7

The Riemann Integral

We now leave differential calculus and enter integral calculus. We construct the Riemann integral, prove the fundamental theorem, and investigate which functions are integrable.

7.1 Darboux Sums

Assume f is a bounded function defined on the closed and bounded interval $[a, b]$. Consider any *partition* $\mathcal{P}$ of $[a, b]$, that is, any finite set of points of $[a, b]$ that includes the two endpoints a, b. We think of this partition

$$\mathcal{P} : x_0 = a < x_1 < x_2 < \cdots < x_n = b \tag{7.1}$$

as dividing the interval into the subintervals $[x_{i-1}, x_i]$ of length $\Delta x_i = x_i - x_{i-1}$. Let the infimum and supremum of f on each subinterval be denoted by

$$m_i = \inf_{[x_{i-1},x_i]} f(x) \quad \text{and} \quad M_i = \sup_{[x_{i-1},x_i]} f(x). \tag{7.2}$$

The sums

$$L(\mathcal{P}) = m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n = \sum_{i=1}^{n} m_i\Delta x_i, \tag{7.3a}$$

$$U(\mathcal{P}) = M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n = \sum_{i=1}^{n} M_i\Delta x_i \tag{7.3b}$$

are called the *lower* and *upper Darboux sums of* f, respectively, *for the partition* $\mathcal{P}$.

Lemma A. *No lower sum can exceed any upper sum.* That is, if $\mathcal{P}_1$ and $\mathcal{P}_2$ are any two partitions of $[a, b]$, then

$$L(\mathcal{P}_1) \ \le U(\mathcal{P}_2). \tag{7.4}$$

Proof. Let $\mathcal{P}_0$ be a common refinement of both partition $\mathcal{P}_1$ and partition $\mathcal{P}_2$. For example, form the refinement $\mathcal{P}_0 = \mathcal{P}_1 \cup \mathcal{P}_2$. But refinements can only increase the lower sum and decrease the upper (exercise 7.1). Thus

$$L(\mathcal{P}_1) \leq L(\mathcal{P}_0) \leq U(\mathcal{P}_0) \leq U(\mathcal{P}_2). \tag{7.5}$$

Thus *each lower sum is a lower bound for the set of all upper sums* and *each upper sum is an upper bound for the set of all lower sums.*

Corollary. *The supremum of all lower sums is at most the infimum of all upper sums.* In symbols,

$$\sup_{\mathcal{P}} L(\mathcal{P}) \ \leq \ \inf_{\mathcal{P}} U(\mathcal{P}). \tag{7.6}$$

Definition. If equality holds in (7.6), then the function f is said to be *Riemann integrable* over the interval $[a, b]$ and the common value of (7.6) is written with the symbol

$$\int_a^b f(x)\, dx \ = \ \sup_{\mathcal{P}} L(\mathcal{P}) \ = \ \inf_{\mathcal{P}} U(\mathcal{P}). \tag{7.7}$$

More informally, if there is only one number trapped between all upper and lower sums, then the Riemann integral exists and equals this unique trapped number.

Example 1. (Lejeune-Dirichlet's example) Let f be defined by the rule

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Consider partitions of $[0, 1]$. Because each subinterval of any partition contains both rationals and irrationals, all $m_i = 0$ and all $M_i = 1$, giving that $L(\mathcal{P}) = 0$ and $U(\mathcal{P}) = 1$ for all partitions $\mathcal{P}$. Thus f is not integrable over $[0, 1]$.

Example 2. Consider the function f given by $f(x) = x$ over the interval $[0, 1]$. For any partition $\mathcal{P} : 0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$, because f is increasing,

$$L(\mathcal{P}) = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

and

$$U(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)\Delta x_i.$$

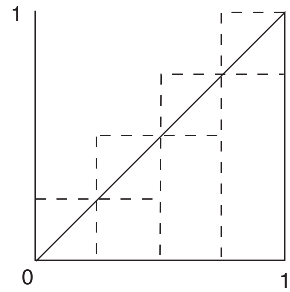

Figure 7.1 Because $f(x) = x$ is increasing, the lower sum is calculated using values of f at left end-point values, while the upper sum is calculated with values to the right.

