Computations of these integrals to 3200-digit precision, combined with searches for relations using the PSLQ algorithm, yielded thousands of unexpected relations among these integrals (see Bailey and Borwein 2012). The scale of the computation was required due to the number of integrals under investigation.

3.7 Snow Crystals

Computational experimentation has even been useful in the study of snowflakes. In a 2007 study, Janko Gravner and David Griffeath used a sophisticated computer-based simulator to study the process of formation of these structures, known in the literature as snow crystals and informally as *snowfakes*. Their model simulated each of the key steps, including diffusion, freezing, and attachment, and thus enabled researchers to study dependence on melting parameters. Snow crystals produced by their simulator vary from simple stars, to six-sided crystals with plate-ends, to crystals with dendritic ends, and they look remarkably similar to natural snow crystals. Among the findings uncovered by their simulator is the fact that these crystals exhibit remarkable overall symmetry, even in the process of dynamically changing parameters. Their simulator is publicly available at http://psoup.math.wisc.edu/Snowfakes.htm.

4 Limits of Computation

Developments such as the above have led to reexamination of the role of computation in formal mathematical work. To begin with, a legitimate question is whether one can truly trust—in the mathematical sense—the result of a computation, since there are many possible sources of errors: unreliable numerical algorithms; bug-ridden computer programs implementing these algorithms; system software or compiler errors; hardware errors, either in processing or storage; insufficient numerical precision; and obscure errors of hardware, software, or programming that surface only in particularly large or difficult computations.

As a single example of the sorts of difficulties that can arise, the present authors found that neither Maple nor Mathematica was able to numerically evaluate constants of the form

$$\frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta,$$

where

$$f(\theta) = \mathrm{Li}_1(\theta)^m\,\mathrm{Li}_1^{(1)}(\theta)^p\,\mathrm{Li}_1(\theta+\pi)^n\,\mathrm{Li}_1^{(1)}(\theta-\pi)^q$$

(for $m, n, p, q \geqslant 0$ integers), to high precision in reasonable run time. In part this was because of the challenge of computing polylogs and polylog derivatives (with respect to order) for complex arguments. The version of Mathematica that we were using was able to numerically compute $\partial\,\mathrm{Li}_s(z)/\partial s$ to high precision, which is required here, but such evaluations were not only many times slower than computation of $\mathrm{Li}_s(z)$ itself but in some cases did not even return a tenth of the requested number of digits correctly.

For such reasons, experienced programmers of mathematical or scientific computations routinely insert validity checks into their code. Typically, such checks take advantage of known high-level mathematical facts, such as the fact that the product of two matrices used in the calculation should always give the identity or that the results of a convolution of integer data, done using a fast Fourier transform, should all be very close to integers.

For instance, Kanada's 2002 computation of π to 1.3 trillion decimal digits involved first computing slightly over one trillion hexadecimal (base-16) digits. He found that the 20 hex digits of π beginning at position $10^{12}+1$ are

B4466E8D21 5388C4E014.

Kanada then calculated these hex digits using the "Bailey–Borwein–Plouffe" algorithm. The result was

B4466E8D21 5388C4E014,

dramatically confirming that both results are almost certainly correct. While one cannot rigorously assign a "probability" to this event, the chances that two random strings of 20 hex digits perfectly agree is one in $16^{20} \approx 1.2089 \times 10^{24}$.

Even so, researchers are well advised to be cautious with experimentation. Consider

$$\int_0^\infty \cos(2x)\prod_{n=1}^{\infty}\cos(x/n)\,\mathrm{d}x = 0.39269908169872415480783042290993786052464\underline{5434187231595926}\ldots. \quad (5)$$

At first glance, this appears to be $\pi/8$, but upon comparison with the numerical value,

$$\pi/8 = 0.39269908169872415480783042290993786052464\underline{6174921888227621}\ldots,$$

the two values disagree after the 42nd digit! Richard Crandall later explained this mystery via a physically motivated analysis of *running out of fuel* random walks.

He found the following very rapidly convergent series expansion, of which formula (5) is the first term:

$$\frac{\pi}{8} = \sum_{m=0}^{\infty} \int_0^{\infty} \cos[2(2m+1)x] \prod_{n=1}^{\infty} \cos(x/n)\,\mathrm{d}x.$$

Two series terms suffice for 500-digit agreement.

As a final sobering example, consider

$$\sigma_p = \sum_{n=-\infty}^{\infty} \operatorname{sinc}(n/2)\operatorname{sinc}(n/3)\cdots\operatorname{sinc}(n/p)\,\mathrm{d}x$$
$$\overset{?}{=} \int_{-\infty}^{\infty} \operatorname{sinc}(x/2)\operatorname{sinc}(x/3)\cdots\operatorname{sinc}(x/p)\,\mathrm{d}x,$$

where in each line the divisors range over *all* primes up to p. Provably, the following is true. The "sum equals integral" identity for σ_p remains valid at least for p among roughly the first 10^{176} primes; but it stops holding after some larger prime, and thereafter the "sum less the integral" is strictly positive, but *they always differ by much less than one part in a googolplex* $= 10^{10^{100}}$. An even stronger estimate is possible assuming the generalized Riemann hypothesis.

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VIII.7 Teaching Applied Mathematics

How can we enthuse the next generation of students about applied mathematics? The four contributors to this group of articles, who have all thought deeply about this question, were asked to give their personal views. The resulting articles provide a variety of perspectives and will be of interest to anyone who wishes to inspire their students to pursue the subject.

I. David Acheson: What's the Big Picture?

Let A and B be two teachers of applied mathematics (at any level) and suppose that, generally speaking, A is a much better teacher than B.

Why is A's teaching so much better? Even without any further information, can we at least hazard a guess?

I wonder, for instance, if you might be prepared to bet that A is more trained in "communication skills"? Or perhaps A knows more mathematics than B or is nearer to the cutting edge of research? Then again, maybe A just has a more lively personality?

All these things can be advantageous, of course, but I would not actually bet on any of them.

In fact, in the absence of any further information, there is only one thing that I would be prepared to bet good money on. I would be prepared to bet that A's teaching is so much better—so inspirational, at best—mainly because A *wants* it to be that way, for reasons that we will probably never learn and that A may not even know.

This is only an opinion, of course, but it comes from thinking back to my own inspirational teachers when I was young. Some were notable for their scholarship, some for their eccentricity, but—so far as I can see—they only really had one thing in common: they had a great story to tell, *and they really wanted to tell it.*