is not ε). Then the Lyapunov exponent $\lambda_{\max}$ defined as

$$\lim_{q\to\infty} \frac{\|x(q)\|_{\max}}{q} = \lambda_{\max}$$

(and analogously for $\lambda_{\min}$) exists for almost all sequences and is independent of the initial (finite) choice of $x(0)$. This result shows that, in principle, it is possible to calculate the efficiency measures discussed above. The main difference is that, although the Lyapunov exponents exist, there is no nice prescription for their calculation. The development of efficient numerical algorithms is an open problem that the interested reader might, perhaps, wish to consider!

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VII.5 Evolving Social Networks, Attitudes, and Beliefs—and Counterterrorism

Peter Grindrod

1 Introduction

The central objects of interest here are

(i) an evolving digital network of peer-to-peer communication,
(ii) the dynamics of information, ideas, and beliefs that can propagate through that digital network, and
(iii) how such networks become important in matters of national security and defense.

The applications of this theory spread far beyond these topics though. For example, more than a quarter of all marketing and advertising spending in the United Kingdom and the United States is now spent online: so digital media marketing ("buzz" marketing), though in its infancy, requires a deeper understanding of the nature of social communication networks. This is an important area of complexity theory, since there is no closed theory (analogous to conservation laws for molecular dynamics or chemical reactions) available at the microscopic "unit" level. Instead, here we must consider irrational, inconsistent, and ever-changing people. Moreover, while the passage of ideas is mediated by the networking behavior, the very existence of such ideas may cause communication to take place: systems can therefore be fully coupled.

In observing peer-to-peer communication in mobile phone networks, messaging, email, and online chats, the size of communities is a substantial challenge. Equally, from a conceptual modeling perspective, it is clear that being able to simulate, anticipate, and infer behavior in real time, or on short timescales, may be critical in designing interventions or spotting sudden aberrations. This field therefore requires and has inspired new ideas in both applied mathematical models and methods.

2 Evolving Networks in Continuous and Discrete Time

Consider a population of N individuals (agents/actors) connected through a dynamically evolving undirected network representing pairwise voice calls or online chats. Let $A(t)$ denote the $N \times N$ binary adjacency matrix for this network at time t, having a zero diagonal. At future times, $A(t)$ is a stochastic object defined by a probability distribution over the set of all possible adjacency matrices. Each edge within this network will be assumed to evolve independently over time, though it is conditionally dependent upon the current network (so any edges conditional on related current substructures may well be highly correlated over time). Rather than model a full probability distribution for future network evolution, conditional on its current structure, say $\mathcal{P}_{\delta t}(A(t+\delta t) \mid A(t))$, it is enough to specify its expected value $E(A(t+\delta t) \mid A(t))$ (a matrix containing all edge probabilities, from which edges may be generated independently). Their equivalence is trivial, since

$$E(A(t+\delta t) \mid A(t)) = \sum_B B\mathcal{P}_{\delta t}(B \mid A(t)),$$

and

$$\mathcal{P}_{\delta t}(B \mid A(t)) = \prod_{i=1,\, j=i+1}^{N-1,N} W_{i,j}^{B_{i,j}} (1 - W_{i,j})^{1-B_{i,j}},$$

where $W = E(A(t + \delta t) \mid A(t))$. Hence we shall specify our model for the stochastic network evolution via

$$E(A(t + \delta t) \mid A(t)) = A(t) + \delta t \mathcal{F}(A(t)), \qquad (1)$$

valid as $\delta t \to 0$. Here the real matrix-valued function $\mathcal{F}$ is symmetric, it has a zero diagonal, and all elements within the right-hand side will be in $[0, 1]$. We write

$$\mathcal{F}(A(t)) = -A(t) \circ \omega(A(t)) + (C - A(t)) \circ \alpha(A(t)).$$

Here C denotes the adjacency matrix for the *clique* where all $\frac{1}{2}N(N-1)$ edges are present (all elements are 1s except for 0s on the diagonal), so $C - A(t)$ denotes the adjacency matrix for the graph complement of $A(t)$; $\omega(A(t))$ and $\alpha(A(t))$ are both real nonnegative symmetric matrix functions containing conditional edge death rates and conditional edge birth rates, respectively; and $\circ$ denotes the Hadamard, or element-wise, matrix product.