See figure 7.1. Note that if the n subintervals of this partition are all of equal length $\Delta x = 1/n$, then

$$U(\mathcal{P}) - L(\mathcal{P}) \;=\; \frac{1}{n},$$

and so, by choosing n arbitrarily large we may force equality in (7.6) and thus f is integrable.

Moreover, the resulting upper sum is

$$\begin{aligned} U_n &= \sum_{i=1}^{n} f(i/n)(1/n) = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}[1+2+3+\cdots+n] \\ &= (\text{exercise 7.2}) = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2}\frac{n+1}{n}. \end{aligned}$$

Hence

$$\inf_n U_n \;=\; \int_0^1 x\,dx \;=\; \frac{1}{2}. \tag{7.8}$$

7.2 The Fundamental Theorem of Calculus

The following stunning result intertwines differential and integral calculus. It is considered one of the milestones of European thought.

Theorem A. (The fundamental theorem of calculus) Suppose

a. f is integrable on $[a, b]$,

b. F is continuous on $[a, b]$,

c. F is differentiable on (a, b), and

d. $F'(x) \;=\; f(x)$ on (a, b).

Then

$$\int_a^b f(x)\,dx \;=\; F(b) - F(a). \tag{7.9}$$

Proof. Consider any partition $\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b$. By the mean value theorem there are $x_i^* \in (x_{i-1}, x_i)$ such that

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^{n} F(x_i) - F(x_{i-1}) \\ &= \sum_{i=1}^{n} F'(x_i^*)(x_i - x_{i-1}) \;=\; \sum_{i=1}^{n} f(x_i^*)\Delta x_i. \end{aligned} \tag{7.10}$$

Hence the number $F(b) - F(a)$ is caught between all upper and lower sums. But since f is integrable, there is only one such number, namely

$$I \;=\; \int_a^b f(x)\,dx.$$

Example 3. The integrals of polynomials are trivial to obtain. For instance,

$$\begin{aligned} \int_{-1}^{2} (3 - 2x + x^2 + x^3)\,dx &= 3x - x^2 + \frac{x^3}{3} + \frac{x^4}{4}\Big|_{-1}^{2} \\ &= \left[3 \cdot 2 - 2^2 + \frac{2^3}{3} + \frac{2^4}{4}\right] \\ &\quad - \left[3(-1) - (-1)^2 + \frac{(-1)^3}{3} + \frac{(-1)^4}{4}\right]. \end{aligned}$$

Example 4.

$$\int_0^{\pi/2} \sin^4 x \cos x\,dx = \frac{\sin^5 x}{5}\Big|_0^{\pi/2} = \frac{\sin^5 \pi/2}{5} - \frac{\sin^5 0}{5} = \frac{1}{5}.$$

7.3 Continuous Integrands

In examples 3 and 4 we have tacitly assumed the integrals exist, as verified by the following not-so-surprising result.

Theorem B. *Continuous functions are integrable.* That is, if f is continuous on $[a, b]$, then f is integrable over $[a, b]$.

Proof. Because the interval $[a, b]$ is compact, f is not only continuous, but is in fact *uniformly continuous* on $[a, b]$; that is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x_0, x \in [a, b]$,

$$|x - x_0| < \delta \text{ implies } |f(x) - f(x_0)| < \epsilon. \tag{7.11}$$

See lemma B below.[1]

Therefore, if we choose a partition $\mathcal{P}$ of $[a, b]$ into n equal subintervals each of length $\Delta x < \delta$, we see that the differences between the maximum and minimum values of f on each subinterval are less than ϵ, and so

$$U(\mathcal{P}) - L(\mathcal{P}) = \sum_{i=1}^{n} (M_i - m_i)\Delta x \le \sum_{i=1}^{n} \epsilon \Delta x = \epsilon(b - a). \tag{7.12}$$

Thus the upper and lower sums are forced together and equality holds in (7.6); that is, f is integrable.

