

3 Stellar Physics

In this chapter, we obtain a physical understanding of main-sequence stars and of their properties, as outlined in the previous chapter. The Sun is the nearest and best-studied star, and its properties provide useful standards to which other stars can be compared. For reference, let us summarize the measured parameters of the Sun. The Earth–Sun distance is

$$d_{\odot} = 1.5 \times 10^8 \text{ km} = 1 \text{ AU}. \quad (3.1)$$

The mass of the Sun is

$$M_{\odot} = 2.0 \times 10^{33} \text{ g}. \quad (3.2)$$

The radius of the Sun is

$$r_{\odot} = 7.0 \times 10^{10} \text{ cm}. \quad (3.3)$$

Using the mass and the radius, we can find the mean density of the Sun:

$$\bar{\rho} = \frac{M_{\odot}}{\frac{4}{3}\pi R_{\odot}^3} = \frac{3 \times 2 \times 10^{33} \text{ g}}{4\pi \times (7 \times 10^{10} \text{ cm})^3} = 1.4 \text{ g cm}^{-3}. \quad (3.4)$$

The Sun's mean density is thus not too different from that of liquid water. The bolometric Solar luminosity is

$$L_{\odot} = 3.8 \times 10^{33} \text{ erg s}^{-1}. \quad (3.5)$$

When divided by $4\pi d_{\odot}^2$, this gives the Solar flux above the Earth's atmosphere, sometimes called the solar constant:

$$f_{\odot} = 1.4 \times 10^6 \text{ erg s}^{-1} \text{ cm}^{-2} = 1.4 \text{ kW m}^{-2}. \quad (3.6)$$

The effective surface temperature is

$$T_{E\odot} = 5800 \text{ K}. \quad (3.7)$$

From Wien's law (Eq. 2.16), the typical energy of a solar photon is then 1.4 eV. When the energy flux is divided by this photon energy, the photon flux is

$$f_{\odot, \text{ph}} \approx \frac{1.4 \times 10^6 \text{ erg s}^{-1} \text{ cm}^{-2}}{1.4 \text{ eV} \times 1.6 \times 10^{-12} \text{ erg eV}^{-1}} = 6.3 \times 10^{17} \text{ s}^{-1} \text{ cm}^{-2} \quad (3.8)$$

(where we have converted between energy units using $1 \text{ eV} = 1.6 \times 10^{-12} \text{ erg}$).

From radioactive dating of Solar System bodies, the Sun's age is about

$$t_{\odot} = 4.5 \times 10^9 \text{ yr.} \quad (3.9)$$

Finally, from a solution of the stellar models that we will develop, the central density and temperature of the Sun are

$$\rho_c = 150 \text{ g cm}^{-3}, \quad (3.10)$$

$$T_c = 15 \times 10^6 \text{ K.} \quad (3.11)$$

3.1 Hydrostatic Equilibrium and the Virial Theorem

A star is a sphere of gas that is held together by its self-gravity, and is balanced against collapse by pressure gradients. To see this, let us calculate the **free-fall timescale** of the Sun, i.e., the time it would take to collapse to a point, if there were no pressure support. Consider a mass element dm at rest in the Sun at a radius r_0 . Its potential energy is

$$dU = -\frac{GM(r_0)dm}{r_0}, \quad (3.12)$$

where $M(r_0)$ is the mass interior to r_0 . From conservation of energy, the velocity of the element as it falls toward the center is

$$\frac{1}{2} \left(\frac{dr}{dt} \right)^2 = \frac{GM(r_0)}{r} - \frac{GM(r_0)}{r_0}, \quad (3.13)$$

where we have assumed that the amount of (also-falling) mass interior to r_0 remains constant. Separating the variables and integrating, we find

$$\begin{aligned} \tau_{\text{ff}} &= \int_0^{\tau_{\text{ff}}} dt = - \int_{r_0}^0 \left[2GM(r_0) \left(\frac{1}{r} - \frac{1}{r_0} \right) \right]^{-1/2} dr \\ &= \left(\frac{r_0^3}{2GM(r_0)} \right)^{1/2} \int_0^1 \left(\frac{x}{1-x} \right)^{1/2} dx. \end{aligned} \quad (3.14)$$

The definite integral on the right equals $\pi/2$, and the ratio $M(r_0)/r_0^3$, up to a factor $4\pi/3$, is the mean density $\bar{\rho}$, so

$$\tau_{\text{ff}} = \left(\frac{3\pi}{32G\bar{\rho}} \right)^{1/2}. \quad (3.15)$$

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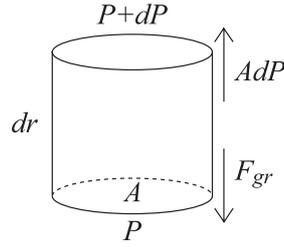


Figure 3.1 Hydrostatic equilibrium. The gravitational force F_{gr} on a mass element of cross-sectional area A is balanced by the force $A dP$ due to the pressure difference between the top and the bottom of the mass element.

For the parameters of the Sun, we obtain

$$\tau_{ff\odot} = \left(\frac{3\pi}{32 \times 6.7 \times 10^{-8} \text{ cgs} \times 1.4 \text{ g cm}^{-3}} \right)^{1/2} = 1800 \text{ s.} \quad (3.16)$$

Thus, without pressure support, the Sun would collapse to a point within half an hour.

This has not happened because the Sun is in **hydrostatic equilibrium**. Consider now a small, cylinder-shaped mass element inside a star, with A the area of the cylinder's base, and dr its height (see Fig. 3.1). If there is a pressure difference dP between the top of the cylinder and its bottom, this will lead to a net force $A dp$ on the mass element, in addition to the force of gravity toward the center. Equilibrium will exist if

$$-\frac{GM(r)dm}{r^2} - A dP = 0. \quad (3.17)$$

But

$$dm = \rho(r)A dr, \quad (3.18)$$

which leads us to the first equation of stellar structure, the **equation of hydrostatic equilibrium**:

$$\boxed{\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}.} \quad (3.19)$$

Naturally, the pressure gradient is negative, because to counteract gravity, the pressure must decrease outward (i.e., with increasing radius).

This simple equation, combined with some thermodynamics, can already provide valuable insight. Let us multiply both sides of Eq. 3.19 by $4\pi r^3 dr$ and integrate from $r = 0$ to r_* , the outer radius of the star:

$$\int_0^{r_*} 4\pi r^3 \frac{dP}{dr} dr = - \int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r}. \quad (3.20)$$

The right-hand side, in the form we have written it, is seen to be the energy that would be gained in constructing the star from the inside out, bringing from infinity shell by shell (each shell with a mass $dM(r) = \rho(r)4\pi r^2 dr$). This is just the gravitational potential self-energy of the star, E_{gr} . On the left-hand side, integration by parts gives

$$[P(r)4\pi r^3]_0^{r_*} - 3 \int_0^{r_*} P(r)4\pi r^2 dr. \quad (3.21)$$

We will define the surface of the star as the radius at which the pressure goes to zero. The first term is therefore zero. The second term is seen to be -3 times the volume-averaged pressure, \bar{P} , up to division by the volume V of the star. Equating the two sides, we obtain

$$\bar{P} = -\frac{1}{3} \frac{E_{\text{gr}}}{V}. \quad (3.22)$$

In words, the mean pressure in a star equals minus one-third of its gravitational energy density. Equation 3.22 is one form of the so-called **virial theorem** for a gravitationally bound system.

To see what Eq. 3.22 implies, consider a star composed of a classical, monoatomic, nonrelativistic, ideal gas of N identical particles.¹ At every point in the star the gas equation of state is

$$PV = NkT, \quad (3.23)$$

and its thermal energy is

$$E_{\text{th}} = \frac{3}{2} NkT. \quad (3.24)$$

Thus,

$$P = \frac{2}{3} \frac{E_{\text{th}}}{V}, \quad (3.25)$$

i.e., the local pressure equals $2/3$ the local thermal energy density. Multiplying both sides by $4\pi r^2$ and integrating over the volume of the star, we find that

$$\bar{P}V = \frac{2}{3} E_{\text{th}}^{\text{tot}}, \quad (3.26)$$

with $E_{\text{th}}^{\text{tot}}$ the total thermal energy of the star. Substituting from Eq. 3.22, we obtain

$$E_{\text{th}}^{\text{tot}} = -\frac{E_{\text{gr}}}{2}, \quad (3.27)$$

¹ *Classical* means that the typical separations between particles are larger than the de Broglie wavelengths of the particles, $\lambda = h/p$, where p is the momentum. *Nonrelativistic* means that the particle velocities obey $v \ll c$. An ideal gas is defined as a gas in which particles experience only short-range (compared to their typical separations) interactions with each other—billiard balls on a pool table are the usual analog.

