CHAPTER 1

Newton

Isaac Newton

Isaac Newton (1642–1727) stands as a seminal figure not just in mathematics but in all of Western intellectual history. He was born into a world where science had yet to establish a clear supremacy over medieval superstition. By the time of his death, the Age of Reason was in full bloom. This remarkable transition was due in no small part to his own contributions.

For mathematicians, Isaac Newton is revered as the creator of calculus, or, to use his name for it, of "fluxions." Its origin dates to the mid-1660s when he was a young scholar at Trinity College, Cambridge. There he had absorbed the work of such predecessors as René Descartes (1596–1650), John Wallis (1616–1703), and Trinity's own Isaac Barrow (1630–1677), but he soon found himself moving into uncharted territory. During the next few years, a period his biographer Richard Westfall characterized as one of "incandescent activity," Newton changed forever the mathematical landscape [1]. By 1669, Barrow himself was describing his colleague as "a fellow of our College and very young . . . but of an extraordinary genius and proficiency" [2].

In this chapter, we look at a few of Newton's early achievements: his generalized binomial expansion for turning certain expressions into infinite series, his technique for finding inverses of such series, and his quadrature rule for determining areas under curves. We conclude with a spectacular consequence of these: the series expansion for the sine of an angle. Newton's account of the binomial expansion appears in his *epistola prior*, a letter he sent to Leibniz in the summer of 1676 long after he had done the original work. The other discussions come from Newton's 1669 treatise *De analysi per aequationes numero terminorum infinitas*, usually called simply the *De analysi*.

Although this chapter is restricted to Newton's early work, we note that "early" Newton tends to surpass the mature work of just about anyone else.

Generalized Binomial Expansion

By 1665, Isaac Newton had found a simple way to expand—his word was "reduce"—binomial expressions into series. For him, such reductions would be a means of recasting binomials in alternate form as well as an entryway into the method of fluxions. This theorem was the starting point for much of Newton's mathematical innovation.

As described in the *epistola prior*, the issue at hand was to reduce the binomial $(P + PQ)^{m/n}$ and to do so whether m/n "is integral or (so to speak) fractional, whether positive or negative" [3]. This in itself was a bold idea for a time when exponents were sufficiently unfamiliar that they had first to be explained, as Newton did by stressing that "instead of $\sqrt{a}$, $\sqrt[3]{a}$, $\sqrt[3]{a^5}$, etc. I write $a^{1/2}$, $a^{1/3}$, $a^{5/3}$, and instead of $1/a$, $1/aa$, $1/a^3$, I write a^{-1}, a^{-2}, a^{-3}" [4]. Apparently readers of the day needed a gentle reminder.

Newton discovered a pattern for expanding not only elementary binomials like $(1 + x)^5$ but more sophisticated ones like $\dfrac{1}{\sqrt[3]{(1+x)^5}} = (1+x)^{-5/3}$. The reduction, as Newton explained to Leibniz, obeyed the rule

$$(P+PQ)^{m/n} = P^{m/n} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \text{etc.}, \qquad (1)$$

where each of A, B, C, . . . represents the previous term, as will be illustrated below. This is his famous binomial expansion, although perhaps in an unfamiliar guise.

Newton provided the example of $\sqrt{c^2+x^2} = [c^2 + c^2(x^2/c^2)]^{1/2}$. Here, $P = c^2$, $Q = \dfrac{x^2}{c^2}$, $m = 1$, and $n = 2$. Thus,

$$\sqrt{c^2+x^2} = (c^2)^{1/2} + \frac{1}{2}A\frac{x^2}{c^2} - \frac{1}{4}B\frac{x^2}{c^2} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \cdots.$$

To identify A, B, C, and the rest, we recall that each is the immediately preceding term. Thus, $A = (c^2)^{1/2} = c$, giving us

$$\sqrt{c^2+x^2} = c + \frac{x^2}{2c} - \frac{1}{4}B\frac{x^2}{c^2} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \cdots.$$

Likewise B is the previous term—i.e., $B = \dfrac{x^2}{2c}$—so at this stage we have

$$\sqrt{c^2+x^2} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} - \frac{1}{2}C\frac{x^2}{c^2} - \frac{5}{8}D\frac{x^2}{c^2} - \cdots.$$

The analogous substitutions yield $C = -\dfrac{x^4}{8c^3}$ and then $D = \dfrac{x^6}{16c^5}$. Working from left to right in this fashion, Newton arrived at

$$\sqrt{c^2+x^2} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \cdots.$$

Obviously, the technique has a recursive flavor: one finds the coefficient of x^8 from the coefficient of x^6, which in turn requires the coefficient of x^4, and so on. Although the modern reader is probably accustomed to a "direct" statement of the binomial theorem, Newton's recursion has an undeniable appeal, for it streamlines the arithmetic when calculating a numerical coefficient from its predecessor.

