^2$ is a curve of degree d with singular points of multiplicity m_i, then this number is

$$g_{\mathrm{app}}(C) := \binom{d-1}{2} - \sum_i \binom{m_i}{2}.$$

Starting with C as above, let $d = \deg C$, $m_0 = \operatorname{mult}_p C$ and $m_i : i > 0$ be the multiplicities of the other multiple points of C. Using (1.55.1) and the above analysis of the singular points of C_1, we obtain that

- $\deg C_1 = 2d - m_0$,
- the other singular points of C give singular points of C_1 with the same multiplicity m_i,
- we get three new ordinary singularities with multiplicities $d, d - m_0, d - m_0$, and
- if $p \in C$ is not ordinary, then there is at least one new singular point corresponding to p.

If we ignore the points of the last type in the computation of the deficiency of C_1, then we get that

$$\begin{aligned} g_{\mathrm{app}}(C_1) &< \tbinom{2d-m_0-1}{2} - \textstyle\sum_{i>0} \tbinom{m_i}{2} - \tbinom{d}{2} - 2\tbinom{d-m_0}{2} \\ &= \tbinom{d-1}{2} - \textstyle\sum_i \tbinom{m_i}{2} = g_{\mathrm{app}}(C), \end{aligned}$$

where the middle equality is by explicit computation. The sequence of quadratic transformations thus stops if we show that $g_{\mathrm{app}}(C) \geq 0$ for any irreducible plane curve C.

Look at the linear system $|A|$ of all curves of degree $d-1$ that have multiplicity $m_i - 1$ at every singular point $p_i \in C$. If $C = (f(x,y,z) = 0)$, then the curve $(\partial f/\partial x = 0)$ is in $|A|$, so $|A|$ is not empty. For any $A \in |A|$, the intersection $A \cap C$ consists of the points p_i, each with multiplicity $\geq m_i(m_i - 1)$ and a residual set R. These form a linear system $|R|$ on C. By Bézout's theorem (cf. [Sha94, IV.2.1]), the degree of $|R|$ is

$$\deg |R| = d(d-1) - \sum_i m_i(m_i - 1) = 2g_{\mathrm{app}}(C) + 2(d-1).$$

On the other hand,

$$\dim |R| = \dim |A| = \binom{d+1}{2} - 1 - \sum_i \binom{m_i}{2} = g_{\mathrm{app}}(C) + 2(d-1).$$

Since $\dim |R| \leq \deg |R|$ by (1.20.7), we conclude that $0 \leq g_{\mathrm{app}}(C)$. □

Remark 1.59. Following the proof of (1.58) easily leads to Noether's genus formula for plane curves:

$$g(C) = \binom{d-1}{2} - \sum_i \binom{m_i}{2},$$

where the sum runs through all infinitely near singular points of C.

Algorithm 1.60 (Bertini algorithm). Let k be an algebraically closed field and $C_0 \subset \mathbb{P}^2_k$ an irreducible plane curve. If $C_i \subset \mathbb{P}^2_k$ is already defined, pick any point $p_1 \in C$, which is neither smooth nor an ordinary node, and five other points $p_2, \dots, p_6 \in \mathbb{P}^2$ in general position.

We use some basic properties of cubic surfaces; see [Rei88, §7] or [Sha94, Sec.IV.2.5].

The blow-up of $\mathbb{P}^2$ at these six points is a cubic surface S. Pick a general point $s \in S$, and let $\pi_s : S \dashrightarrow \mathbb{P}^2$ be the projection from s. Let

$$\sigma : \mathbb{P}^2 \dashrightarrow S \xrightarrow{\pi_s} \mathbb{P}^2$$

be the composite of the inverse of the blow-up followed by projection from s. Note that $\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ has degree 2. In coordinates, σ is given by three general cubics through the six points $p_1, \dots, p_6$.

Set $C_{i+1} := \sigma_* C_i$.

The following result is proved in [Ber94], but an added remark of F. Klein says that it was already known to Clebsch in 1869.

Theorem 1.61 (Clebsch, 1869; Bertini, 1894). *Let k be an algebraically closed field and $C \subset \mathbb{P}^2_k$ an irreducible plane curve. Then the algorithm*

(1.60) eventually stops with a curve $C_m \subset \mathbb{P}^2$*, which has only ordinary nodes.*

Proof. As in the proof of (1.58) we again follow the resolution algorithm (1.52), and we only need to check the following lemma.