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which is another form of the virial theorem. Equation 3.22 says that when a star contracts and loses energy, i.e., its gravitational self-energy becomes more negative, then its thermal energy rises. This means that stars have negative heat capacity—their temperatures rise when they lose energy. As we will see, this remarkable fact is at the crux of all of stellar evolution.

A third form of the virial theorem is obtained by considering the total energy of a star, both gravitational and thermal:

$$E_{\text{total}} = E_{\text{th}}^{\text{tot}} + E_{\text{gr}} = -E_{\text{th}}^{\text{tot}} = \frac{E_{\text{gr}}}{2}. \quad (3.28)$$

Thus, the total energy of a star that is composed of a classical, nonrelativistic, ideal gas is negative, meaning the star is bound. (To see what happens in the case of a relativistic gas, solve Problem 1.) Since all stars constantly radiate away their energy (and hence E_{total} becomes more negative), they are doomed to collapse (E_{gr} becomes more negative), eventually. We will see in chapter 4 that an exception to this occurs when the stellar gas moves from the classical to the quantum regime.

We can also use Eq. 3.22 to get an idea of the typical pressure and temperature inside a star, as follows. The right-hand side of Eq. 3.20 permits evaluating E_{gr} for a choice of $\rho(r)$. For example, for a constant density profile, $\rho = \text{const.}$,

$$E_{\text{gr}} = - \int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r} = - \int_0^{r_*} \frac{G \frac{4\pi}{3} r^3 \rho^2 4\pi r^2 dr}{r} = - \frac{3}{5} \frac{GM_*^2}{r_*}. \quad (3.29)$$

A density profile, $\rho(r)$, that falls with radius will give a somewhat more negative value of E_{gr} . Taking a characteristic $E_{\text{gr}} \sim -GM^2/r$, we find that the mean pressure in the Sun is

$$\bar{P}_{\odot} \sim \frac{1}{3} \frac{GM_{\odot}^2}{\frac{4}{3}\pi r_{\odot}^3 r_{\odot}} = \frac{GM_{\odot}^2}{4\pi r_{\odot}^4} \approx 10^{15} \text{ dyne cm}^{-2} = 10^9 \text{ atm}. \quad (3.30)$$

To find a typical temperature, which we will call the **virial temperature**, let us assume again a classical, nonrelativistic ideal gas, with particles of mean mass \bar{m} . Equation 3.27 then applies, and

$$\frac{3}{2} NkT_{\text{vir}} \sim \frac{1}{2} \frac{GM_{\odot}^2}{r_{\odot}} = \frac{1}{2} \frac{GM_{\odot} N\bar{m}}{r_{\odot}}. \quad (3.31)$$

The mass of an electron is negligibly small, only $\approx 1/2000$ compared to the mass of a proton. For an ionized hydrogen gas, consisting of an equal number of protons and electrons, the mean mass \bar{m} ,

$$\bar{m} = \frac{m_e + m_p}{2} = \frac{m_{\text{H}}}{2}, \quad (3.32)$$

is therefore close to one-half the mass of the proton or exactly one-half of the hydrogen atom, $m_{\text{H}} = 1.7 \times 10^{-24}$ g. The typical thermal energy is then

$$kT_{\text{vir}} \sim \frac{GM_{\odot}m_{\text{H}}}{6r_{\odot}} = \frac{6.7 \times 10^{-8} \text{ cgs} \times 2 \times 10^{33} \text{ g} \times 1.7 \times 10^{-24} \text{ g}}{6 \times 7 \times 10^{10} \text{ cm}} \\ = 5.4 \times 10^{-10} \text{ erg} = 0.34 \text{ keV.} \quad (3.33)$$

With $k = 1.4 \times 10^{-16} \text{ erg K}^{-1} = 8.6 \times 10^{-5} \text{ eV K}^{-1}$, this gives a virial temperature of about $4 \times 10^6 \text{ K}$. As we will see, at temperatures of this order of magnitude, nuclear reactions can take place, and thus replenish the thermal energy that the star radiates away, halting the gravitational collapse (if only temporarily).

Of course, in reality, just like $P(r)$, the density $\rho(r)$ and the temperature $T(r)$ are also functions of radius and they grow toward the center of a star. To find them, we need to define additional equations. We will see that the equation of hydrostatic equilibrium is one of four coupled differential equations that determine stellar structure.

3.2 Mass Continuity

In the hydrostatic equilibrium equation (Eq. 3.19), we have $M(r)$ and $\rho(r)$, which are easily related to each other:

$$dM(r) = \rho(r)4\pi r^2 dr, \quad (3.34)$$

or

$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r). \quad (3.35)$

Although this is, in essence, merely the definition of density, in the context of stellar structure this equation is often referred to as the **equation of mass continuity** or the **equation of mass conservation**.

3.3 Radiative Energy Transport

The radial gradient in $P(r)$ that supports a star is produced by a gradient in $\rho(r)$ and $T(r)$. In much of the volume of most stars, $T(r)$ is determined by the rate at which radiative energy flows in and out through every radius, i.e., the luminosity $L(r)$. To find the equation that determines $T(r)$, we need to study some of the basics of *radiative transfer*, the passage of radiation through matter. In some of the volume of some stars, the energy transport mechanism that dominates is convection, rather than radiative transport. We discuss convection in section 3.12. Energy transport by means of conduction plays a role only in dense stellar remnants—white dwarfs and neutron stars—which are discussed in chapter 4.

Photons in stars can be absorbed or scattered out of a beam via interactions with molecules, with atoms (either neutral or ions), and with electrons. If a photon traverses a

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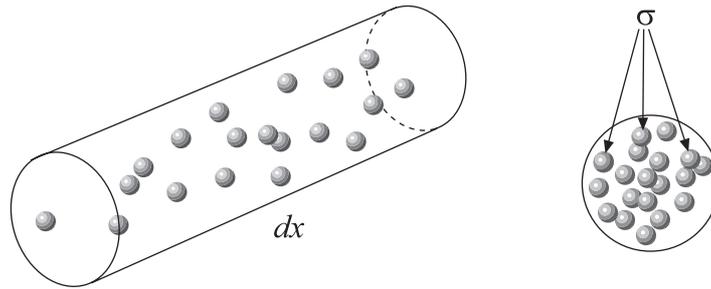


Figure 3.2 A volume element in a field of targets as viewed in perspective (*left*). The target number density is n , and each target presents a cross section σ . From the base of the cylindrical volume, $n dx$ targets per unit area are seen in projection (*right*). A straight line along the length of the volume will therefore intercept, on average, $n\sigma dx$ targets.

path dx filled with “targets” with a number density n (i.e., the number of targets per unit volume), then the projected number of targets per unit area lying in the path of the photon is $n dx$ (see Fig. 3.2). If each target poses an effective **cross section**² σ for absorption or scattering, then the fraction of the area covered by targets is $\sigma n dx$. Thus, the number of targets that will typically be intersected by a straight line traversing the path dx , or, in other words, the number of interactions the photon undergoes, will be

$\# \text{ of interactions} = n\sigma dx. \tag{3.36}$

Equation 3.36 actually defines the concept of cross section. (Cross section can be defined equivalently as the ratio between the interaction rate per target particle and the incoming flux of projectiles.) Setting the left-hand side equal to 1, the typical distance a photon will travel between interactions is called the **mean free path**:

$$l = \frac{1}{n\sigma}. \tag{3.37}$$

More generally, the stellar matter will consist of a variety of absorbers and scatterers, each with its own density n_i and cross section σ_i . Thus,

$$l = \frac{1}{\sum n_i \sigma_i} \equiv \frac{1}{\rho \kappa}, \tag{3.38}$$

where we have used the fact that all the particle densities will be proportional to the mass density ρ , to define the **opacity** κ . The opacity obviously has cgs units of $\text{cm}^2 \text{g}^{-1}$, and will depend on the local density, temperature, and element abundance.