For the record, it is a simple matter to replace A, B, C, . . . by their equivalent expressions in terms of P and Q, then factor the common

$P^{m/n}$ from both sides of (1), and so arrive at the result found in today's texts:

$$(1+Q)^{m/n} = 1 + \frac{m}{n}Q + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2\times 1}Q^2 + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{3\times 2\times 1}Q^3 + \cdots. \qquad (2)$$

Newton likened such reductions to the conversion of square roots into infinite decimals, and he was not shy in touting the benefits of the operation. "It is a convenience attending infinite series," he wrote in 1671,

> that all kinds of complicated terms . . . may be reduced to the class of simple quantities, i.e., to an infinite series of fractions whose numerators and denominators are simple terms, which will thus be freed from those difficulties that in their original form seem'd almost insuperable. [5]

To be sure, freeing mathematics from insuperable difficulties is a worthy undertaking.

One additional example may be helpful. Consider the expansion of $\frac{1}{\sqrt{1-x^2}}$, which Newton put to good use in a result we shall discuss later in the chapter. We first write this as $(1-x^2)^{-1/2}$, identify $m = -1$, $n = 2$, and $Q = -x^2$, and apply (2):

$$\begin{aligned}\frac{1}{\sqrt{1-x^2}} &= 1 + \left(-\frac{1}{2}\right)(-x^2) + \frac{(-1/2)(-3/2)}{2\times 1}(-x^2)^2 \\ &\quad + \frac{(-1/2)(-3/2)(-5/2)}{3\times 2\times 1}(-x^2)^3 \\ &\quad + \frac{(-1/2)(-3/2)(-5/2)(-7/2)}{4\times 3\times 2\times 1}(-x^2)^4 + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots. \qquad (3)\end{aligned}$$

Newton would "check" an expansion like (3) by *squaring* the series and examining the answer. If we do the same, restricting our attention to terms of degree no higher than x^8, we get

$$\begin{aligned}&\left[1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\frac{35}{128}x^8+\cdots\right]\\&\quad\times\left[1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\frac{35}{128}x^8+\cdots\right]\\&=1+x^2+x^4+x^6+x^8+\cdots,\end{aligned}$$

where all of the coefficients miraculously turn out to be 1 (try it!). The resulting product, of course, is an infinite geometric series with common ratio x^2 which, by the well-known formula, sums to $\frac{1}{1-x^2}$. But if the *square* of the series in (3) is $\frac{1}{1-x^2}$, we conclude that that series itself must be $\frac{1}{\sqrt{1-x^2}}$. *Voila*!

Newton regarded such calculations as compelling evidence for his general result. He asserted that the "common analysis performed by means of equations of a finite number of terms" may be extended to such infinite expressions "albeit we mortals whose reasoning powers are confined within narrow limits, can neither express nor so conceive all the terms of these equations, as to know exactly from thence the quantities we want" [6].

Inverting Series

Having described a method for reducing certain binomials to infinite series of the form $z = A + Bx + Cx^2 + Dx^3 + \cdots$, Newton next sought a way of finding the series for x in terms of z. In modern terminology, he was seeking the inverse relationship. The resulting technique involves a bit of heavy algebraic lifting, but it warrants our attention for it too will appear later on. As Newton did, we describe the inversion procedure by means of a specific example.

Beginning with the series $z = x - x^2 + x^3 - x^4 + \cdots$, we rewrite it as

$$(x - x^2 + x^3 - x^4 + \cdots) - z = 0 \qquad (4)$$

and discard all powers of x greater than or equal to the quadratic. This, of course, leaves $x - z = 0$, and so the inverted series begins as $x = z$.