LEMMA 1.62. *Let* $C \subset S \subset \mathbb{P}^3$ *be a reduced curve on a smooth cubic surface and* $\pi : S \dashrightarrow \mathbb{P}^2$ *the projection from a general point* $s \in S$*. Then* $\pi_* C \subset \mathbb{P}^2$ *has the same singularities as* C*, plus a few ordinary nodes.*

Proof. We have to find a point $s \in S$ such that

(1) s is not on the tangent plane of C at any singular point of C,
(2) s is not on the tangent line of C at any smooth point of C,
(3) s is not on any secant line of C connecting a singular point of C with another point,
(4) s is not on any secant line of C connecting two smooth points of C with coplanar tangent lines, and
(5) s is not on a trisecant line of C.

The last condition would mean that the trisecant line of C is also a quadrisecant of S and hence one of the twenty-seven lines on S. For the first four cases we prove that the points in $\mathbb{P}^3$ where they fail is a subset of a union of linearly ruled surfaces. Since S is not ruled by lines, a general point on S satisfies all five conditions.

For (1) we have to avoid finitely many planes, for (2) the union of all tangent lines and for (3) the cones over C with vertex a singular point of C.

Condition (4) needs a little extra work, and I do it only in characteristic zero. The general case is left to the reader. For any smooth point $c \in C$, let L_c be the tangent line. Projecting C from L_c is separable, and hence there are only finitely many other points $c'_j \in C$ such that the tangent line at c'_j is coplanar with L_c. Thus s has to be outside the union of all lines $\langle c, c'_j \rangle$ as c runs through all smooth points of C. □

If $C \subset \mathbb{P}^2$ is defined over a field k, in general the singular points are not defined over k, and so the steps of the algorithm (1.57) are not defined over k. A suitable modification works over any infinite perfect field. The proof uses some basic results about ruled surfaces.

THEOREM 1.63. *Let* k *be an infinite perfect field and* $C \subset \mathbb{P}^2_k$ *an irreducible plane curve. Then there is a birational map* $\Phi : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ *such that* $\Phi_*(C) \subset \mathbb{P}^2_k$ *has only ordinary multiple points.*

Proof. Pick a point $p \in \mathbb{P}^2 \setminus C$ that does not lie on any of the singular tangent cones of C and such that projection from p is separable.

As a first step, we blow up p and then we blow up all the nonordinary singular points of C. Then we contract the birational transforms of all lines passing through p and any of the nonordinary singular points of C. If there are m nonordinary singular points, we end up with a minimal ruled surface $\mathbb{F}_{m+1}$ and $C' \subset \mathbb{F}_{m+1}$ such that all the nonordinary singular points of C have been replaced by the singularities in their first infinitesimal neighborhoods.

If $E \subset \mathbb{F}_{m+1}$ denotes the negative section and $F \subset \mathbb{F}_{m+1}$ the fiber, then $|E + (m+1)F|$ is very ample outside E, and hence a general member $D \in |E + (m+1)F|$ intersects C' transversally.

Pick $m+2$ general fibers $F_1, \dots, F_{m+2}$ that intersect C' transversally, and blow up the intersection points $F_i \cap D$.

The birational transforms of F_i and of D are now all -1-curves, thus they can be contracted, and we get $\mathbb{P}^2$ after contraction.

Thus we get $C_1 \subset \mathbb{P}^2$ such that all the nonordinary singular points of C have been replaced by the singularities in their first infinitesimal neighborhoods, and we have a number of new ordinary singular points. Moreover, everything we did is defined over the ground field k.

Iterating this procedure eventually gives a curve with only ordinary singular points. □

EXAMPLE 1.64. Let $C \subset \mathbb{P}^2_{\mathbb{F}_q}$ be an irreducible curve with ordinary multiple points. There are only $q+1$ different tangent directions defined over $\mathbb{F}_q$ at any point of $\mathbb{P}^2$, and hence we conclude that the normalization of C has at most $(q+1)(q^2+q+1)$ points in $\mathbb{F}_q$.

On the other hand, every irreducible curve C over $\mathbb{F}_q$ is birational to a plane curve. Indeed, its function field $\mathbb{F}_q(C)$ is separable over some $\mathbb{F}_q(x)$ (cf. [vdW91, 19.7]), and so it can be given by two generators $\mathbb{F}_q(C) \cong \mathbb{F}_q(x,y)$. This provides the birational map of C to the affine plane over $\mathbb{F}_q$. There are curves over $\mathbb{F}_q$ with an arbitrary large number of $\mathbb{F}_q$-points, and so (1.63) fails for finite fields.

The following example, due to B. Poonen, gives very nice explicit curves with this property.