We will return later to the various processes that produce opacity. However, to get an idea of the magnitude of the scattering process, let us consider one of the important interactions—**Thomson scattering** of photons on free electrons (see Fig. 3.3).

² The cross section of a particle, which has units of area, quantifies the degree to which the particle is liable to take part in a particular interaction (e.g., a collision or a reaction) with some other particle.

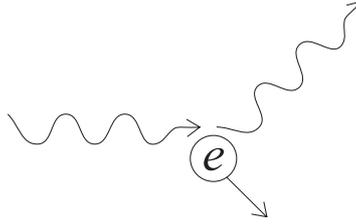


Figure 3.3 Thomson scattering of a photon on a free electron.

The Thomson cross section is

$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2} \right)^2 = 6.7 \times 10^{-25} \text{ cm}^2. \quad (3.39)$$

It is independent of temperature and photon energy.³ In the hot interiors of stars, the gas is fully ionized and therefore free electrons are abundant. If we approximate, for simplicity, that the gas is all hydrogen, then there is one electron per atom of mass m_H , and

$$n_e \approx \frac{\rho}{m_H}. \quad (3.40)$$

The mean free path for electron scattering is then

$$l_{es} = \frac{1}{n_e \sigma_T} \approx \frac{m_H}{\rho \sigma_T} \approx \frac{1.7 \times 10^{-24} \text{ g}}{1.4 \text{ g cm}^{-3} \times 6.7 \times 10^{-25} \text{ cm}^2} \approx 2 \text{ cm}, \quad (3.41)$$

where we have used the mean mass density of the Sun calculated previously. In reality, the density of the Sun is higher than average in regions where electron scattering is the dominant source of opacity, while in other regions other processes, apart from electron scattering, are important. As a result, the actual typical photon mean free path is even smaller, and is $l \approx 1 \text{ mm}$.

Thus, photons can travel only a tiny distance inside the Sun before being scattered or absorbed and reemitted in a new direction. Since the new direction is random, the emergence of photons from the Sun is necessarily a **random walk** process. The vector \mathbf{D} describing the change in position of a photon after N steps, each described by a vector \mathbf{l}_i having length l and random orientation (see Fig. 3.4), is

$$\mathbf{D} = \mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 + \cdots + \mathbf{l}_N. \quad (3.42)$$

The square of the linear distance covered is

$$\mathbf{D}^2 = |\mathbf{l}_1|^2 + |\mathbf{l}_2|^2 + \cdots + |\mathbf{l}_N|^2 + 2(\mathbf{l}_1 \cdot \mathbf{l}_2 + \mathbf{l}_1 \cdot \mathbf{l}_3 + \cdots), \quad (3.43)$$

and its expectation value is

$$\langle \mathbf{D}^2 \rangle = Nl^2. \quad (3.44)$$

³ Note the inverse square dependence of the Thomson cross section on the electron mass. For this reason, protons and nuclei, which are much heavier, are much less effective photon scatterers. Similarly, the relevant mass for electrons bound in atoms is the mass of the entire atom, and hence bound electrons pose a very small Thomson-scattering cross section.

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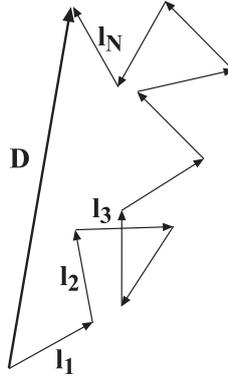


Figure 3.4 The net advance of a photon performing a random walk consisting of N steps, each described by a vector l_i , is the vector D .

The expectation value of the term in parentheses in Eq. 3.43 is zero because it is a sum over many vector dot products, each with a random angle, and hence with both positive and negative cosines contributing equally. The linear distance covered in a random walk is therefore

$$\langle D^2 \rangle^{1/2} = D = \sqrt{N} l. \quad (3.45)$$

To gain some intuition, it is instructive to calculate how long it takes a photon to travel from the center of the Sun, where most of the energy is produced, to the surface.⁴ From Eq. 3.45, traveling a distance r_\odot will require $N = r_\odot^2/l^2$ steps in the random walk. Each step requires a time l/c . Thus, the total time for the photon to emerge from the Sun is

$$\begin{aligned} \tau_{\text{rw}} &\approx \frac{l}{c} \frac{r_\odot^2}{l^2} = \frac{r_\odot^2}{lc} = \frac{(7 \times 10^{10} \text{ cm})^2}{10^{-1} \text{ cm} \times 3 \times 10^{10} \text{ cm s}^{-1}} \\ &= 1.6 \times 10^{12} \text{ s} = 52,000 \text{ yr.} \end{aligned} \quad (3.46)$$

Thus, if the nuclear reactions powering the Sun were to suddenly switch off, we would not notice⁵ anything unusual for 50,000 years.

With this background, we can now derive the equation that relates the temperature profile, $T(r)$, to the flow of radiative energy through a star. The small mean free path of photons inside the Sun and the very numerous scatterings, absorptions, and reemissions every photon undergoes reaffirms that locally, every volume element inside the Sun radiates as a blackbody to a very good approximation. However, there is a net flow of radiation energy outward, meaning there is some small anisotropy (a preferred direction), and implying there is a higher energy density at smaller radii than at larger radii. The net flow

⁴ In reality, of course, it is not the same photon that travels this path. In every interaction, the photon can transfer energy to the particle it scatters on, or distribute its original energy among several photons that emerge from the interaction. Hence, the photon energy is strongly “degraded” during the passage through the Sun.

⁵ In fact, even then, nothing dramatic would happen. As we will see in section 3.9, a slow contraction of the Sun would begin, with a timescale of $\sim 10^7$ yr. Over $\sim 10^5$ yr, the solar radius would shrink only by $\sim 1\%$, which is small but discernible.

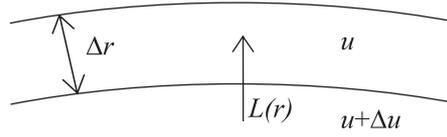


Figure 3.5 Radiative diffusion of energy between volume shells in a star, driven by the gradient in the thermal energy density.

of radiation energy through a mass shell at radius r , per unit time, is just $L(r)$ (see Fig. 3.5). This must equal the excess energy in the shell, compared to a shell at larger radius, divided by the time it takes this excess energy to flow across the shell's width Δr . The excess energy density is Δu , which multiplied by the shell's volume gives the total excess energy of the shell, $4\pi r^2 \Delta r \Delta u$. The time for the photons to cross the shell in a random walk is $(\Delta r)^2/lc$. Thus,

$$L(r) \approx -\frac{4\pi r^2 \Delta r \Delta u}{(\Delta r)^2/lc} = -4\pi r^2 lc \frac{\Delta u}{\Delta r}. \quad (3.47)$$

A more rigorous derivation of this equation adds a factor $1/3$ on the right-hand side, which comes about from an integration of $\cos^2 \theta$ over solid angle (see the derivation of the equation of radiation pressure, Eq. 3.74). Including this factor and replacing the differences with differentials, we obtain

$$\frac{L(r)}{4\pi r^2} = -\frac{cl}{3} \frac{du}{dr}. \quad (3.48)$$

Note that this is, in effect, a **diffusion equation**, describing the outward flow of energy. The left-hand side is the energy flux. On the right-hand side, du/dr is the gradient in energy density, and $-cl/3$ is a diffusion coefficient that sets the proportionality relating the energy flow and the energy density gradient. The opacity, as reflected in the mean free path l , controls the flow of radiation through the star. For low opacity (large l), the flow will be relatively unobstructed, and hence the luminosity will be high, and vice versa.