Newton was aware that discarding all those higher degree terms rendered the solution inexact. The *exact* answer would have the form $x = z + p$, where p is a series yet to be determined. Substituting $z + p$ for x in (4) gives

$$[(z+p) - (z+p)^2 + (z+p)^3 - (z+p)^4 + \cdots] - z = 0,$$

which we then expand and rearrange to get

$$\begin{aligned}&[-z^2 + z^3 - z^4 + z^5 - \cdots] + [1 - 2z + 3z^2 - 4z^3 + 5z^4 - \cdots]p\\ &\quad + [-1 + 3z - 6z^2 + 10z^3 - \cdots]p^2 + [1 - 4z + 10z^2 - \cdots]p^3\\ &\quad + [-1 + 5z - \cdots]p^4 + \cdots = 0.\end{aligned} \tag{5}$$

Next, jettison the quadratic, cubic, and higher degree terms in p and solve to get

$$p = \frac{z^2 - z^3 + z^4 - z^5 + \cdots}{1 - 2z + 3z^2 - 4z^3 + \cdots}.$$

Newton now did a second round of weeding, as he tossed out all but the lowest power of z in numerator and denominator. Hence p is approximately $\frac{z^2}{1}$, so the inverted series at this stage looks like $x = z + p = z + z^2$.

But p is not *exactly* z^2. Rather, we say $p = z^2 + q$, where q is a series to be determined. To do so, we substitute into (5) to get

$$\begin{aligned}&[-z^2 + z^3 - z^4 + z^5 - \cdots] + [1 - 2z + 3z^2 - 4z^3 + 5z^4 - \cdots](z^2 + q)\\ &\quad + [-1 + 3z - 6z^2 + 10z^3 - \cdots](z^2 + q)^2 + [1 - 4z + 10z^2 - \cdots]\\ &\quad (z^2 + q)^3 + [-1 + 5z - \cdots](z^2 + q)^4 + \cdots = 0.\end{aligned}$$

We expand and collect terms by powers of q:

$$\begin{aligned}&[-z^3 + z^4 - z^6 + \cdots] + [1 - 2z + z^2 + 2z^3 - \cdots]q\\ &\quad + [-1 + 3z - 3z^2 - 2z^3 + \cdots]q^2 + \cdots.\end{aligned} \tag{6}$$

As before, discard terms involving powers of q above the first, solve to get $q = \frac{z^3 - z^4 + z^6 - \cdots}{1 - 2z + z^2 + 2z^3 + \cdots}$, and then drop all but the lowest degree terms top and bottom to arrive at $q = \frac{z^3}{1}$. At this point, the series looks like $x = z + z^2 + q = z + z^2 + z^3$.

The process would be continued by substituting $q = z^3 + r$ into (6). Newton, who had a remarkable tolerance for algebraic monotony, seemed able to continue such calculations *ad infinitum* (almost). But eventually even he was ready to step back, examine the output, and seek a pattern. Newton put it this way: "Let it be observed here, by the bye, that when 5 or 6 terms . . . are known, they may be continued at pleasure for most part, by observing the analogy of the progression" [7].

For our example, such an examination suggests that $x = z + z^2 + z^3 + z^4 + z^5 + \cdots$ is the inverse of the series $z = x - x^2 + x^3 - x^4 + \cdots$ with which we began.

In what sense can this be trusted? After all, Newton discarded most of his terms most of the time, so what confidence remains that the answer is correct?

Again, we take comfort in the following "check." The original series $z = x - x^2 + x^3 - x^4 + \cdots$ is geometric with common ratio $-x$, and so in closed form $z = \dfrac{x}{1+x}$. Consequently, $x = \dfrac{z}{1-z}$, which we recognize to be the sum of the geometric series $z + z^2 + z^3 + z^4 + z^5 + \cdots$. This is precisely the result to which Newton's procedure had led us. Everything seems to be in working order.

The techniques encountered thus far—the generalized binomial expansion and the inversion of series—would be powerful tools in Newton's hands. There remains one last prerequisite, however, before we can truly appreciate the master at work.

Quadrature Rules from the *De Analysi*

In his *De analysi* of 1669, Newton promised to describe the method "which I had devised some considerable time ago, for measuring the quantity of curves, by means of series, infinite in the number of terms" [8]. This was not Newton's first account of his fluxional discoveries, for he had drafted an October 1666 tract along these same lines. The *De analysi* was a revision that displayed the polish of a maturing thinker. Modern scholars find it strange that the secretive Newton withheld this manuscript from all but a few lucky colleagues, and it did not appear in print until 1711, long after many of its results had been published by others. Nonetheless, the early date and illustrious authorship justify its description as "perhaps the most celebrated of all Newton's mathematical writings" [9].