Claim 1.64.1. Let C_m be any curve over $\mathbb{F}_q$ of the form

$$\prod_{\deg f \le m-1} \bigl(y - xf(x)\bigr)\bigl(\text{unit at } (0,0)\bigr) + x^n\bigl(\text{unit at } (0,0)\bigr) = 0.$$

Then the normalization of C_m has q^m points over $(0,0)$ if $n > q^m + q^{m-1} + \cdots + 1$.

Indeed, let us see what happens under one blow-up. In the chart $y_1 = y/x, x_1 = x$ we get

$$\prod_{\deg f \le m-1} \bigl(y_1 - f(x_1)\bigr)\bigl(\text{unit at } (0,0)\bigr) + x_1^{n-q^m}\bigl(\text{unit at } (0,0)\bigr).$$

The terms where $f(0) \neq 0$ can be absorbed into the unit, so we have q^{m-1} points sitting over $(x_1, y_1) = (0, 0)$ and the same happens over any other point $(x_1, y_1) = (0, c)$.

Claim 1.64.2. The polynomial

$$\prod_{\deg f \leq m-1} \big(y - xf(x)\big) + x^n \big(y^{q^{m-1}} + x - 1\big) \quad \text{is irreducible.}$$

We use the Eisenstein criterion [vdW91, 5.5] over $\bar{\mathbb{F}}_q(x)[y]$ and the prime $x - 1$. Setting $x = 1$ we get

$$\prod_{\deg f \leq m-1} \big(y - f(1)\big) + y^{q^{m-1}} = (y^q - y)^{q^{m-1}} + y^{q^{m-1}} = y^{q^m},$$

so all but the leading coefficient (in y) is divisible by $x - 1$. The constant term (in y) is $x^n(x - 1)$, which is not divisible by $(x - 1)^2$.

Thus

$$C_m := \Big(\prod_{\deg f \leq m-1} \big(y - xf(x)\big) + x^n \big(y^{q^{m-1}} - x + 1\big) = 0 \Big) \subset \mathbb{A}^2$$

is an irreducible curve defined over $\mathbb{F}_q$ whose normalization has at least q^m points in $\mathbb{F}_q$.

1.65 (Aside on birational transforms of $(\mathbb{P}^2, C)$). In both of the above methods we have considerable freedom to choose the birational transformations, and a given curve C has many models with only ordinary multiple points. It is natural to ask if the method can be sharpened to get a nodal curve using birational maps of $\mathbb{P}^2$.

In the rest of the section we show that this is usually impossible. More precisely, we prove that if $C \subset \mathbb{P}^2$ is a curve of degree at least 7 such that every point has multiplicity $< \frac{1}{3} \deg C$ and $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is any birational map, then $\phi_* C$ is not nodal, save when C itself has only nodes and ϕ is an isomorphism.

The methods of the proof have more to do with the birational geometry of surfaces, so one may prefer to come back to this part after reading Section 2.2. See [KSC04, Chap.V] for further applications of this method.

Theorem 1.65.1. Let $C_1, C_2 \subset \mathbb{P}^2$ be two plane curves, and assume that $\operatorname{mult}_p C_i < \frac{1}{3} \deg C_i$ for every $p \in C_i$.

Then every birational map $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $\Phi_* C_1 = C_2$ is an isomorphism.

Proof. Assuming that $(\mathbb{P}^2, C_1)$ is birational to $(\mathbb{P}^2, C_2)$, by resolution for surfaces there is a common resolution

$$\begin{array}{ccc} & (S, C) & \\ {\scriptstyle q_1}\swarrow & & \searrow{\scriptstyle q_2} \\ (\mathbb{P}^2, C_1) & & (\mathbb{P}^2, C_2), \end{array}$$

where $C = (q_i)_*^{-1} C_i$. Set $d_i = \deg C_i$, and assume that $d_1 \geq d_2$. Consider the linear system $|d_1 d_2 K_S + 3d_1 C|$.

We can first view this as

$$|d_1 d_2 K_S + 3d_1 C| = d_1 |d_2 K_S + 3(q_2)_*^{-1} C_2|$$

and use (1.65.2) to conclude that it has a unique effective member whose support is exactly the exceptional curves of q_2.

On the other hand, we can also view this as

$$|d_1 d_2 K_S + 3d_1 C| \supset d_2 |d_1 K_S + 3(q_1)_*^{-1} C_1| + 3(d_1 - d_2) C,$$

and thus again by (1.65.2) it has a member that contains C if $d_1 > d_2$, a contradiction. If $d_1 = d_2$, then $|d_1 d_2 K_S + 3d_1 C|$ has a unique effective member whose support is exactly the exceptional curves of q_1.