Since at every radius the energy density is close to that of blackbody radiation (Eq. 2.9), then

$$u = aT^4, \quad (3.49)$$

and

$$\frac{du}{dr} = \frac{du}{dT} \frac{dT}{dr} = 4aT^3 \frac{dT}{dr}. \quad (3.50)$$

Substituting in Eq. 3.48, and expressing l as $(\kappa\rho)^{-1}$, we obtain the **equation of radiative energy transport**,

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$$\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT^3(r)}. \quad (3.51)$$

From Eqs. 3.48 and 3.49, together with an estimate of the mean free path, we can make an order-of-magnitude prediction of the Sun's luminosity. Approximating $-du/dr$ with $\sim u/r_\odot = aT^4/r_\odot$, we have

$$L_\odot \sim 4\pi r_\odot^2 \frac{cl}{3} \frac{aT^4}{r_\odot}. \quad (3.52)$$

Based on the virial theorem and the Sun's mass and radius, we obtained in Eq. 3.33 an estimate of the Sun's internal temperature, $T_{\text{vir}} \sim 4 \times 10^6$ K. Using this as a typical temperature and taking $l = 0.1$ cm, we find

$$\begin{aligned} L_\odot &\sim \frac{4\pi}{3} 7 \times 10^{10} \text{ cm} \times 3 \times 10^{10} \text{ cm s}^{-1} \times 10^{-1} \text{ cm} \times 7.6 \times 10^{-15} \text{ cgs} \times (4 \times 10^6 \text{ K})^4 \\ &= 2 \times 10^{33} \text{ erg s}^{-1}, \end{aligned} \quad (3.53)$$

in reasonable agreement with the observed solar luminosity, $L_\odot = 3.8 \times 10^{33} \text{ erg s}^{-1}$. (We have abbreviated above the units of the radiation constant, a , as cgs.) The above estimate can also be used to argue that, based on its observed luminosity, the Sun must be composed primarily of ionized hydrogen. If the Sun were composed of, say, ionized carbon, the mean particle mass would be $\bar{m} \approx 12m_{\text{H}}/7 \approx 2m_{\text{H}}$, rather than $m_{\text{H}}/2$. Equation 3.33 would then give a virial temperature that is 4 times as high, resulting in a luminosity prediction in Eq. 3.52 that is too high by two orders of magnitude.

3.4 Energy Conservation

We will see that the luminosity of a star is produced by nuclear reactions, with output energies that depend on the local conditions (density and temperature) and hence on r . Let us define $\epsilon(r)$ as the power produced per unit mass of stellar material. Energy conservation means that the addition to a star's luminosity due to the energy production in a thin shell at radius r is

$$dL = \epsilon dm = \epsilon \rho 4\pi r^2 dr, \quad (3.54)$$

or

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r), \quad (3.55)$$

which is the **equation of energy conservation**.

3.5 The Equations of Stellar Structure

We have derived four coupled first-order differential equations describing stellar structure. Let us rewrite them here:

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}, \quad (3.56)$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r), \quad (3.57)$$

$$\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT(r)^3}, \quad (3.58)$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r). \quad (3.59)$$

We can define four boundary conditions for these equations, for example,

$$M(r = 0) = 0, \quad (3.60)$$

$$L(r = 0) = 0, \quad (3.61)$$

$$P(r = r_*) = 0, \quad (3.62)$$

$$M(r = r_*) = M_*, \quad (3.63)$$

where M_* is the total mass of the star. (In reality, at the radius r_* of the photosphere of the star, P does not really go completely to zero, nor do T and ρ , and more sophisticated boundary conditions are required, which account for the processes in the photosphere.)

To these four differential equations we need to add three equations connecting the pressure, the opacity, and the energy production rate of the gas with its density, temperature, and composition:

$$P = P(\rho, T, \text{composition}), \quad (3.64)$$

$$\kappa = \kappa(\rho, T, \text{composition}), \quad (3.65)$$

$$\epsilon = \epsilon(\rho, T, \text{composition}). \quad (3.66)$$

The function $P(\rho, T)$ is usually called **the equation of state**. Each of these three functions will depend on the composition through the **element abundances** and the ionization states of each element in the gas. It is common in astronomy to parametrize the mass abundances of hydrogen, helium, and all the heavier elements (the latter are often referred to collectively by the term “metals”) as

$$X \equiv \frac{\rho_{\text{H}}}{\rho}, \quad Y \equiv \frac{\rho_{\text{He}}}{\rho}, \quad Z \equiv \frac{\rho_{\text{metals}}}{\rho}. \quad (3.67)$$

We have thus ended up with seven coupled equations defining the seven unknown functions: $P(r)$, $M(r)$, $\rho(r)$, $T(r)$, $\kappa(r)$, $L(r)$, and $\epsilon(r)$. As there are four boundary conditions

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for the four first-order differential equations, if there is a solution, it is unique. This is usually expressed in the form of the *Vogt–Russell conjecture*, which states that the properties and evolution of an isolated star are fully determined by its initial mass and its chemical abundances. These determine the star’s observable parameters: its surface temperature, radius, and luminosity. Two variables that we have neglected in this treatment, and that have minor influence on stellar structure, are stellar rotation and magnetic fields. To proceed, we need to define the three functions, P , κ , and ϵ .

3.6 The Equation of State

Different equations of state $P(\rho, T, X, Y, Z)$ apply for different ranges of gas density, temperature, and abundance. Under the conditions in most normal stars, the equation of state of a classical, nonrelativistic, ideal gas, provides a good description. Consider, for example, such a gas, composed of three different kinds of particles, each with its own mass m_i and density n_i . The mean particle mass will be

$$\bar{m} = \frac{n_1 m_1 + n_2 m_2 + n_3 m_3}{n_1 + n_2 + n_3} = \frac{\rho}{n}. \quad (3.68)$$

The gas pressure will then be

$$P_g = nkT = \frac{\rho}{\bar{m}} kT. \quad (3.69)$$

The mean mass will depend on the chemical abundance and ionization state of the gas. As we have already seen, for completely ionized pure hydrogen,

$$\bar{m} = \frac{m_H}{2}, \quad (3.70)$$

and therefore $\bar{m}/m_H = 0.5$.

More generally, the number densities of hydrogen, helium, or an element of atomic mass number A (i.e., an element with a total of A protons and neutrons in each atomic nucleus) will be

$$n_H = \frac{X\rho}{m_H}, \quad n_{\text{He}} = \frac{Y\rho}{4m_H}, \quad n_A = \frac{Z_A\rho}{Am_H}, \quad (3.71)$$

where Z_A is the mass abundance of an element of atomic mass number A . Complete ionization of hydrogen results in two particles (an electron and a proton); of helium, three particles (two electrons and a nucleus); and of an atom with atomic number Z (i.e., with Z protons or electrons), $Z + 1$ particles, which for heavy enough atoms is always close to $A/2$. Thus, for an ionized gas we will have

$$\begin{aligned} n &= 2n_H + 3n_{\text{He}} + \sum \frac{A}{2} n_A = \frac{\rho}{m_H} \left(2X + \frac{3}{4}Y + \frac{1}{2}Z \right) \\ &= \frac{\rho}{2m_H} \left(3X + \frac{Y}{2} + 1 \right), \end{aligned} \quad (3.72)$$

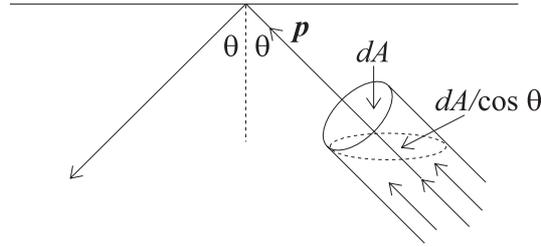


Figure 3.6 Calculation of radiation pressure. A beam of photons with blackbody intensity B strikes the wall of a container at an angle θ to its perpendicular. The projected area of the beam, dA , is increased by $1/\cos \theta$, and therefore the power reaching the wall per unit area is decreased to $B \cos \theta$. Since a photon's momentum, p , is its energy divided by c , the momentum flux in the beam is B/c . Every reflection of a photon transfers twice its perpendicular component of momentum, and therefore the momentum transfer per unit time and per unit area, i.e., the pressure, is $2B \cos^2 \theta/c$. The total pressure is obtained by integrating over all angles of the beams that approach the wall.

where we have used the fact that $X + Y + Z = 1$. Thus,

$$\frac{\bar{m}}{m_{\text{H}}} = \frac{\rho}{nm_{\text{H}}} = \frac{2}{1 + 3X + 0.5Y} \quad (3.73)$$

for a totally ionized gas. For solar abundances, $X = 0.71$, $Y = 0.27$, $Z = 0.02$, and therefore $\bar{m}/m_{\text{H}} = 0.61$. In the central regions of the Sun, about half of the hydrogen has already been converted into helium by nuclear reactions, and as a result $X = 0.34$, $Y = 0.64$, and $Z = 0.02$, giving $\bar{m}/m_{\text{H}} = 0.85$.