The treatise began with a statement of the three rules for "the quadrature of simple curves." In the seventeenth century, *quadrature* meant determination of area, so these are just integration rules.

> Rule 1. The quadrature of simple curves: If $y = ax^{m/n}$ is the curve AD, where a is a constant and m and n are positive integers, then the area of region ABD is $\dfrac{an}{m+n}x^{(m+n)/n}$ (see figure 1.1).

A modern version of this would identify A as the origin, B as $(x, 0)$, and the curve as $y = at^{m/n}$. Newton's statement then becomes $\int_0^x at^{m/n}\,dt = \dfrac{ax^{(m/n)+1}}{(m/n)+1} = \dfrac{an}{m+n}x^{(m+n)/n}$, which is just a special case of the power rule from integral calculus.

Only at the end of the *De analysi* did Newton observe, almost as an afterthought, that "an attentive reader" would want to see a proof for Rule 1 [10]. Attentive as always, we present his argument below.

Again, let the curve be AD with $AB = x$ and $BD = y$, as shown in figure 1.2. Newton assumed that the *area ABD* beneath the curve was given by an expression z written in terms of x. The goal was to find a corresponding

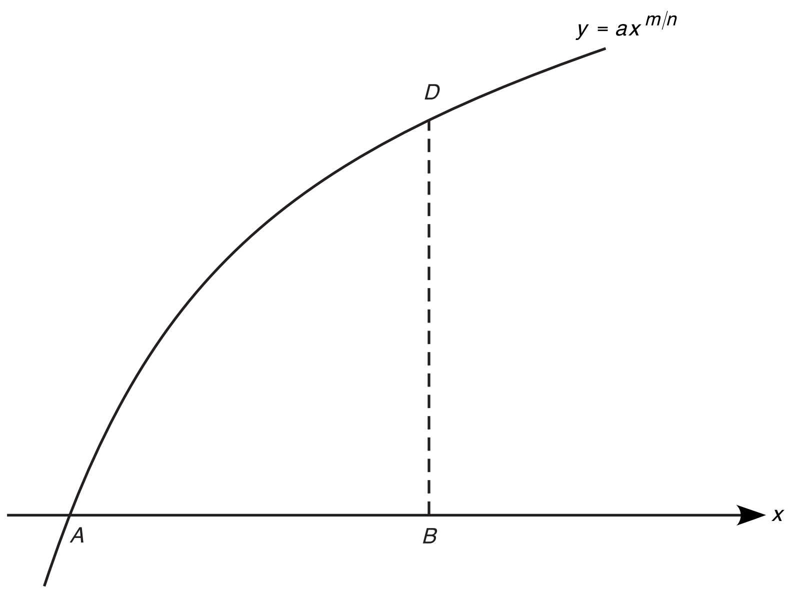


Figure 1.1

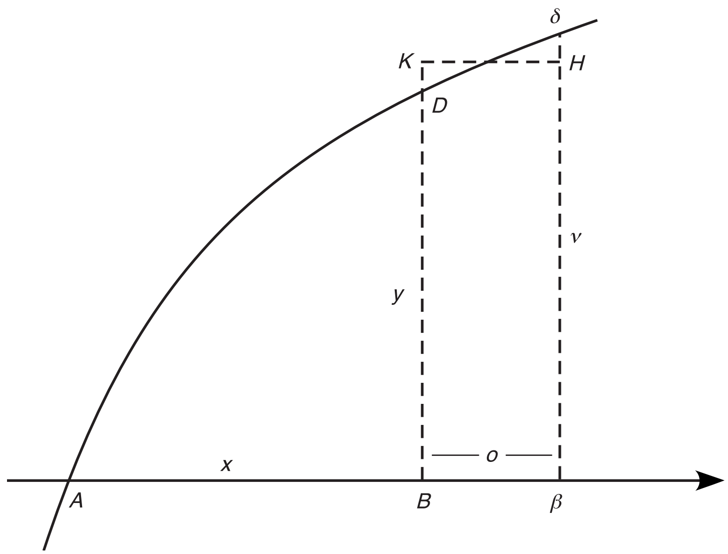


Figure 1.2

formula for y in terms of x. From a modern vantage point, he was beginning with $z = \int_0^x y(t)dt$ and seeking $y = y(x)$. His derivation blended geometry, algebra, and fluxions before ending with a few dramatic flourishes.