Thus q_1 and q_2 have the same exceptional curves, and so S is the blow-up of the graph of an isomorphism. □

Lemma 1.65.2. Let $C \subset \mathbb{P}^2$ be a plane curve such that $\operatorname{mult}_p C < \frac{1}{3} \deg C$ for every $p \in C$. Let $q : S \to \mathbb{P}^2$ be any birational morphism, S smooth. Then the linear system $m|(\deg C) K_S + 3q_*^{-1} C|$ has a unique effective member for $m \geq 1$. Its support is exactly the exceptional curves of q.

Proof. For any curve C, the intersection number of $(\deg C) K_S + 3q_*^{-1} C$ with a general line in $\mathbb{P}^2$ is zero, and thus every effective member of $m|(\deg C) K_S + 3q_*^{-1} C|$ is supported on the exceptional curves of q. Exceptional 1-cycles cannot move in a linear system, so the linear system $m|(\deg C) K_S + 3q_*^{-1} C|$ is at most zero-dimensional.

The effectiveness of $m|(\deg C) K_S + 3q_*^{-1} C|$ follows from basic general results on singularities of pairs (see, for instance, [KSC04, Chap.VI]) applied to $(S, \frac{3}{\deg C} C)$. Here is a short proof.

Lemma 1.65.3. Let S be a smooth surface and Δ an effective $\mathbb{Q}$-divisor such that $\operatorname{mult}_p \Delta \leq 1$ for every $p \in S$. Let $f : S' \to S$ be a proper birational morphism, S' smooth, and let $\Delta' \subset S'$ denote the birational transform of Δ. Then the divisor $K_{S'} + \Delta' - f^*(K_S + \Delta)$ is effective. Furthermore, if $\operatorname{mult}_p \Delta < 1$ for every $p \in S$, then its support is the whole exceptional divisor.

Proof. By induction (and using (2.13)) it is enough to prove this for one blow-up. Let $\pi : B_p S \to S$ be the blow-up of the point $p \in S$ with exceptional curve E. Then

$$K_{B_p S} = \pi^* K_S + E \quad \text{and} \quad \pi^* \Delta = \Delta' + (\operatorname{mult}_p \Delta) \cdot E.$$

Thus

$$K_{S'} + \Delta' = \pi^*(K_S + \Delta) + (1 - \operatorname{mult}_p \Delta) E.$$

To continue with the induction we also need to show that

$$\operatorname{mult}_q\bigl(\Delta' + (1 - \operatorname{mult}_p \Delta)E\bigr) \le 1 \quad \text{for every } q \in B_pS.$$

This is obvious for $q \notin E$. If $q \in E$, then by (1.40)

$$\begin{aligned} \operatorname{mult}_q\bigl(\Delta' + (1 - \operatorname{mult}_p \Delta)E\bigr) &\le (E \cdot \Delta') + (1 - \operatorname{mult}_p \Delta) \\ &= \operatorname{mult}_p \Delta + (1 - \operatorname{mult}_p \Delta) = 1. \quad \square \end{aligned}$$

Example 1.65.4. Let $C_1 \subset \mathbb{P}^2$ be a general degree 6 curve with three nodes p, q, r. The Cremona transformation with base points p, q, r creates another degree 6 curve with three nodes.

The methods of the proof of (1.65.1) have been developed much further, and they lead to a general understanding of birational maps between varieties that are rational or are close to being rational. See [KSC04] for an elementary introduction to these techniques and results.

1.8. Embedded resolution, II: Local methods

Here we study the resolution of embedded curve singularities by direct local computations. We explicitly compute the chain of blow-ups and deduce a normal form for the singularity in case the multiplicity has not dropped in a sequence of blow-ups.

With each successive blow-up, these normal forms become more and more special, but they all require a suitable coordinate change. The final analysis is then to show that eventually such a coordinate change is impossible. The last step is actually quite delicate.

One could say that this approach relies on brute force, rather than a nice general idea. It is, however, extensions of this approach that have proved most successful in dealing with higher-dimensional resolution problems, so a careful study of it is very worthwhile.

1.66 (Idea of proof). Assume for simplicity that $S = \mathbb{A}^2_k$, the affine plane over a field k, and C is given by an equation $f(x, y) = 0$. We can write f as a sum of homogeneous terms

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots .$$

If the lowest term f_m is not an mth power of a linear form, then by (1.40) the multiplicity drops after a single blow-up. Thus we can complete the proof by induction on the multiplicity.