In addition to the kinetic gas pressure, the photons in a star exert **radiation pressure**. Let us digress briefly, and derive the equation of state for this kind of pressure. Consider photons inside a blackbody radiator with an intensity given by the Planck function, $I_{\nu} = B_{\nu}$, which, when integrated over wavelength, we denoted as B . As illustrated in Fig. 3.6, the energy arriving at the surface of the radiator per unit time, per unit area, at some angle θ to the perpendicular to the surface, is $B \cos \theta$, because the area of the beam, when projected onto the wall of the radiator, is increased by $1/\cos \theta$. Now, consider the photons in the beam, which strike the fully reflective surface of the radiator at the angle θ . Every photon of energy E has momentum $p = E/c$. When reflected, it transmits to the surface a momentum $\Delta p = (2E/c) \cos \theta$. Therefore, there is a second factor of $\cos \theta$ that must be applied to the incoming beam. The rate of momentum transfer per unit area will be obtained by integrating over all angles at which the photons hit the surface. But, the rate of momentum transfer (i.e., the force) per unit area is, by definition, the pressure. Thus,

$$P = \frac{F}{A} = \frac{dp/dt}{A} = \frac{2}{c} \int_{\pi/2}^{\pi} B \cos^2 \theta \sin \theta d\theta \int_0^{2\pi} d\phi = \frac{4\pi}{3c} B = \frac{1}{3} u, \quad (3.74)$$

where in the last equality we have used the previously found relation (Eq. 2.4) between intensity and energy density. Note that the derivation above applies not only to photons,

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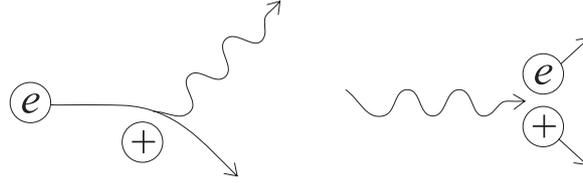


Figure 3.7 *Left:* A free electron accelerating in the Coulomb potential of an ion emits bremsstrahlung, or “free–free” radiation. *Right:* In the inverse process of free–free absorption, a photon is absorbed by a free electron. The process is possible only if a neighboring ion, which can share some of the photon’s momentum, is present.

but to any particles with an isotropic velocity distribution, and with kinetic energies large compared to their rest-mass energies, so that the relation $p \approx E/c$ (which is exact for photons) is a good approximation. Thus, the equation of state in which the radiation pressure equals one-third of the thermal energy density holds for any ultrarelativistic gas.

Returning to the case of the pressure due to a thermal photon gas inside a star, we can write

$$P_{\text{rad}} = \frac{1}{3}u = \frac{1}{3}aT^4, \quad (3.75)$$

which under some circumstances can become important or dominant (see Problems 2 and 3). The full equation of state for normal stars will therefore be

$$P = P_g + P_{\text{rad}} = \frac{\rho kT}{\bar{m}} + \frac{1}{3}aT^4. \quad (3.76)$$

We will see in chapter 4 that the conditions in white dwarfs and in neutron stars dictate equations of state that are very different from this form.

3.7 Opacity

Like the equation of state, the opacity, κ , at every radius in the star will depend on the density, the temperature, and the chemical composition at that radius. We have already mentioned one important source of opacity: Thomson scattering of photons off free electrons. Let us now calculate correctly the electron density for an ionized gas of arbitrary abundance (rather than pure hydrogen, as before):

$$n_e = n_{\text{H}} + 2n_{\text{He}} + \sum \frac{A}{2}n_A = \frac{\rho}{m_{\text{H}}} \left(X + \frac{2}{4}Y + \frac{1}{2}Z \right) = \frac{\rho}{2m_{\text{H}}} (1 + X), \quad (3.77)$$

where we have again assumed that the number of electrons in an atom of mass number A is $A/2$. Therefore,

$$\kappa_{\text{es}} = \frac{n_e \sigma_{\text{T}}}{\rho} = \frac{\sigma_{\text{T}}}{2m_{\text{H}}} (1 + X) = (1 + X) 0.2 \text{ cm}^2 \text{ g}^{-1}. \quad (3.78)$$

In regions of a star with relatively low temperatures, such that some or all of the electrons are still bound to their atoms, three additional processes that are important sources of opacity are **bound–bound**, **bound–free** (also called **photoionization**), and **free–free** absorption. In bound–bound and bound–free absorption, which we have already discussed in the context of photospheric absorption features, an atom or ion is excited to a higher energy level, or ionized to a higher degree of ionization, by absorbing a photon. Free–free absorption is the inverse process of free–free emission, often called **bremstrahlung** (“braking radiation” in German). In free–free emission (see Fig. 3.7), a free electron is accelerated by the electric potential of an ion, and as a result radiates. Thus, in free–free absorption, a photon is absorbed by a free electron and an ion, which share the photon’s momentum and energy. All three processes depend on photon wavelength, in addition to gas temperature, density, and composition.

When averaged over all wavelengths, the *mean opacity* due to both bound–free absorption and free–free absorption behaves approximately as

$$\bar{\kappa}_{\text{bf,ff}} \sim \frac{\rho}{T^{3.5}}, \quad (3.79)$$

which is called a Kramers opacity law. This behavior holds only over limited ranges in temperature and density. For example, free–free absorption actually *increases* with temperature at low temperature and density, with the increase in free electron density. Similarly, bound–free opacity cuts off at high temperatures at which the atoms are fully ionized. Additional sources of opacity, significant especially in low-mass stars, are molecules and H^- ions.⁶

3.8 Scaling Relations on the Main Sequence

From the equations we have derived so far, we can already deduce and understand the observed functional forms of the mass–luminosity relation, $L \sim M^\alpha$, and the effective-temperature–luminosity relation, $L \sim T_{\text{E}}^8$, that are observed for main-sequence stars. Let us assume, for simplicity, that the functions $P(r)$, $M(r)$, $\rho(r)$, and $T(r)$ are roughly power laws, i.e., $P(r) \sim r^\beta$, $M(r) \sim r^\gamma$, etc. If so, we can immediately write the first three differential equations (Eqs. 3.56, 3.57, and 3.58) as scaling relations,

$$P \sim \frac{M\rho}{r}, \quad (3.80)$$

$$M \sim r^3\rho, \quad (3.81)$$

and

$$L \sim \frac{T^4 r}{\kappa\rho} \quad (3.82)$$

⁶ The negative H^- ion forms when a second electron attaches (with a quite weak bond) to a hydrogen atom.

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(just as, instead of solving a differential equation, say, $df/dx = x^4$, we can write directly $f \sim x^5$). For moderately massive stars, the pressure will be dominated by the kinetic gas pressure, and the opacity by electron scattering. Therefore,

$$P \sim \rho T \quad (3.83)$$

and (Eq. 3.78)

$$\kappa = \text{const.} \quad (3.84)$$

Equating 3.80 and 3.83, we find

$$T \sim \frac{M}{r} \quad (3.85)$$

(which is basically just the virial theorem again, for a nonrelativistic, classical, ideal gas—Eq. 3.27). Substituting this into 3.82, and using 3.81 to express $r^3\rho$, we find

$$L \sim M^3, \quad (3.86)$$

as observed for main-sequence stars more massive than the Sun.

Equation 3.85 also suggests that $r \sim M$ on the main sequence. To see this, consider a star that is forming from a mass M that is contracting under its own gravity and heating up (star formation is discussed in some detail in chapter 5). The contraction will stop, and an equilibrium will be set up, once the density and the temperature in the core are high enough for the onset of nuclear reactions. We will see that the nuclear power density depends mainly on temperature. Thus, for any initial mass, r will stop shrinking when a particular core temperature is reached. Therefore, the internal temperature T is comparable in all main-sequence stars (i.e., it is weakly dependent on mass, and hence approximately constant), and

$$r \sim M. \quad (3.87)$$

Detailed models confirm that the core temperature varies only by a factor ≈ 4 over a range of ~ 100 in mass on the main sequence. With $r \sim M$, we see from Eq. 3.81 that the density of a star decreases as M^{-2} , so that more massive stars will have low density, and low-mass stars will have high density.