At the outset, Newton let β be a point on the horizontal axis a tiny distance o from B. Thus, segment $A\beta$ has length $x + o$. He let z be the area ABD, although to emphasize the functional relationship we shall take the liberty of writing $z = z(x)$. Hence, $z(x + o)$ is the area $A\beta\delta$ under the curve. Next he introduced rectangle $B\beta HK$ of height $v = BK = \beta H$, the area of which he stipulated to be *exactly* that of region $B\beta\delta D$ beneath the curve. In other words, the area of $B\beta\delta D$ was to be ov.

At this point, Newton specified that $z(x) = \dfrac{an}{m+n} x^{(m+n)/n}$ and proceeded to find the instantaneous rate of change of z. To do so, he examined the change in z divided by the change in x as the latter becomes small. For notational ease, he temporarily let $c = an/(m + n)$ and $p = m + n$ so that $z(x) = cx^{p/n}$ and

$$[z(x)]^n = c^n x^p. \tag{7}$$

Now, $z(x + o)$ is the area $A\beta\delta$, which can be decomposed into the area of ABD and that of $B\beta\delta D$. The latter, as noted, is the same as rectangular area ov and so Newton concluded that $z(x+o) = z(x) + ov$. Substituting into (7), he got

$$[z(x) + ov]^n = [z(x+o)]^n = c^n(x+o)^p,$$

and the binomials on the left and right were expanded to

$$[z(x)]^n + n[z(x)]^{n-1}ov + \frac{n(n-1)}{2}[z(x)]^{n-2}o^2v^2 + \cdots$$
$$= c^n x^p + c^n p x^{p-1}o + c^n \frac{p(p-1)}{2}x^{p-2}o^2 + \cdots.$$

Applying (7) to cancel the leftmost terms on each side and then dividing through by o, Newton arrived at

$$n[z(x)]^{n-1}v + \frac{n(n-1)}{2}[z(x)]^{n-2}ov^2 + \cdots$$
$$= c^n p x^{p-1} + c^n \frac{p(p-1)}{2}x^{p-2}o + \cdots. \qquad (8)$$

At that point, he wrote, "If we suppose $B\beta$ to be diminished infinitely and to vanish, or o to be nothing, v and y in that case will be equal, and the terms which are multiplied by o will vanish" [11]. He was asserting that, as o becomes zero, so do all terms in (8) that contain o. At the same time, v becomes equal to y, which is to say that the height BK of the rectangle in Figure 1.2 will equal the ordinate BD of the original curve. In this way, (8) transforms into

$$n[z(x)]^{n-1}y = c^n p x^{p-1}. \qquad (9)$$

A modern reader is likely to respond, "Not so fast, Isaac!" When Newton divided by o, that quantity most certainly was *not* zero. A moment later, it *was* zero. There, in a nutshell, lay the rub. This zero/nonzero dichotomy would trouble analysts for the next century and then some. We shall have much more to say about this later in the book.

But Newton proceeded. In (9) he substituted for $z(x)$, c, and p and solved for

$$y = \frac{c^n p x^{p-1}}{n[z(x)]^{n-1}} = \frac{\left[\frac{an}{(m+n)}\right]^n (m+n)x^{m+n-1}}{n\left[\frac{an}{(m+n)}x^{(m+n)/n}\right]^{n-1}} = ax^{m/n}.$$

Thus, starting from his assumption that the area *ABD* is given by $z(x) = \frac{an}{m+n} x^{(m+n)/n}$. Newton had deduced that curve *AD* must satisfy the equation $y = ax^{m/n}$. He had, in essence, differentiated the integral. Then, without further justification, he stated, "Wherefore conversely, if $ax^{m/n} = y$, it shall be $\frac{an}{m+n} x^{(m+n)/n} = z$." His proof of rule 1 was finished [12].

This was a peculiar twist of logic. Having derived the equation of *y* from that of its area accumulator *z*, Newton asserted that the relationship went the other way and that the area under $y = ax^{m/n}$ is indeed $\frac{an}{m+n} x^{(m+n)/n}$. Such an argument tends to leave us with mixed feelings, for it features some gaping logical chasms. Derek Whiteside, editor of Newton's mathematical papers, aptly characterized this quadrature proof as "a brief, scarcely comprehensible appearance of fluxions" [13]. On the other hand, it is important to remember the source. Newton was writing at the very beginning of the long calculus journey. Within the context of his time, the proof was groundbreaking, and his conclusion was correct. Something rings true in Richard Westfall's observation that, "however briefly, *De analysi* did indicate the full extent and power of the fluxional method" [14].