The hard case is when f_m is an mth power, $f_m(x, y) = c(y - a_1x)^m$. If k is a perfect field, then $a_1 \in k$ and by a linear change of coordinates we may assume that $f_m = y^m$. We call such an f to be in *normal form.*

(If the field k is not perfect and m is a power of the characteristic, it may happen that a_1 is not in k but in some inseparable extension of it. We will not deal with this case.)

To get the blow-up, set $y = y_1x_1, x = x_1$. The birational transform of C is given by the equation

$$\bigl(f^1(x_1, y_1) = 0\bigr) \quad \text{where } f^1(x_1, y_1) := x_1^{-m} f(x_1, y_1x_1).$$

At the next blow-up we have a multiplicity drop unless f^1 also has multiplicity m and its degree m part is the mth power of a linear form. We are now faced with a linear change of coordinates

$$(x_1, y_1) \mapsto (x_1, y_1 - a_2x_1).$$

It turns out that if we make a quadratic coordinate change

$$(x, y) \mapsto (x, y - a_1x - a_2x^2),$$

then f and f^1 are both in normal form.

More generally, if the multiplicity does not drop for k blow-ups in a row, then there is a degree k coordinate change

$$(x, y) \mapsto (x, y - a_1x - a_2x^2 - \cdots - a_kx^k)$$

such that the inductively defined $f, f^1, \dots, f^{k-1}$ are all in normal form.

If the multiplicity stays m for infinitely many blow-ups, then we would need a coordinate change

$$(x, y) \mapsto (x, y - a_1x - a_2x^2 - \cdots).$$

This is not a polynomial change, and hence we have to work now in the power series ring $\mathbb{C}[[x, y]]$.

It is easy to see that if $f, f^1, f^2, \dots$ are all in normal form then

$$f = (y - a_1x - a_2x^2 - \cdots)^m(\text{unit}),$$

this of course only in $\mathbb{C}[[x, y]]$.

We are thus left with the question, is it possible that f is divisible by an mth power in $\mathbb{C}[[x, y]]$ but not divisible by an mth power in $\mathbb{C}[x, y]$?

This is easily settled using the differential criterion of smoothness (1.72), but it is precisely this question that causes the most trouble when we try to replace a surface over k with the spectrum of a 2-dimensional regular local ring; see (1.103).

THEOREM 1.67. *Let S_0 be a smooth surface over a perfect field k and $C_0 \subset S_0$ a reduced curve. Then the embedded resolution algorithm (1.42) terminates with $C_m \subset S_m$, where C_m is smooth*

Proof. The question is local, so we may assume that $p_0 \in C_0$ is the only singular point.

As a first step, we use induction on the multiplicity. Let $\operatorname{mult}_{p_0} C_0 = m$. As we saw in (1.40), $\sum_{p \in C_1 \cap E} \operatorname{mult}_p C_1 \le \operatorname{mult}_0 C$, where summation is over all infinitely near singularities in the first infinitesimal neighborhood of $p_0 \in C_0$. Thus we can use induction, unless there is only one singularity after blow-up and it also has multiplicity m.

We repeat the procedure. If the multiplicity drops after some blow-ups, then we can use induction. Thus the proof of (1.67) is complete if we can prove the following.

LEMMA 1.68. *Let the notation be as above. There is no infinite sequence of blow-ups*

$$(p_0 \in C_0 \subset S_0) \leftarrow (p_1 \in C_1 \subset S_1) \leftarrow (p_2 \in C_2 \subset S_2) \leftarrow \cdots$$

where

(1) $\operatorname{mult}_{p_i} C_i = m \geq 2$ *for every* i,
(2) C_{i+1} *is obtained from* C_i *by blowing up* p_i, *and*
(3) p_{i+1} *is the unique singular point of* C_{i+1} *lying above* p_i.

A sequence as in (1.68)

$$(p_0 \in C_0 \subset S_0) \leftarrow \cdots \leftarrow (p_k \in C_k \subset S_k)$$

is called a *length k blow-up sequence* of multiplicity m.

Let us now pass to the algebraic side. Let R_0 be the local ring of S_0 at p_0 and $f^0 \in R_0$ a local equation of C_0.

As we noted in (1.40), if the multiplicity does not drop then one can choose local parameters $x, y \in R_0$ such that the leading term of f^0 is y^m. That is, $f^0 \in (y^m) + (x, y)^{m+1}$.

The following is the key computation.

PROPOSITION 1.69. *With the above notation, there is a length k blow-up sequence of multiplicity m iff there are local parameters (x, y_k) such that*

$$f^0 \in (x^{k+1}, y_k)^m.$$

Moreover, if the above length k blow-up sequence extends to a length $k+1$ blow-up sequence of multiplicity m, then we can choose $y_{k+1} = y_k - a_k x^k$ for some $a_k \in R_0$ such that $f^0 \in (x^{k+2}, y_{k+1})^m$.