In many cases we can usefully consider a discrete-time version of the above evolution. Let $\{A_k\}_{k=1}^{K}$ denote an ordered sequence of adjacency matrices (binary, symmetric with zero diagonals) representing a discrete-time evolving network with value A_k at time step t_k. We shall then assume that edges evolve independently from time step to time step with each new network conditionally dependent on the previous one. A first-order model is given by a Markov process

$$E(A_{k+1} \mid A_k) = A_k \circ (C - \tilde{\omega}(A_k)) + (C - A_k) \circ \tilde{\alpha}(A_k). \quad (2)$$

Here $\tilde{\omega}(A_k)$ is a real nonnegative symmetric matrix function containing conditional death probabilities, each in $[0, 1]$, and $\tilde{\alpha}(A_k)$ is a real nonnegative symmetric matrix function containing conditional edge birth probabilities, each in $[0, 1]$.

As before, the edge independence assumption implies that $P(A_{k+1} \mid A_k)$ can be reconstructed from $E(A_{k+1} \mid A_k)$.

A generalization of KATZ [IV.18 §3.4] centrality for such discrete-time evolving networks can be obtained. In particular, if $0 < \mu < 1/\max\{\rho(A_k)\}$, then the communicability matrix

$$\mathcal{Q} = (I - \mu A_1)^{-1}(I - \mu A_2)^{-1} \cdots (I - \mu A_K)^{-1}$$

provides a weighted count of all possible dynamic paths between all pairs of vertices. It is nonsymmetric (due to time's arrow) and its row sums represent the abilities of the corresponding people to send messages to others, while its column sums represent the abilities of the corresponding people to receive messages from others. Such performance measures are useful in identifying influential people within evolving networks. This idea has recently been extended so as to successively discount the older networks in order to produce better inferences.

3 Nonlinear Effects: Seen and Unseen

In the sociology literature the simplest form of nonlinearity occurs when people introduce their friends to each other. So, in (2), if two nonadjacent people are connected to a common friend at step k, then it is more likely that those two people will be directly connected at step $k+1$. To model this *triad closure* dynamic we may use $\tilde{\omega}(A_k) = \gamma C$, so all edges have the same step-to-step death probability, $\gamma \in [0, 1]$, and

$$\tilde{\alpha}(A_k) = \delta C + \varepsilon A_k^2.$$

Here δ and ε are positive and such that $\delta + \varepsilon(N-2) < 1$. The element $(A^2)_{i,j}$ counts the number of mutual connections that person i and person j have at step k. This equation is ergodic and yet it is destined to spend most of its time close to states where the density of edges means that there is a balance between edge births and deaths. A mean-field approach can be applied, approximating A_k with its expectation, which may be assumed to be of the form $p_k C$ (an ERDŐS–RÉNYI RANDOM GRAPH [IV.18 §4.1] with edge density p_k). In the mean-field dynamic one obtains

$$p_{k+1} = p_k(1 - \gamma) + (1 - p_k)(\delta + (N - 2)\varepsilon p_k^2). \quad (3)$$

If δ is small and $\omega < \frac{1}{4}\varepsilon(N - 2)$, then this nonlinear iteration has three fixed points: two stable ones, at $\delta/\gamma + O(\delta^2)$ and $\frac{1}{2} + (\frac{1}{4} - \gamma/(\varepsilon(N - 2)))^{1/2} + O(\delta)$, and one unstable one in the middle. Thus the extracted mean-field behavior is bistable. In practice, one might observe the edge density of such a network approaching one or other stable mean-field equilibrium and jiggling around it for a very long time, without any awareness that another type of orbit or pseudostable edge density could exist. Direct comparisons of transient orbits from (2), incorporating triad closure, with their mean-field approximations in (3) are very good over short to medium timescales. Yet though we have captured the nonlinear effects well in (3), the stochastic nature of (2) must *eventually* cause orbits to diverge from the deterministic stability seen in (3).