Whatever the modern verdict, Newton was satisfied. His other two rules, for which the *De analysi* contained no proofs, were as follows:

> Rule 2. The quadrature of curves compounded of simple ones: If the value of *y* be made up of several such terms, the area likewise shall be made up of the areas which result from every one of the terms. [15]

> Rule 3. The quadrature of all other curves: But if the value of *y*, or any of its terms be more compounded than the foregoing, it must be reduced into more simple terms . . . and afterwards by the preceding rules you will discover the [area] of the curve sought. [16]

Newton's second rule affirmed that the integral of the sum of finitely many terms is the sum of the integrals. This he illustrated with an example or two. The third rule asserted that, when confronted with a more complicated expression, one was first to "reduce" it into an infinite series, integrate each term of the series by means of the first rule, and then sum the results.

This last was an appealing idea. More to the point, it was the final prerequisite Newton would need to derive a mathematical blockbuster: the infinite series for the sine of an angle. This great theorem from the *De analysi* will serve as the chapter's climax.

Newton's Derivation of the Sine Series

Consider in figure 1.3 the quadrant of a circle centered at the origin and with radius 1, where as before $AB = x$ and $BD = y$. Newton's initial objective was to find an expression for the length of arc αD [17].

From D, draw DT tangent to the circle, and let BK be "the moment of the base AB." In a notation that would become standard after Newton, we let $BK = dx$. This created the "indefinitely small" right triangle DGH, whose hypotenuse DH Newton regarded as the moment of the arc αD. We write $DH = dz$, where $z = z(x)$ stands for the length of arc αD. Because all of this is occurring within the unit circle, the radian measure of $\angle \alpha AD$ is z as well.

Under this scenario, the infinitely small triangle DGH is similar to triangle DBT so that $\frac{GH}{DH} = \frac{BT}{DT}$. Moreover, radius AD is perpendicular to tangent line DT, and so altitude BD splits right triangle ADT into similar pieces: triangles DBT and ABD. It follows that $\frac{BT}{DT} = \frac{BD}{AD}$, and from these two proportions we conclude that $\frac{GH}{DH} = \frac{BD}{AD}$. With the differential notation above, this amounts to $\frac{dx}{dz} = \frac{y}{1}$, and hence $dz = \frac{dx}{y}$.

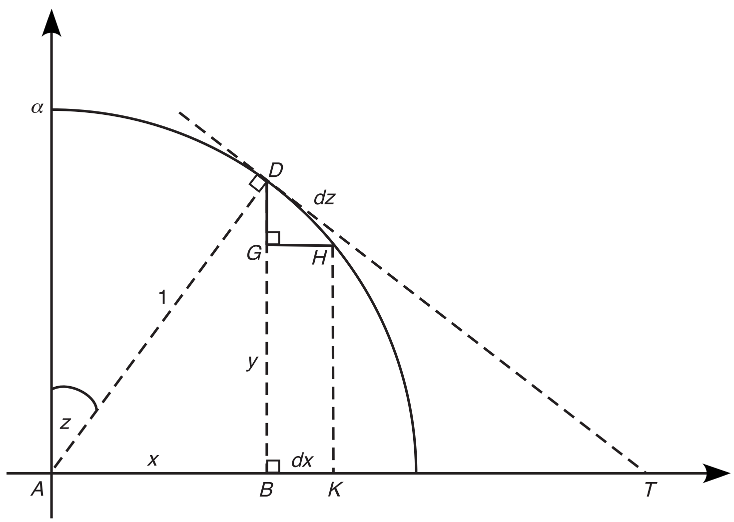


Figure 1.3

Newton's next step was to exploit the circular relationship $y = \sqrt{1 - x^2}$ to conclude that $dz = \frac{dx}{y} = \frac{dx}{\sqrt{1 - x^2}}$. Expanding $\frac{1}{\sqrt{1 - x^2}}$ as in (3) led to

$$dz = \left[1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \frac{35}{128}x^8 + \cdots\right]dx,$$

and so

$$z = z(x) = \int_0^x dz = \int_0^x \left[1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \frac{5}{16}t^6 + \frac{35}{128}t^8 + \cdots\right]dt.$$

Finding the quadratures of these individual powers and summing the results by Rule 3, Newton concluded that the arclength of αD was

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots. \tag{10}$$

Referring again to Figure 1.3, we see that z is not only the radian measure of $\angle \alpha AD$, but the measure of $\angle ADB$ as well. From triangle ABD, we know that $\sin z = x$ and so

$$\arcsin x = z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \cdots.$$

Thus, beginning with the *algebraic* expression $\frac{1}{\sqrt{1 - x^2}}$, Newton had used his generalized binomial expansion and basic integration to derive the series for arcsine, an intrinsically more complicated relationship.