Lemma B. *A function continuous on a compact set is uniformly continuous.*

Proof. Let us prove the result for any function $f : X \to Z$ from the metric space X to the metric space Z. Suppose f is continuous on the compact subset K of X. Fix $\epsilon > 0$. Since f is continuous on K, for each $x \in K$ there is a $\delta(x) > 0$ so that for all $y \in K$,

$$d(x, y) < \delta(x) \quad \text{implies} \quad d(f(x), f(y)) < \epsilon/2. \tag{7.12}$$

The open balls with center x of radius $\delta(x)/2$ cover K, a cover that possesses a finite subcover of balls $B_1, B_2, \ldots, B_p$ centered at $x_1, x_2, \ldots, x_p$, respectively. Choose

$$\delta = \min_{1 \le i \le p} \delta(x_i)/2. \tag{7.13}$$

Suppose $x, y \in K$ and $d(x, y) < \delta$. Since the B_i cover K, the point $y \in B_1$ (say) and hence $d(y, x_1) < \delta(x_1)/2$. But then, since $d(x, y) < \delta \le \delta(x_1)/2$, the point x must be at distance $d(x, x_1) < \delta(x_1)$. Hence by (7.12), $d(f(x), f(y)) \le d(f(x), f(x_1)) + d(f(x_1), f(y)) < \epsilon/2 + \epsilon/2 = \epsilon$. That is, for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in K$,

$$d(x, y) < \delta \quad \text{implies} \quad d(f(x), f(y)) < \epsilon. \tag{7.14}$$

[1]Notice the subtle difference between continuity and uniform continuity. Borrowing the standard quantification symbols from logic, the statement of continuity begins with $\forall\epsilon > 0\ \forall x_0\ \exists\delta > 0\ \forall x \ldots$, while uniform continuity has the inner two quantifiers reversed: $\forall\epsilon > 0\ \exists\delta > 0\ \forall x_0\ \forall x \ldots$. In words, the same δ works everywhere, for every x_0.

7.4 Properties of Integrals

Theorem C. If f and g are integrable over $[a, b]$, then so are $f + g$ and cf for any constant c. Moreover,

$$\int_a^b (f(x) + g(x))\, dx \;=\; \int_a^b f(x)\, dx \;+\; \int_a^b g(x)\, dx \tag{7.15}$$

and

$$\int_a^b cf(x)\, dx \;=\; c \int_a^b f(x)\, dx. \tag{7.16}$$

Proof. Exercise 7.4.

Theorem D. If f is integrable over $[a, b]$ then f is integrable over any subinterval $[a_1, b_1] \subset [a, b]$.

Proof. Let $\epsilon > 0$ be given. Let $\mathcal{P}$ be any partition of $[a, b]$ where the Darboux sums for f satisfy $U(\mathcal{P}) - L(\mathcal{P}) < \epsilon$. Refine this partition by adding the endpoints a_1, b_1. Let $\mathcal{P}_1$ be the partition of $[a_1, b_1]$ obtained by discarding points in the refinement not in $[a_1, b_1]$. Then the Darboux sums of f over $[a_1, b_1]$ for this partition satisfy $U(\mathcal{P}_1) - L(\mathcal{P}_1) < \epsilon$, since we have discarded nonnegative contributions.

Corollary. Suppose f is integrable over $[a, b]$. If $a < c < b$, then

$$\int_a^b f(x)\, dx \;=\; \int_a^c f(x)\, dx \;+\; \int_c^b f(x)\, dx. \tag{7.17}$$

Proof. Exercise 7.5.

Theorem E. If f and g are integrable over $[a, b]$ and $f(x) \le g(x)$ on $[a, b]$, then

$$\int_a^b f(x)\, dx \;\le\; \int_a^b g(x)\, dx. \tag{7.18}$$

Proof. Exercise 7.6.

Theorem F. If f is continuous on $[a, b]$, then

$$\int_a^b f(x)^2\, dx = 0 \quad \text{implies} \quad f(x) = 0 \;\text{ for all } x \in [a, b]. \tag{7.19}$$

Proof. Exercise 7.7.

Notation. We extend the integral symbol by setting

$$\int_a^a f(x)\,dx = 0 \tag{7.20}$$

and

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx. \tag{7.21}$$

7.5 Variable Limits of Integration

Some refer to the following result as the "baby fundamental theorem of calculus."