The phenomenon seen here explains the events of a new undergraduate's first week at university are so important in forming high-density connected social networks among student year groups. If we do not perturb them with a mix of opportunities to meet, they may be condemned to remain close to the low-density (few-friends) state for a very long time.

4 Fully Coupled Systems

There is a large literature within psychology that is based on individuals' attitudes and behaviors being in a tensioned equilibrium between excitory (activating) processes and inhibiting processes. Typically, the state of an individual is represented by a set of state variables, some measuring activating elements and some measuring the inhibiting elements.

Activator–inhibitor systems have had an impact within mathematical models where a uniformity equilibrium across a population of individual systems becomes destabilized by the very act of simple "passive" coupling between them. Such Turing instabilities can sometimes seem counterintuitive.

Homophily is a term that describes how associations are more likely to occur between people who have similar attitudes and views. Here we show how individuals' activator–inhibitor dynamics coupled through a homophilic evolving network produce systems that have pseudoperiodic consensus and fractionation.

Consider a population of N identical individuals, each described by a set of m state variables that are continuous functions of time t. Let $x_i(t) \in \mathbb{R}^m$ denote the ith individual's attitudinal state. Let $A(t)$ denote the adjacency matrix for the communication network, as it does in (1). Then consider

$$\dot{x}_i = f(x_i) + D\sum_{j=1}^{N} A_{ij}(x_j - x_i), \quad i = 1, \dots, N. \tag{4}$$

Here f is a given smooth field over $\mathbb{R}^m$, drawn from a class of activator–inhibitor systems, and is such that $f(x^*) = 0$ for some x^*, and the Jacobian there, $\mathrm{d}f(x^*)$, is a stability matrix (that is, all its eigenvalues have negative real parts). D is a real diagonal nonnegative matrix containing the maximal transmission coefficients (diffusion rates) for the corresponding attitudinal variables between adjacent neighbors. Let $X(t)$ denote the $m \times N$ matrix with ith column given by $x_i(t)$, and let $F(X)$ be the $m \times N$ matrix with ith column given by $f(x_i(t))$. Then (4) may be written as

$$\dot{X} = F(X) - DX\Delta. \tag{5}$$

Here $\Delta(t)$ denotes the graph Laplacian for $A(t)$, given by $\Delta(t) = \Gamma(t) - A(t)$, where $\Gamma(t)$ is the diagonal matrix containing the degrees of the vertices. This system has an equilibrium at $X = X^*$, say, where the ith column of X^* is given by x^* for all $i = 1, \dots, N$.

Now consider an evolution equation for $A(t)$, in the form of (1), coupled to the states X:

$$\begin{aligned} E(A(t+\delta t) \mid A(t)) &\\ = A(t) + \delta t(-A(t) \circ (C - \Phi(X(t)))\gamma &\\ + (C - A(t)) \circ \Phi(X(t))\delta). & \quad (6) \end{aligned}$$

Here δ and γ are positive constants representing the maximum birth rate and the maximum death rate, respectively; and the homophily effects are governed by the *pairwise similarity* matrix, $\Phi(X(t))$, such that each term $\Phi(X(t))_{i,j} \in [0,1]$ is a monotonically decreasing function of a suitable seminorm $\|x_j(t) - x_i(t)\|$. We shall assume that $\Phi(X(t))_{i,j} \sim 1$ for $\|x_j(t) - x_i(t)\| < \varepsilon$, and $\Phi(X(t))_{i,j} = 0$ otherwise, for some suitably chosen $\varepsilon > 0$.

