provided an indication of both the plate separation distance and how much antibody and labeled antigen were required to be affixed to the plate surfaces. In short, the tool obviated a great deal of experimentation, thus saving time and allowing Unilever Research, through a company that was then called Unimed, to bring the product to market early.

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VII.19 Airport Baggage Screening with X-Ray Tomography

W. R. B. Lionheart

1 The Security Screening Problem

In airport security, baggage carried in an aircraft's hold needs to be screened for explosive devices. Traditionally, an X-ray machine is used that gives a single two-dimensional projection of the X-ray attenuation of the contents of the bag. In some systems, several views are used to reveal threats that may be obscured by large dense objects. An extension of this idea is to use a large number of projections to reconstruct a three-dimensional volume image of the luggage. This is the X-ray computed tomography (CT) technology that is familiar from medical applications (see MEDICAL IMAGING [VII.9]). The image can then be viewed by an operator from any desired angle and threat-detection software can be used that, for example, segments the volume image, identifying objects that have a similar X-ray attenuation to explosives. In some airports, a two-stage system is used in which bags that cannot be cleared by an automatic system analyzing a two-dimensional projection are then passed to a much slower X-ray tomography system.

Airport baggage handling systems operate using conveyor belts traveling at around 0.5 m/s. Medical CT machines use a gantry supporting the X-ray source and an array of detectors that rotates in a horizontal plane while the patient is translated in the direction of the rotation axis. Relative to the patient, the source describes a helical trajectory. By contrast, small laboratory CT machines rotate and translate the sample while the source and detector remain fixed. Neither of these is practical for scanning luggage at the desired speed: the mass of the gantry is too great to rotate fast enough and rotating the bag would displace the contents.

2 Real-Time Tomography

The company Rapiscan Systems has developed a system called real-time tomography (RTT) that uses multiple X-ray sources fixed in a circular configuration that can be switched electronically, removing the need for a rotating gantry. A cylindrical array of detectors is used, and this is coaxial with the sources, but the X-rays cannot penetrate the detectors, so the detectors are offset relative to the sources (see figure 1).

A reasonable mathematical model of X-ray tomography is that the line integrals of the linear attenuation are measured for all lines joining sources to detectors. For a helical source trajectory and detectors covering a set called the *Tam–Danielson window*, there is an exact reconstruction algorithm due to Katsevich expressed in terms of derivatives and integral operators applied to the data. Most medical and industrial CT machines use an approximation of this using overdetermined data.

The RTT presents several mathematical challenges.

- The data is incomplete, resulting in an ill-posed INVERSE PROBLEM [IV.15] to solve.
- The reconstruction must be completed quickly to ensure the desired throughput of luggage.
- The sources can be fired in almost any order. In fact, sequential firing that approximates a rotating gantry and a single-threaded helix trajectory is the most difficult due to heat dissipation issues in clusters of sources. What is the optimal firing order?

3 Sampling Data and Sufficiency

As the RTT was originally conceived as a fast helical scan machine, it is natural to think of the sources firing sequentially at equal time intervals as a discrete approximation to a curve. A firing sequence in which a fixed number of sources is skipped at each time step approximates a multithreaded helix. Figure 2 illustrates

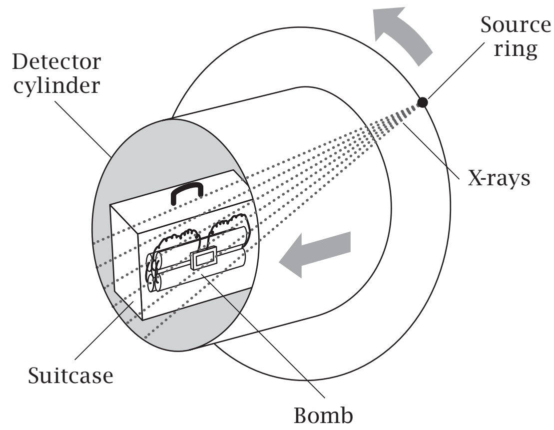

Figure 1 A cartoon of an RTT scanner showing the source circle and detector cylinder.

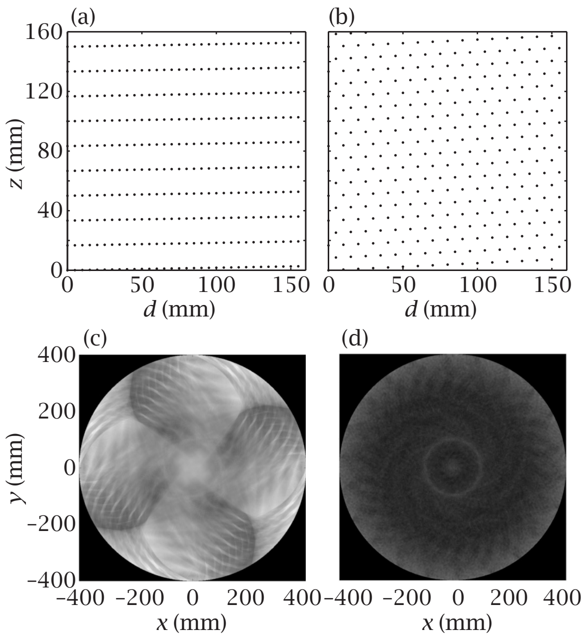

Figure 2 (a), (b) Two multithreaded helix source firing sequences; (b) a lattice on the cylinder of source positions. (c), (d) The number of rays intersecting each voxel for the firing orders above. The second sequence ((b), (d)) has more uniform coverage of the cylinder and produces a more uniform distribution of rays.

two possible firing orders, the first approximating a multithreaded helix. The second could be interpreted as a multithreaded helix (with two different pitches), but it is more natural to view it as a triangular lattice that samples the two-dimensional surface of a cylinder rather than a curve.