Proof. If $f^0 \in (x^{k+1}, y_k)^m$, then after k blow-ups C_k lies on the affine chart $\operatorname{Spec} R_k$, where $R_k = R_0[\frac{y_k}{x^k}]$. Indeed,

$$(x^{k+1}, y_k)^m = x^{km}\Bigl(x, \frac{y_k}{x^k}\Bigr)^m;$$

thus we can write $f^0 = x^{mk} f^k$, and the unique singular point of $C_k = (f^k = 0)$ is at the maximal ideal $(x, \frac{y_k}{x^k})$. What happens at the next blow-up?

Here we get the ring $R_{k+1} = R_0[\frac{y_k}{x^{k+1}}]$ and by assumption C_{k+1} has multiplicity m at some point of the exceptional curve, say, at the maximal ideal $(x, \frac{y_k}{x^{k+1}} - a_{k+1})$. Thus $f^{k+1} \in x^m(x, \frac{y_k}{x^{k+1}} - a_{k+1})^m$, which gives

$$f^0 \in x^{m(k+1)}\Bigl(x, \frac{y_k}{x^{k+1}} - a_{k+1}\Bigr)^m = (x^{k+2}, y_k - a_{k+1}x^{k+1})^m.$$

This is exactly what we wanted, but sloppy notation helped us here. Indeed, we need that $f^0 \in (x^{k+2}, y_k - a_{k+1}x^{k+1})^m R_0$, but we proved that

$$f^0 \in (x^{k+2}, y_k - a_{k+1}x^{k+1})^m R_0\Big[\frac{y_k}{x^{k+1}}\Big].$$

Fortunately, these two are equivalent by (1.70). □

LEMMA 1.70. *Let R be a unique factorization domain and $x, y \in R$ distinct primes. Then*

$$R \cap (x^b, y)R\Big[\frac{y}{x^a}\Big] = (x^b, y).$$

Proof. Every element of $(x^b, y)R[\frac{y}{x^a}]$ can be written as

$$s = \frac{y^c}{x^{ac}} r_1 x^b + \frac{y^d}{x^{ad}} r_2 y,$$

where $x^a \nmid r_1$ if $c \geq 1$ and $x^a \nmid r_2$ if $d \geq 1$.

Assume that $s \in R$. If $c = 0$, then $\frac{y^d}{x^{ad}} r_2 y \in R$, which happens only for $d = 0$, and then $s \in (x^b, y)$.

If $c \geq 1$, then we can write $s = ys_1$, where

$$s_1 = \frac{y^{c-1}}{x^{ac}} r_1 x^b + \frac{y^d}{x^{ad}} r_2 \in R\Big[\frac{1}{x}\Big].$$

Multiplying by y cannot cancel out a denominator that is a power of x; thus $s_1 \in R$, and so $s \in (y) \subset (x^b, y)$. □

These coordinate changes can all be put together if we pass from the ring R to its *completion*, denoted by $\hat{R}$. See [AM69, Chap.10] for the definition and its basic properties.

COROLLARY 1.71. *Let S_0 be a smooth surface over a perfect field k and R_0 its local ring at a point p_0 with completion $\hat{R}_0$. There is an infinite blow-up sequence of multiplicity m iff there are local parameters x and $y_\infty = y - \sum_{k \geq 1} a_k x^k \in \hat{R}_0$ such that*

$$f^0 \in (y_\infty)^m.$$

Proof. Applying (1.69) we get that $f^0 \in (x^{k+1}, y_\infty)^m$ for every k. Thus, working in $\hat{R}_0/(y_\infty)^m$, we obtain that $f^0 \in \cap_k (x^{k+1})$, and the latter intersection is zero by Krull's intersection theorem (cf. [AM69, 10.17]). □

1.72 (Going back from $\hat{R}_0$ to R_0). By assumption S_0 is a smooth surface and C_0 a reduced curve; thus it has an isolated singularity at p_0. That is, p_0 is an isolated solution of $f^0 = \partial f^0/\partial x = \partial f^0/\partial y = 0$. Thus

$$\hat{R}/\big(f^0, \tfrac{\partial f^0}{\partial x}, \tfrac{\partial f^0}{\partial y}\big) = R/\big(f^0, \tfrac{\partial f^0}{\partial x}, \tfrac{\partial f^0}{\partial y}\big)$$

is finite dimensional. On the other hand, if $f^0 = y_\infty^m(\text{unit})$, then

$$\left(f^0, \tfrac{\partial f^0}{\partial x}, \tfrac{\partial f^0}{\partial y}\right) \subset (y_\infty^{m-1}),$$

a contradiction. $\square$

It is straightforward to see that the method of the proof gives the following stronger result.