There are equilibria at $X = X^*$ with either $A = 0$ or $A = C$ (the full clique). To understand their stability, let us assume that δ and $\gamma \to 0$. Then $A(t)$ evolves very slowly via (6). Let $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N$ be the eigenvalues of Δ. Then it can be shown that X^* is asymptotically stable only if all N matrices, $\mathrm{d}f(x^*) - D\lambda_i$, are simultaneously stability matrices; and conversely, it is unstable in the ith mode of Δ if $\mathrm{d}f(x^*) - D\lambda_i$ has an eigenvalue with positive real part.

Now one can see the possible tension between homophily and the attitude dynamics.

Consider the spectrum of $\mathrm{d}f(x^*) - D\lambda$ as a function of λ. If λ is small then this is dominated by the stability of the uncoupled system, $\mathrm{d}f(x^*)$. If λ is large, then this is again a stability matrix, since D is positive-definite. The situation, dependent on some collusion between choices of D and $\mathrm{d}f(x^*)$, where there is a *window of instability* for an intermediate range of λ, is known as a Turing instability. Note that, as $A(t) \to C$, we have $\lambda_i \to N$, for $i > 1$. So if N lies within the window of instability, we are assured that the systems can never reach a stable consensual fully connected equilibrium. Instead, Turing instabilities can drive the breakup (weakening) of the network into relatively well-connected subnetworks. These in turn may restabilize the equilibrium dynamics (as the eigenvalues leave the window of instability), and then the whole process can begin again as homophily causes any absent edges to reappear. Thus we expect a pseudocyclic emergence and diminution of patterns, representing transient variations in attitudes. In simulations, by projecting the network $A(t)$ onto two dimensions using the Frobenius matrix inner product, one may observe directly the cyclic nature of consensus and division.

Even if the stochastic dynamics in (6) are replaced by deterministic dynamics for a weighted communication adjacency matrix, one obtains a system that exhibits aperiodic, wandering, and also sensitive dependence. In such cases the orbits are chaotic: we know that they will oscillate, but we cannot predict whether any specific individuals will become relatively inhibited or relatively activated within future cycles. This phenomenon even occurs when $N = 2$.

These models show that, when individuals, who are each in a dynamic equilibrium between their activational and inhibitory tendencies, are coupled in a homophilic way, we should expect a relative lack of global social convergence to be the norm. Radical and conservative behaviors can coexist across a population and are in a constant state of flux. While the macroscopic situation is predictable, the journeys for individuals are not, within both deterministic and stochastic versions of the model. There are some commentators in socioeconomic fields who assert that divergent attitudes, beliefs, and social norms require leaders and are imposed on populations; or else they are driven by partial experiences and events. But here we can see that the transient existence of locally clustered subgroups, holding diverse views, can be an emergent behavior within fully coupled systems. This can be the normal state of affairs within societies, even without externalities and forcing terms.

Sociology studies have in the past focused on rather small groups of subjects under experimental conditions. Digital platforms and modern applied mathematics will transform this situation: computation and social science can use vast data sets from very large numbers of users of online platforms (Twitter, Facebook, blogs, group discussions, multiplayer online games) to analyze how norms, opinions, emotions, and collective action emerge and spread through local interactions.

5 Networks on Security and Defense

"It takes a network to defeat a network" is the mantra expressed by the most senior U.S. command in Afghanistan and Iraq. This might equally be said of the threats posed by terrorists, or in post-conflict peacekeeping (theaters of asymmetric warfare), and even by the recent summer riots and looting within U.K. cities. But what type of networks must be defeated, and what type of network thinking will be required?

So far we have discussed peer-to-peer networks in general terms. But we are faced with some specific challenges that stress the importance of social and communications networks in enabling terrorist threats:

- the analysis of very large communications networks, in real time;
- the identification of influential individuals;
- inferring how such networks *should* evolve in the future (and thus spotting aberrant behavior); and
- recognizing that fully coupled systems may naturally lead to diverse views, and pattern formation.