Proceeding to low-mass stars, the high density means there is a dominant role for bound-free and free-free opacity,

$$\kappa \sim \frac{\rho}{T^{3.5}}. \quad (3.88)$$

Since $T \sim \text{const.}$, $r \sim M$, and $\rho \sim M^{-2}$, then $\kappa \sim \rho \sim M^{-2}$, and Eq. 3.82 gives

$$L \sim \frac{T^4 r}{\kappa \rho} \sim \frac{r}{\rho^2} \sim M^5, \quad (3.89)$$

as seen in low-mass stars.

For the most massive stars, the low gas density will make radiation pressure dominant in the equation of state (see Problem 3),

$$P \sim T^4, \quad (3.90)$$

and electron scattering, with $\kappa = \text{const.}$, will again be the main source of opacity. Equating with 3.80 and substituting for T^4 in Eq. 3.82, we find

$$L \sim M, \quad (3.91)$$

a flattening of the mass–luminosity relationship that is, in fact, observed for the most massive stars.

Finally, we can also reproduce the functional dependence of the main sequence in the H-R diagram. We saw that $L \sim M^5$ for low-mass stars and $L \sim M^3$ for moderately massive stars. Let us then take an intermediate slope, $L \sim M^4$, as representative. Since $r \sim M$, then

$$\sigma T_E^4 = \frac{L}{4\pi r_*^2} \sim \frac{M^4}{M^2} \sim M^2 \sim L^{1/2}, \quad (3.92)$$

so

$$L \sim T_E^8, \quad (3.93)$$

as observed.

We have thus seen that the mass vs. luminosity relation and the surface-temperature vs. luminosity relation of main-sequence stars are simply consequences of the different sources of pressure and opacity in stars of different masses, and of the fact that the onset of nuclear hydrogen burning keeps the core temperatures of all main-sequence stars in a narrow range. The latter fact is elucidated below.

3.9 Nuclear Energy Production

The last function we still need to describe is the power density $\epsilon(\rho, T, X, Y, Z)$. To see that the energy source behind ϵ must be nuclear burning, we consider the alternatives. Suppose that the source of the Sun's energy were gravitational, i.e., that the Sun had radiated until now the potential energy liberated by contracting from infinity to its present radius. From the virial theorem, we saw that the thermal energy resulting from such a contraction is minus one-half the gravitational energy,

$$E_{\text{gr}} = -2E_{\text{th}}. \quad (3.94)$$

Therefore, the other half of the gravitational energy released by the contraction, and which the Sun could have radiated, is

$$E_{\text{rad}} \sim \frac{1}{2} \frac{GM_{\odot}^2}{r_{\odot}}. \quad (3.95)$$

To see how long the Sun could have shined at its present luminosity with this energy source, we divide this energy by the solar luminosity. This gives the so-called **Kelvin–Helmholtz timescale**,

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$$\begin{aligned}\tau_{\text{kh}} &\sim \frac{1}{2} \frac{GM_{\odot}^2}{r_{\odot}} \frac{1}{L_{\odot}} = \frac{6.7 \times 10^{-8} \text{ cgs} \times (2 \times 10^{33} \text{ g})^2}{2 \times 7 \times 10^{10} \text{ cm} \times 3.8 \times 10^{33} \text{ erg s}^{-1}} \\ &= 5 \times 10^{14} \text{ s} = 1.6 \times 10^7 \text{ yr.}\end{aligned}\quad (3.96)$$

The geological record shows that the Earth and Moon have existed for over 4 billion years, and that the Sun has been shining with about the same luminosity during all of this period. A similar calculation shows that chemical reactions (e.g., if the Sun were producing energy by combining hydrogen and oxygen into water) are also not viable for producing the solar luminosity for so long.

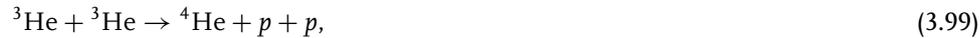
A viable energy source for the Sun and other main-sequence stars is nuclear fusion of hydrogen into helium. Most of the nuclear energy of the Sun comes from a chain of reactions called the **p-p chain**. The first step is the reaction



where d designates a deuteron, composed of a proton and a neutron. As we will see in section 3.10, the timescale for this process inside the Sun is 10^{10} yr. The timescale is so long mainly because the reaction proceeds via the weak interaction (as is evidenced by the emission of a neutrino). The positron, the deuteron, and the neutrino share an energy of 0.425 MeV. Once the reaction occurs, the positron quickly annihilates with an electron, producing two 0.511-MeV γ -ray photons. The neutrino, having a weak interaction with matter, escapes the Sun and carries off its energy, which has a mean of 0.26 MeV. The remaining kinetic energy and photons quickly thermalize by means of frequent matter–matter and matter–photon collisions. Typically, within 1 s, the deuteron will merge with another proton to form ${}^3\text{He}$:



with a total energy release (kinetic + the γ -ray photon) of 5.49 MeV. Finally, on a timescale of 300,000 years, we have



with a kinetic energy release of 12.86 MeV. Every time this three-step chain occurs twice, four protons are converted into a ${}^4\text{He}$ nucleus, two neutrinos, photons, and kinetic energy. The total energy released per ${}^4\text{He}$ nucleus is thus

$$(4 \times 0.511 + 2 \times 0.425 + 2 \times 5.49 + 12.86) \text{ MeV} = 26.73 \text{ MeV.} \quad (3.100)$$

Deducting the 2×0.511 MeV from the annihilation of two preexisting electrons, we find that this is just the rest-mass difference between four free protons and a ${}^4\text{He}$ nucleus:

$$[m(4p) - m({}^4\text{He})]c^2 = 25.71 \text{ MeV} = 0.7\% m(4p)c^2. \quad (3.101)$$

Thus, the rest-mass-to-energy conversion efficiency of the p-p chain is 0.7%. The time for the Sun to radiate away just 10% of the energy available from this source is

$$\begin{aligned}
 \tau_{\text{nuc}} &= \frac{0.1 \times 0.007 \times M_{\odot} c^2}{L_{\odot}} \\
 &= \frac{0.1 \times 0.007 \times 2 \times 10^{33} \text{ g} \times (3 \times 10^{10} \text{ cm s}^{-1})^2}{3.8 \times 10^{33} \text{ erg s}^{-1}} \\
 &= 3.3 \times 10^{17} \text{ s} = 10^{10} \text{ yr.}
 \end{aligned} \tag{3.102}$$

In other words, in terms of energy budget, hydrogen fusion can easily produce the solar luminosity over the age of the Solar System.

Next, we need to see whether the conditions in the Sun are suitable for these reactions to actually take place. Consider two nuclei with atomic numbers (i.e., number of protons per nucleus) Z_A and Z_B . The strong interaction produces a short-range attractive force between the nuclei on scales smaller than

$$r_0 \approx 1.4 \times 10^{-13} \text{ cm.} \tag{3.103}$$

The strong interaction goes to zero at larger distances, and the Coulomb repulsion between the nuclei takes over. The Coulomb energy barrier is

$$E_{\text{coul}} = \frac{Z_A Z_B e^2}{r}, \tag{3.104}$$

which at r_0 is of order

$$E_{\text{coul}}(r_0) \approx Z_A Z_B \text{ MeV.} \tag{3.105}$$

Figure 3.8 shows schematically the combined nuclear (strong) and electrostatic (Coulomb) potential. In the reference frame of one of the nuclei, the other nucleus, with kinetic energy E , can classically approach only to a distance

$$r_1 = \frac{Z_A Z_B e^2}{E}, \tag{3.106}$$

where it will be repelled away. At a typical internal stellar temperature of 10^7 K, the kinetic energy of a nucleus is $1.5kT \sim 1$ keV. The characteristic kinetic energy is thus of order 10^{-3} of the energy required to overcome the Coulomb barrier. Typical nuclei will approach each other only to a separation $r_1 \sim 10^{-10}$ cm, 1000 times larger than the distance at which the strong nuclear binding force operates. Perhaps those nuclei that are in the high-energy tail of the Maxwell–Boltzmann distribution can overcome the barrier? The fraction of nuclei with such energies is

$$e^{-E/kT} \approx e^{-1000} \approx 10^{-434}. \tag{3.107}$$

The number of protons in the Sun is

$$N_p \approx \frac{M_{\odot}}{m_{\text{H}}} = \frac{2 \times 10^{33} \text{ g}}{1.7 \times 10^{-24} \text{ g}} \approx 10^{57}. \tag{3.108}$$

Thus, there is not a single nucleus in the Sun (or, for that matter, in all the stars in the observable Universe) with the kinetic energy required classically to overcome the Coulomb barrier and undergo nuclear fusion with another nucleus.