THEOREM 1.73. *Let S be a 2-dimensional regular scheme over a field and $C \subset S$ a reduced curve. Assume that C has finitely many singular points $c_i \in C$ and the residue fields $k(c_i)$ are perfect. Then*

(1) *either the weak embedded resolution algorithm terminates with $C_m \subset S_m$, where C_m is smooth,*
(2) *or the completion $\widehat{\mathcal{O}}_{c_i,C}$ is nonreduced for some i.*

We will use this result when $S = \operatorname{Spec} R$ is the spectrum of a 2-dimensional complete regular local ring with perfect residue field. Then $\widehat{\mathcal{O}}_{c_i,C} = \mathcal{O}_{c_i,C}$, so the second case never happens, and we always get a strong embedded resolution.

1.9. Principalization of ideal sheaves

So far we have dealt with a single curve $C \subset S$, but now we consider linear systems of curves on S. Equivalently, we are considering a finite-dimensional vector space of sections $V \subset H^0(S, L)$ of a line bundle L on S. Let $L_V \subset L$ be the subsheaf that they generate. Then $I_V := L_V \otimes L^{-1}$ is an ideal sheaf, and the main result says that after some blow-ups the pullback of any ideal sheaf becomes locally principal.

THEOREM 1.74 (Principalization of ideal sheaves). *Let S be a smooth surface over a perfect field k and $I \subset \mathcal{O}_S$ an ideal sheaf. Then there is a sequence of point blow-ups $\pi : S_m \to S$ and a simple normal crossing divisor $F_m \subset S_m$ such that*

$$\pi^* I = \mathcal{O}_{S_m}(-F_m),$$

where $\pi^* I \subset \mathcal{O}_{S_m}$ denotes the ideal sheaf generated by the pullback of local sections of I. (This is denoted by $\pi^* I \cdot \mathcal{O}_{S_m}$ or by $I \cdot \mathcal{O}_{S_m}$ in [Har77, p.163]*.)*

Applied to $V \subset H^0(S, L)$ and $I = I_V$ as above, we get that V, as a subspace of $H^0(S, \pi^* L) = H^0(S, L)$, generates the subsheaf $\pi^* L(-F_m)$, which itself is locally free. Thus V, as a subspace of $H^0(S, \pi^* L(-F_m))$, gives a base point–free linear system, and we obtain the following.

COROLLARY 1.75 (Elimination of base points). *Let $|D|$ be a finite-dimensional linear system of curves on a smooth surface S defined over a* *perfect field k. Then there is a sequence of point blow-ups* $\pi : S_m \to S$ *such that*

$$\pi^*|D| = F_m + |M|,$$

where $F_m \subset S_m$ *is a simple normal crossing divisor and* $|M|$ *is a base point–free linear system.* □

Let $f : S \dashrightarrow \mathbb{P}^n$ be a rational map and $|H|$ the linear system of hyperplanes. The base points of $f^*|H|$ are precisely the points where f is not a morphism. Thus using (1.75) for $f^*|H|$ we conclude the following.

COROLLARY 1.76 (Elimination of indeterminacies). *Let* S *be a smooth surface defined over a perfect field* k *and* $f : S \dashrightarrow \mathbb{P}^n$ *a rational map. Then there is a sequence of point blow-ups* $\pi : S_m \to S$ *such that the composite*

$$f \circ \pi : S_m \to S \dashrightarrow \mathbb{P}^n \quad \text{is a morphism.} \qquad \square$$

1.77 (Proof of (1.74)). For a projective surface over an infinite field, a simple proof is given in [Sha94, IV.3.3]. Here we give a local argument in the spirit of Section 1.8.

There are only finitely many points where I is not of the form $\mathcal{O}_S(-D)$ for some simple normal crossing divisor D. The problem is local at these points, so we may assume that S is affine. Write $I = (g_1, \dots, g_t)$ and set $D_i = (g_i = 0)$.

First we apply the strong embedded resolution theorem (1.47) to $D_1 + \cdots + D_t \subset S$. (To avoid too many subscripts, we keep denoting the blown-up surface again by S.) Thus we are reduced to the case where $D_1 + \cdots + D_t \subset S$ is a simple normal crossing divisor.