The manifold of lines in three-dimensional space is four dimensional and data from the X-ray transform of a function of three variables satisfies a consistency condition called John's ultrahyperbolic equation. In conventional helical CT, the lines through the helix in the Tam–Danielson window are sufficient data to solve the reconstruction problem (and consequently solve the Dirichlet problem for John's equation). We can interpret the RTT data in figure 1 intersected with the detector array as a discrete sampling scheme for an open subset of the four-dimensional space of lines. Using the Fourier slice theorem for the Radon plane transform along with the Payley–Wiener theorem, we can see that for continuum data the inverse problem has a unique solution, but an inversion using this method would be highly unstable. In a practical problem with a discrete sampling scheme and noisy data, we would expect to need some regularization for a numerically stable solution.

4 Inversion

The inversion of the RTT data can be considered as the solution of a sparse linear system of equations relating the attenuation coefficients in voxels within the region of interest to the measured data. As such we can employ the standard techniques of numerical linear algebra and linear inverse problems, including iterative solution methods and regularization. The matrix of the system we solve is generally too large for direct solution, but one method that works well for this problem is conjugate gradient least squares applied to the generalized Tikhonov regularized system.

This approach allows us to apply a systematic choice of regularization penalty, equivalent to an assumption about the covariance of the prior distribution in Bayesian terms. The matrix of the system we solve is generally too large for direct solution with the computing hardware available, but looking at scaled-down systems we have found that the condition number of the (unregularized) matrix is smallest for firing orders that uniformly sample space. Another way to assess the merits of a firing order is to look at the distribution of directions of rays intersecting each voxel, and this criterion leads to the same conclusion as the studies of condition number (see figure 2). The resulting reconstruction results also demonstrate an improvement over the lattice source firing.

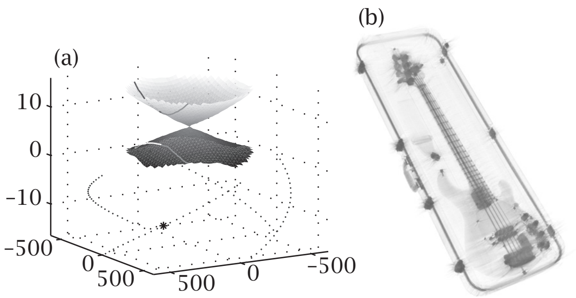

Figure 3 (a) Optimal two-sheet surface with an exaggerated axial scale. (b) Reconstruction of a bass guitar from data collected using a prototype RTT80 baggage scanner.

Even using contemporary graphics processing units, iterative methods are too slow to solve the reconstruction problem in real time. Rebinning methods interpolate data from a (multithreaded) helix to approximate data taken on a plane. Radon transform inversion on the planes can be performed very quickly using filtered back projection. A variation of this method is surface rebinning. For each source location λ the image values on a surface $z = \zeta_\lambda(x, y)$ (where z is the axial direction) are approximated using two-dimensional Radon transform inversion of the data corresponding to rays close to that surface (see figure 3). For conventional helical CT, the optimal surface to use is close to a plane, but for the RTT geometry a good solution is to find a surface with two sheets and reconstruct the sum of the attenuation coefficients at points on those sheets with the same (x, y) coordinates. There is then a separate very sparse linear system to solve to find the values of the attenuation coefficient at a voxel, which will depend on values on the upper and lower sheet for different source positions. The optimal surface for a given geometry is conveniently found as a fixed point of a contraction mapping.

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VII.20 Mathematical Economics
Ivar Ekeland

Economic theory is traditionally divided into microeconomics and macroeconomics. Microeconomics deal with individuals, macroeconomics with society. At the present time, microeconomics is seen as foundational, meaning that society is understood as no more than a contract (implicit or explicit) between individuals, so that macroeconomics should be derived from the behavior of individuals, just as all the laws of physics should be derived from the behavior of atoms. Let me hasten to add that we are no closer to this grand unification in economics than we are in physics. Macroeconomics and microeconomics are largely separate fields, with the latter being much more conceptually mature, whereas the basic principles of the former are still under discussion.

1 Individuals

Individuals are seen as utility maximizers. Each of us lives in an environment in which decisions are to be made: choosing a point $x \in A$, say, where A is a closed subset of Euclidean space. My preferences are characterized by a continuous function $U : A \to R$: I prefer x to y if $U(x) > U(y)$, and I am indifferent if $U(x) = U(y)$. Note that the preference relation thus defined is transitive: if I prefer x to y and y to z, then I prefer x to z. As we shall see later on, this transitivity holds only for individuals, not for groups. The utility function characterizes the individual: I have mine; you have yours; they are different. Note that it is taken as given that some of us are selfish, some are not, some of us are drug addicts, some of us prefer guns—it will all show up in our utility. At this point, economic theory is *positive*, meaning that it does not tell people what they should aim for; it tells them how best to reach their goals.

It is assumed that each individual chooses the point that maximizes his or her utility function U over the set of possible choices A. Of course, we need this maximizer to be unique; otherwise we would have to choose between all possible maximizers, and the decision problem would not be solved. In order to achieve uniqueness, the set A is assumed to be convex and the function U concave, one of them strictly so. The admissible set A is usually defined by budgetary constraints. For instance, we may consider an economy