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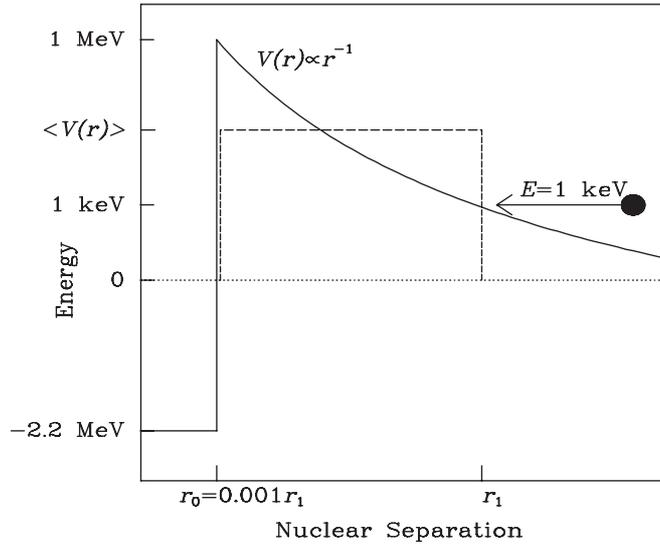


Figure 3.8 Schematic illustration of the potential energy $V(r)$ between two nuclei as a function of separation r . For two protons, the Coulomb repulsion reaches a maximum, with $V(r) \sim 1$ MeV at $r = r_0$, at which point the short-range nuclear force sets in and binds the nuclei (negative potential energy). Two nuclei with relative kinetic energy of ~ 1 keV, typical for the temperatures in stellar interiors, can classically approach each other only to within a separation r_1 , 1000 times greater than r_0 . The dashed rectangle is a *rectangular barrier* of height $\langle V(r) \rangle \approx 3E/2$, which we use to approximate the Coulomb barrier in our calculation of the probability for quantum tunneling through the potential.

Fortunately, **quantum tunneling** through the barrier allows nuclear reactions to take place after all. To see this, let us describe this two-body problem by means of the time-independent Schrödinger equation, for a wave function Ψ in a spherically symmetric potential $V(r)$:

$$\frac{\hbar^2}{2\mu} \nabla^2 \Psi = [V(r) - E] \Psi, \quad (3.109)$$

where the reduced mass of the two nuclei, of masses m_A and m_B , is

$$\mu \equiv \frac{m_A m_B}{m_A + m_B}. \quad (3.110)$$

In our case, the potential is

$$V(r) = \frac{Z_A Z_B e^2}{r}, \quad (3.111)$$

and E is the kinetic energy. Let us obtain an order-of-magnitude solution to the Schrödinger equation. By our definition of r_1 , the radius of closest classical approach, we have $V(r_1) = E$.

We can then write $V(r) = Er_1/r$, and the mean, volume-averaged height of the potential between r_1 and $r_0 \ll r_1$ is

$$\langle V(r) \rangle = \frac{\int_{r_0}^{r_1} 4\pi r^2 V(r) dr}{\int_{r_0}^{r_1} 4\pi r^2 dr} \approx \frac{3}{2} E. \quad (3.112)$$

Approximating $V(r)$ with a constant function of this height (a “rectangular barrier”), the radial component of the Schrödinger equation becomes

$$\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{d^2(r\Psi)}{dr^2} \approx \frac{E}{2} \Psi, \quad (3.113)$$

which has a solution

$$\Psi = A \frac{e^{\beta r}}{r}, \quad \beta = \frac{\sqrt{\mu E}}{\hbar}. \quad (3.114)$$

(The second independent solution of the equation, with an amplitude that rises with decreasing radius, is unphysical.) The wave function amplitude, squared, is proportional to the probability density for a particle to be at a given location. Multiplying the ratio of the probability densities by the ratio of the volume elements, $4\pi r_0^2 dr$ and $4\pi r_1^2 dr$, thus gives the probability that a nucleus will tunnel from r_1 to within $r_0 \ll r_1$ of the other nucleus:

$$\begin{aligned} \frac{|\Psi(r_0)|^2 r_0^2}{|\Psi(r_1)|^2 r_1^2} &= \frac{e^{2\beta r_0}}{e^{2\beta r_1}} \approx e^{-2\beta r_1} = \exp\left(-\frac{2\sqrt{\mu E}}{\hbar} \frac{Z_A Z_B e^2}{E}\right) \\ &= \exp\left(-\frac{2\sqrt{\mu}}{\hbar} Z_A Z_B e^2 \frac{1}{\sqrt{E}}\right). \end{aligned} \quad (3.115)$$

A full solution of the Schrödinger equation gives the same answer, but with an additional factor $\pi/\sqrt{2}$ in the exponential. If we recall the definition of the fine-structure constant,

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}, \quad (3.116)$$

and define an energy

$$E_G = (\pi \alpha Z_A Z_B)^2 2\mu c^2, \quad (3.117)$$

then the probability of penetrating the Coulomb barrier simplifies to the function

$$g(E) = e^{-\sqrt{E_G/E}}. \quad (3.118)$$

The quantity E_G is called the **Gamow energy** and $g(E)$ is called the **Gamow factor**. For two protons,

$$E_G = \left(\pi \frac{1}{137} \times 1 \times 1\right)^2 2 \frac{1}{2} m_p c^2 \approx 500 \text{ keV}. \quad (3.119)$$

(It is convenient to remember that the rest energy of a proton, $m_p c^2$, is 0.94 GeV.) Thus, for the typical kinetic energy of particles in the Sun’s core, $E \sim 1$ keV, we find

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$g(E) \sim e^{-22} \sim 10^{-10}$. While this probability, for a given pair of protons, is still small, it is considerably larger than the classical probability we found in Eq. 3.107.

3.10 Nuclear Reaction Rates

Even if tunneling occurs, and two nuclei are within the strong force's interaction range, the probability of a nuclear reaction will still depend on a nuclear cross section, which will generally depend inversely on the kinetic energy. Thus, the total cross section for a nuclear reaction involving ingredient nuclei A and B is

$$\sigma_{AB}(E) = \frac{S_0}{E} e^{-\sqrt{E_G/E}}, \quad (3.120)$$

where S_0 is a constant, or a weak function of energy, with units of [area]×[energy]. For a given nuclear reaction, S_0 is generally derived from accelerator experiments, or is calculated theoretically.

The number of reactions per nucleus A as it traverses a distance dx in a field with a density n_B of “target” nuclei B is

$$dN_A = n_B \sigma_{AB} dx. \quad (3.121)$$

If we divide both sides by dt , then the number of reactions per nucleus A per unit time is

$$\frac{dN_A}{dt} = n_B \sigma_{AB} v_{AB}, \quad (3.122)$$

where v_{AB} is the relative velocity between the nuclei.