At a smooth point of $D_1 + \cdots + D_t$ with local equation $x = 0$, the pullbacks of the g_i are locally given as x^{m_i}(unit), so the pullback ideal is locally principal with generator x^m, where $m = \min\{m_i\}$.

We now have to look more carefully at the finitely many nodes of $\operatorname{red}(D_1 + \cdots + D_t)$.

If $s \in S$ is such a node, we can choose local coordinates x, y and natural numbers A_i, C_i such that $x^{A_i} y^{C_i} = 0$ is a local equation for D_i. I is locally principal at s iff there is a j such that

$$A_j \le A_i \quad \forall i \quad \text{and} \quad C_j \le C_i \quad \forall i. \tag{1.77.1}$$

After one blow-up, we get two new nodes, and the new local equations are

$$\left(\tfrac{x}{y}\right)^{A_i} y^{A_i + C_i} = 0, \quad \text{resp.,} \quad x^{A_i + C_i} \left(\tfrac{y}{x}\right)^{C_i} = 0.$$

We need to prove that in any sequence of blow-ups we eventually get a locally principal ideal. The condition (1.77.1) shows that this is a purely combinatorial question, which is settled by (1.79) when we use $k_s = 0$ for every s. □

REMARK 1.78. This proof of (1.74) works for any 2-dimensional regular scheme S, where strong embedded resolution holds. In particular, by (1.73), it holds when $S = \operatorname{Spec} R$ is the spectrum of a 2-dimensional complete regular local ring with perfect residue field.

Hironaka proposed some combinatorial games that aim to describe the above process where player one chooses a subvariety to blow up and player two decides which chart of the blow-up to consider next [Hir72]. In the simplest case we need, we have only the origin to blow up, so player one has no choices at all, and the game is not very exciting. (The numbers k_s are added with later applications in mind (2.68).)

LEMMA 1.79 (2-dimensional Hironaka game). *Let k_s be a sequence of rational numbers and (A_i, C_i) finitely many pairs of rational numbers. We define inductively a sequence $(A_i(s), C_i(s))$, where $A_i(0) = A_i, C_i(0) = C_i$ and*

$$\big(A_i(s+1), C_i(s+1)\big) = \begin{cases} \textit{either} & \big(A_i(s) + C_i(s) + k_s, C_i(s)\big) \quad \forall i, \\ \textit{or} & \big(A_i(s), A_i(s) + C_i(s) + k_s\big) \quad \forall i. \end{cases}$$

Then, for $s \gg 1$, there is a j such that for very i, $A_j(s) \le A_i(s)$ and $C_j(s) \le C_i(s)$.

Proof. The simplification under blow-ups is more transparent if we set $a(s) := \min_i\{A_i(s)\}$, $c(s) := \min_i\{C_i(s)\}$ and write $a_i(s) = A_i(s) - a(s)$, $c_i(s) = C_i(s) - c(s)$. If we set $m(s) = \min\{a_i(s) + c_i(s)\}$, then the transformation rules are

$$\big(a_i(s+1), c_i(s+1)\big) = \begin{cases} \text{either} & \big(a_i(s) + c_i(s) - m_s, c_i(s)\big) \quad \forall i, \\ \text{or} & \big(a_i(s), a_i(s) + c_i(s) - m_s\big) \quad \forall i. \end{cases}$$

We see that $\min_i\{a_i(s+1) + c_i(s+1)\} \le \min_i\{a_i(s) + c_i(s)\}$, and strict inequality holds unless the minimal value m_s is achieved only for a pair $(0, m_s)$ if we use the first rule and for a pair $(m_s, 0)$ if we use the second rule. By our definitions, there is also a pair of the form $(a_k(s), 0)$ or $(0, c_k(s))$. For these the above rules give

$$\begin{array}{rcl} \big(a_k(s+1), c_k(s+1)\big) & = & \big(a_k(s) - m_s, 0\big) \quad \text{or} \\ \big(a_k(s+1), c_k(s+1)\big) & = & \big(0, c_k(s) - m_s\big). \end{array}$$

Thus we see that for $\min_i\{a_i(s) + c_i(s)\} > 0$ the pair

$$\Big(\min_i\{a_i(s) + c_i(s)\}, \min_{i:c_i(s)=0}\{a_i(s)\} + \min_{i:a_i(s)=0}\{c_i(s)\}\Big)$$

decreases lexicographically at each step. As we have positive rational numbers with bounded denominators, we get that $(a_j(s), c_j(s)) = (0, 0)$ for some j for $s \gg 1$. Thus $A_j(s) \le A_i(s)$ and $C_j(s) \le C_i(s)$ for every i. □

--continued--