But Newton had one other trick up his sleeve. Instead of a series for arclength (z) in terms of its coordinate (x), he sought to reverse the process. He wrote, "If, from the Arch αD given, the Sine AB was required, I extract the root of the equation found above" [18]. That is, Newton would apply his inversion procedure to convert the series for $z = \arcsin x$ into one for $x = \sin z$.

Following the technique described earlier, we begin with $x = z$ as the first term. To push the expansion to the next step, substitute $x = z + p$ into (10) and solve to get

$$p = \frac{-\frac{1}{6}z^3 - \frac{3}{40}z^5 - \frac{5}{112}z^7 - \cdots}{1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \frac{5}{16}z^6 + \cdots},$$

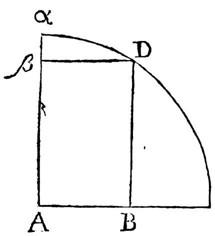
from which we retain only $p = -\frac{1}{6}z^3$. This extends the series to $x = z - \frac{1}{6}z^3$. Next introduce $p = -\frac{1}{6}z^3 + q$ and continue the inversion process, solving for

$$q = \frac{\frac{1}{120}z^5 + \frac{1}{56}z^7 - \frac{1}{72}z^8 + \cdots}{1 + \frac{1}{2}z^2 + \frac{3}{8}z^4 + \cdots},$$

or simply $q = \frac{1}{120}z^5$. At this stage $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5$, and, as Newton might say, we "continue at pleasure" until discerning the pattern and writing down one of the most important series in analysis:

$$\sin z = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 - \cdots$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}.$$

To find the Baſe from the Length of the Curve given.



45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, *viz.* $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being ſuppoſed that AB $= x$, αD $= z$, and Aα $= 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9$ &c.

46. And moreover if the Coſine Aβ were required from that Arch given, make Aβ $(= \sqrt{1 - xx}) = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \frac{1}{720}z^6 + \frac{1}{40320}z^8 - \frac{1}{3628800}z^{10}$, &c.

Newton's series for sine and cosine (1669)

For good measure, Newton included the series for $\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}$ In the words of Derek Whiteside, "These series for the sine and cosine . . . here appear for the first time in a European manuscript" [19].

To us, this development seems incredibly roundabout. We now regard the sine series as a trivial consequence of Taylor's formula and differential calculus. It is so natural a procedure that we expect it was always so. But Newton, as we have seen, approached this very differently. He applied rules of integration, not of differentiation; he generated the sine series from the (to our minds) incidental series for the arcsine; and he needed his complicated inversion scheme to make it all work.

This episode reminds us that mathematics did not necessarily evolve in the manner of today's textbooks. Rather, it developed by fits and starts and odd surprises. Actually that is half the fun, for history is most intriguing when it is at once significant, beautiful, and *unexpected*.

On the subject of the unexpected, we add a word about Whiteside's qualification in the passage above. It seems that Newton was not the first to discover a series for the sine. In 1545, the Indian mathematician Nilakantha (1445–1545) described this series and credited it to his even more remote predecessor Madhava, who lived around 1400. An account of these discoveries, and of the great Indian tradition in mathematics, can be found in [20] and [21]. It is certain, however, that these results were unknown in Europe when Newton was active.

We end with two observations. First, Newton's *De analysi* is a true classic of mathematics, belonging on the bookshelf of anyone interested in how calculus came to be. It provides a glimpse of one of history's most fertile thinkers at an early stage of his intellectual development.

Second, as should be evident by now, a revolution had begun. The young Newton, with a skill and insight beyond his years, had combined infinite series and fluxional methods to push the frontiers of mathematics in new directions. It was his contemporary, James Gregory (1638–1675), who observed that the elementary methods of the past bore the same relationship to these new techniques "as dawn compares to the bright light of noon" [22]. Gregory's charming description was apt, as we see time and again in the chapters to come. And first to travel down this exciting path was Isaac Newton, truly "a man of extraordinary genius and proficiency."