Theorem G. If f is continuous on $[a, b]$, then

$$\frac{d}{dx}\int_a^x f(t)\,dt \;=\; f(x) \tag{7.22}$$

at each $x \in [a, b]$.

Proof. Fix $x_0 \in [a, b]$ and set[2]

$$F(x) = \int_a^x f(t)\,dt. \tag{7.23}$$

Then the difference quotient

$$\begin{aligned}\frac{F(x) - F(x_0)}{x - x_0} &= \frac{1}{x - x_0}\left[\int_a^x f(x)\,dx - \int_a^{x_0} f(t)\,dt\right]\\ &= \frac{1}{x - x_0}\int_{x_0}^x f(t)\,dt.\end{aligned}$$

When $x_0 < x$, using a partition of $[x_0, x]$ of one subinterval,

$$f(x_m) \;\le\; \frac{1}{x - x_0}\int_{x_0}^x f(x)\,dx \;\le f(x_M),$$

where x_m and x_M are points between x_0 and x where f achieves its minimum and maximum values, respectively. (When $x_0 > x$ the inequalities reverse.) But in the limit (exercise 7.8),

$$\lim_{x\to x_0} f(x_m) = f(x_0) = \lim_{x\to x_0} f(x_M). \tag{7.24}$$

[2]The logicians insist that we must use a "dummy variable" such as t for the bound variable of the integrand in (7.23) because the upper limit of integration is free to vary.

Corollary. Suppose f is continuous on an interval I containing all values of the differentiable functions u, v on $[a, b]$. Then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt \;=\; f(v(x))v'(x) \;-\; f(u(x))u'(x) \tag{7.25}$$

at each $x \in [a, b]$.

Proof. Fix $y_0 \in I$ and set

$$F(y) = \int_{y_0}^{y} f(t)\,dt.$$

Then by the chain rule, $F(v(x)) - F(u(x))$ has derivative $F'(v(x))v'(x) - F'(u(x))u'(x)$, which is our desired result.

Example 5.

$$\frac{d}{dx}\int_{\tan x}^{x^3} \cos t^2\,dt \;=\; 3x^2\cos x^6 \;-\; \sec^2 x \cos \tan^2 x.$$

7.6 Integrability

Which functions are integrable? We have proved that continuous functions are integrable. Dirichlet's example 1 is everywhere discontinuous and not integrable. So where between continuous everywhere and continuous nowhere is the dividing line for integrability? A finite number of discontinuities is not serious.

Theorem H. *A bounded function with at most a finite number of discontinuities is integrable.*

Proof. Suppose the bounded function f is continuous on $[a, b]$ except at a finite number of points. We immediately reduce to the case of exactly one discontinuity by dividing up the interval $[a, b]$ into subintervals each containing only one discontinuity; for if f is integrable over each subinterval, it is integrable over the whole interval, by exercise 7.10, the converse to theorem D.

Let x^* be the point of discontinuity of f on $[a, b]$ but suppose $x^* \neq a, b$. Let $\epsilon > 0$ be given, and let $m = \inf_{[a,b]} f(x)$ and $M = \sup_{[a,b]} f(x)$. Choose a subinterval $I = (a_1, b_1) \subset [a, b]$ containing x^* satisfying

$$(M - m)(b_1 - a_1) < \epsilon/2.$$

The complement $[a, b] \setminus I$ consists of two disjoint closed intervals where f is continuous with a combined partition $\mathcal{P}$ such that in total,

$U(\mathcal{P}) - L(\mathcal{P}) < \epsilon/2$. But then the points of $\mathcal{P}$ form a partition of all of $[a, b]$ where

$$U(\mathcal{P}) - L(\mathcal{P}) < \epsilon.$$

A similar argument obtains when $x^* = a$ or b.