All of these things become ever more essential. Population-wide data from digital platforms requires efficient and effective applicable mathematics.

Modern adversaries may be most likely to be

- organized through an *actor network* of transient affiliations appropriate for (i) time-limited opportunities and *trophy* or *inspired* goals; (ii) procurement, intelligence, reconnaissance and planning; and (iii) empowering individuals and encouraging both innovation and replication through competition;
- employing an operational digital *communication network* that enables and empowers action while maximizing agility (self-adaptation and reducing the time to act) through the flow of information, ideas, and innovations; and
- reliant upon a third-party *dissemination network* within the public and media space (social media, broadcast media, and so forth) so as to maximize the impact of their actions.

There are thus at least three independent networks operating on the side of those who would threaten security.

Here we have set out a framework for analyzing the form and dynamics of large evolving peer-to-peer communications networks. It seems likely that the challenge of modeling their behavior may lead us to develop new models and methods in the future.

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VII.6 Chip Design

Stephan Held, Stefan Hougardy, and Jens Vygen

1 Introduction

An *integrated circuit* or *chip* contains a collection of electronic circuits—composed of *transistors*—that are connected by wires to fulfill some desired functionality. The first integrated circuit was built in 1958 by Jack Kilby. It contained a single transistor. As predicted by Gordon Moore in 1965, the number of transistors per chip doubles roughly every two years. The process of creating chips soon became known as *very-large-scale integration* (VLSI). In 2014 the most complex chips contain billions of transistors on a few square centimeters.

In this article we concentrate on the design of digital logic chips. Analog integrated circuits have many fewer transistors and more complex design rules and are therefore still largely designed manually. In a memory chip, the transistors are packed in a very regular structure, which makes their design rather easy. In contrast, the design of VLSI digital logic chips is impossible without advanced mathematics.

New technological challenges, exponentially increasing transistor counts, and shifting objectives like decreased power consumption or increased yield constantly create new and challenging mathematical problems. This has made chip design one of the most interesting application areas for mathematics during the last forty years, and we expect this to continue to be the case for at least the next two decades, although technology scaling might slow down at some point.

1.1 Hierarchical Chip Design

Due to its enormous complexity, the design of VLSI chips is usually done hierarchically. A hierarchical design makes it possible to distribute the design task to different teams. Moreover, it can reduce the overall effort, and it makes the design process more predictable and more manageable.

For hierarchical design, a chip is subdivided into logical units, each of which may be subdivided into several levels of smaller units. An obvious advantage of hierarchical design is that components that are used multiple times need to be designed only once. In particular, almost all chips are designed based on a *library* of so-called *books*, predesigned integrated circuits that realize simple logical functions such as AND or NOT or a simple memory element. A chip often contains many instances of the same book; these instances are often called *circuits*.

The books are composed of relatively few transistors and are predesigned at an early stage. For their design one needs to work at the *transistor level* and hence follow more complicated rules. Once a book (or any hierarchical unit) is designed, the properties it has that are relevant for the design of the next higher level (e.g., minimum-distance constraints, timing behavior, power consumption) are computed and stored. Most books are designed so that they have a rectangular shape and the same height, making it easier to place them in rows or columns.

1.2 The Chip-Design Process

The first step in chip design is the specification of the desired functionality and the technology that will be used. In *logic design*, this functionality is made precise using some hardware description language. This hardware description is converted into a *netlist* that specifies which circuits have to be used and how they have to be connected to achieve the required functionality.

The *physical-design* step takes this netlist as input and outputs the physical location of each circuit and each wire on the chip. It will also change the netlist (in a logically equivalent way) in order to meet timing constraints.

Before fabricating the chip (or fixing a hierarchical unit for later use on the next level up), one conducts *physical verification* to confirm that the physical layout meets all constraints and implements the desired