Remark. The crux of the proof of theorem H was to isolate the dicontinuities of f into subintervals so small that their contributions to the Darboux sums were negligible. Informally, if the set D of points of discontinuity can be contained in sets of arbitrarily small size, then f is integrable. That in fact is the complete answer: *A bounded Borel-measurable*[3] *function is Riemann integrable over a bounded closed interval* $[a, b]$ *if and only if the set of its discontinuities in* $[a, b]$ *has Lebesgue measure zero.*

Allow me to explain. Consider the collection $\mathcal{B}$ of *Borel* sets, the smallest collection of subsets of **R** that is closed under countable union and complementation that contains all open sets. It is a fundamental result of graduate analysis that there exists a natural *measure* on the Borel sets; that is, there is a real- or infinite-valued function μ with the three properties

I. $\mu(E) \geq 0$ for every Borel set E.

II. For any countable disjoint collection of Borel sets E_k,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

III. $\mu([a, b]) = b - a$.

Thus there is a concept of "length" of Borel sets that agrees with the ordinary notion of length on intervals. If the "length" of the set of discontinuities of f is 0, then f is integrable and conversely. For the details of this famous result see [Bruckner et al.].

Exercises

7.1 Prove that when a partition is refined, (i.e., more points are added), the lower sum either increases or remains the same while the upper sum either decreases or remains the same.

7.2 Prove that $1 + 2 + 3 + \cdots + n = n(n + 1)/2$.

[3] A function f is *Borel measurable* if the inverse image under f of every Borel set is again a Borel set—a very mild requirement.

7.3 **(Project)** Trace the history of the Riemann integral. Does Cavalieri deserve credit for the idea? Do not recent documents show Archimedes employed the concept? Who is credited with first uncovering the fundamental theorem? Why is this integral named after the much-later Georg Riemann?

7.4 Prove theorem C.
Hint:

$$\inf_E f(x) + \inf_E g(x) \le \inf_E (f(x) + g(x)) \le \sup_E (f(x)+g(x)) \le \sup_E f(x) + \sup_E g(x).$$

7.5 Prove the corollary to theorem D.

7.6 Prove theorem E.

7.7 Prove theorem F.
Hint: If f is nonzero at a point it is nonzero on a neighborhood of that point.

7.8 Verify (7.24).

7.9 Show that $f(x) = x^2$ is integrable on $[0, 1]$ directly using Darboux sums. Find the value of the integral directly (without using the fundamental theorem).
Hint: Show $1 + 4 + 9 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.

7.10 Prove that if f is integrable over $[a, c]$ and over $[c, b]$, where $a < c < b$, then f is integrable over $[a, b]$.
Outline: For $\epsilon > 0$, there are partitions $\mathcal{P}_1$ of $[a, c]$ with $U(\mathcal{P}_1) - L(\mathcal{P}_1) < \epsilon/2$. Likewise there are partitions $\mathcal{P}_2$ of $[c, b]$ where $U(\mathcal{P}_2) - L(\mathcal{P}_2) < \epsilon/2$. Thus for the partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ of $[a, b]$, we have $U(\mathcal{P}) - L(\mathcal{P}) < \epsilon$.

7.11* **(Riemann sums)** Let f be integrable over $[a, b]$. For any partition $\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b$ and any choices $x_i^* \in [x_{i-1}, x_i]$, a sum of the form

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

is called a *Riemann sum* of f. Prove that the Riemann sums of f eventually cluster about the integral of f; that is, prove that for every

$\epsilon > 0$ there exists a $\delta > 0$ such that

$$\max_{1\le i\le n} \Delta x_i < \delta \quad \text{implies} \quad \left|\sum_{i=1}^{n} f(x_i^*)\Delta x_i \; - \; \int_a^b f(x)\, dx\right| < \epsilon.$$

7.12 Occasionally a sum can be recognized as a Riemann sum of a familiar integral. Calculate the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{3n^2 + 2k^2}{n^3}.$$

Answer: 11/3.

7.13** Construct an example of a bounded function that is a derivative on $[a, b]$ yet is not integrable over $[a, b]$.

7.14* Prove that if f is differentiable at x_0, then its *Lanczos derivative* at x_0 given by the limit

$$LDf(x_0) \; = \; \lim_{h\to 0} \frac{3}{2h^3} \int_{-h}^{h} tf(x_0 + t)\, dt$$

exists and equals $f'(x_0)$. In contrast, although $f(x) = |x|$ is not differentiable at $x_0 = 0$, prove that its Lanczos derivative does exist at $x_0 = 0$ and equals 0.