From Eq. 3.122, we can proceed to find the power density function, $\epsilon(\rho, T, X, Y, Z)$, needed to solve the equations of stellar structure. Multiplying by the density n_A of nuclei A will give the number of reactions per unit time and per unit volume, i.e., the reaction rate per unit volume,

$$\mathcal{R}_{AB} = n_A n_B \sigma_{AB} v_{AB}. \quad (3.123)$$

If every reaction releases an energy Q , multiplying by Q gives the power per unit volume. Dividing by ρ then gives the power per unit mass, rather than per unit volume:

$$\epsilon = n_A n_B \sigma_{AB} v_{AB} Q / \rho. \quad (3.124)$$

Recalling that

$$n_A = \frac{\rho X_A}{A_A m_H}, \quad n_B = \frac{\rho X_B}{A_B m_H}, \quad (3.125)$$

with X_A and X_B symbolizing the mass abundances, and A_A and A_B the atomic mass numbers of the two nucleus types, ϵ can be expressed as

$$\epsilon = \frac{\rho X_A X_B}{m_H^2 A_A A_B} \sigma_{AB} v_{AB} Q. \quad (3.126)$$

In reality, the nuclei in a gas will have a distribution of velocities, so every velocity has some probability of occurring. Hence, ϵ can be obtained by averaging $v_{AB}\sigma_{AB}$ over all velocities, with each velocity weighted by its probability, $P(v_{AB})$:

$$\epsilon = \frac{\rho X_A X_B}{m_H^2 A_A A_B} \langle \sigma_{AB} v_{AB} \rangle Q, \quad (3.127)$$

with

$$\langle \sigma_{AB} v_{AB} \rangle = \int_0^\infty \sigma_{AB} v_{AB} P(v_{AB}) dv_{AB}. \quad (3.128)$$

A classical, nonrelativistic gas will have a distribution of velocities described by the Maxwell–Boltzmann distribution. The *relative* velocities of nuclei A and B will also follow a Maxwell–Boltzmann distribution,

$$P(v)dv = 4\pi \left(\frac{\mu}{2\pi kT} \right)^{3/2} v^2 \exp\left(-\frac{\mu v^2}{2kT}\right) dv, \quad (3.129)$$

but with a mass represented by the reduced mass of the particles, μ . For brevity, we have omitted here the subscript AB from the velocities.

Inserting 3.120 and 3.121 into 3.128, and changing the integration variable from velocity to kinetic energy using $E = \frac{1}{2}\mu v^2$, $dE = \mu v dv$, we obtain

$$\langle \sigma v \rangle = \left(\frac{8}{\pi \mu} \right)^{1/2} \frac{S_0}{(kT)^{3/2}} \int_0^\infty e^{-E/kT} e^{-\sqrt{E_G/E}} dE. \quad (3.130)$$

The integrand in this expression,

$$f(E) = e^{-E/kT} e^{-\sqrt{E_G/E}}, \quad (3.131)$$

is composed of the product of two exponential functions, one (from the Boltzmann distribution) falling with energy, and the other (due to the Gamow factor embodying the Coulomb repulsion) rising with energy. Obviously, $f(E)$ will have a narrow maximum at some energy E_0 , at which most of the reactions take place (see Fig. 3.9). The maximum of $f(E)$ is easily found by taking its derivative and equating to zero. It is at

$$E_0 = \left(\frac{kT}{2} \right)^{2/3} E_G^{1/3}. \quad (3.132)$$

A Taylor expansion of $f(E)$ around E_0 shows $f(E)$ can be approximated by a Gaussian with a width parameter (i.e., the “ σ ,” or standard deviation, of the Gaussian $e^{-x^2/2\sigma^2}$) of

$$\Delta = \frac{2^{1/6}}{3^{1/2}} E_G^{1/6} (kT)^{5/6}. \quad (3.133)$$

The value of the integral can therefore be approximated well with the area of the Gaussian, $\sqrt{2\pi}f(E_0)\Delta$ (see Problem 7). Replacing in 3.127, we obtain the final expression for the power density due to a given nuclear reaction:

$$\epsilon = \frac{2^{5/3}\sqrt{2}}{\sqrt{3}} \frac{\rho X_A X_B}{m_H^2 A_A A_B \sqrt{\mu}} Q S_0 \frac{E_G^{1/6}}{(kT)^{2/3}} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]. \quad (3.134)$$

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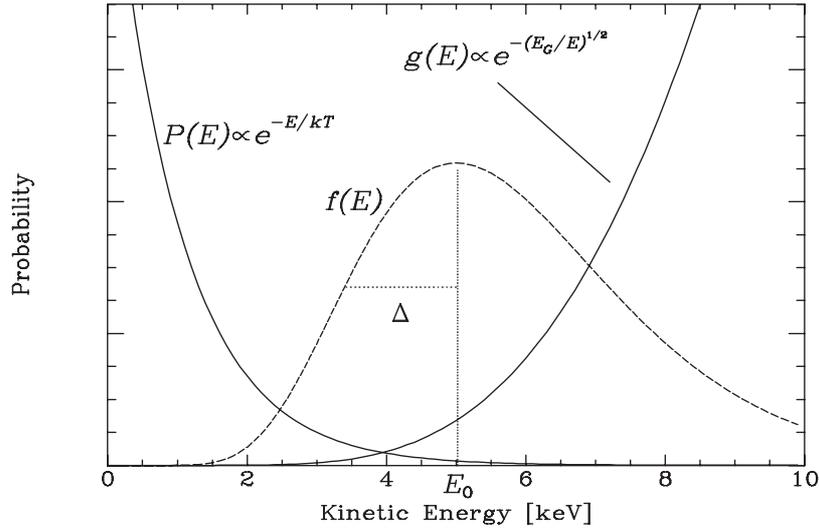


Figure 3.9 The Boltzmann probability distribution, $P(E)$, the Gamow factor, $g(E)$, and their product, $f(E)$. $P(E)$ is shown for $kT = 1$ keV, $g(E)$ is for the case of two protons, with $E_G = 500$ keV, and both $g(E)$ and $f(E)$ have been scaled up by large factors for display purposes. The rarity of protons with large kinetic energies, as described by $P(E)$, combined with the Coulomb barrier, embodied by $g(E)$, limit the protons taking part in nuclear reactions to those with energies near E_0 , where $f(E)$ peaks. $f(E)$ can be approximated as a Gaussian centered at E_0 ($E_0 \approx 5$ keV for the case shown), with width parameter Δ .

Equation 3.134 can tell us, for example, the luminosity produced by the p-p chain in the Sun. For a rough estimate, let us take for the mass density in the core of the Sun the central density, $\rho = 150 \text{ g cm}^{-3}$. As already noted in section 3.6, in the central regions of the Sun, some of the hydrogen has already been converted into helium by nuclear reactions. Let us assume a typical hydrogen abundance of $X = 0.5$, which we can use for X_A and X_B . The first step in the p-p chain, the $p + p \rightarrow d + e^+ + \nu_e$ reaction, is by far the slowest of the three steps in the chain, and it is therefore the bottleneck that sets the rate of the entire p-p process. The constant S_0 for this reaction is calculated theoretically to be $\approx 4 \times 10^{-46} \text{ cm}^2 \text{ keV}$, which is characteristic of weak interactions. For Q , let us take the entire thermal energy release of each p-p chain completion, since once the first step occurs, on a timescale of 10^{10} yr, the following two reactions, with timescales of order 1 s and 300,000 yr, respectively, are essentially instantaneous. We saw that every completion of the chain produces 26.73 MeV of energy and two neutrinos. Subtracting the 0.52 MeV carried off, on average, by the two neutrinos, the thermal energy released per p-p chain completion is $Q = 26.2$ MeV. As already noted, $E_G = 500$ keV for two protons, and the typical core temperature is $kT = 1$ keV. The atomic mass numbers are, of course, $A_A = A_B = 1$, and the reduced mass is $\mu = m_p/2$. Finally, since we are considering a reaction between identical particles (i.e., protons on protons) we need to divide the collision rate by 2, to avoid double counting. With these numbers, Eq. 3.134 gives a power density of

$$\epsilon = 10 \text{ erg s}^{-1} \text{ g}^{-1}. \quad (3.135)$$

Multiplying this by the mass of the core of the Sun, say, $0.2M_{\odot} = 4 \times 10^{32}$ g, gives a luminosity of $\sim 4 \times 10^{33}$ erg s^{-1} , in good agreement with the observed solar luminosity of 3.8×10^{33} erg s^{-1} .

Reviewing the derivation of Eqs. 3.122–3.134, we see that we can also recover the reaction rate per nucleus, dN_A/dt , by dividing back from ϵ a factor $(X_A Q)/(m_H A_A)$. For the $p + p$ reaction, this gives a rate of 1.6×10^{-18} s^{-1} per proton. The reciprocal of this rate is the typical time a proton has to wait until it reacts with another proton, and indeed equals

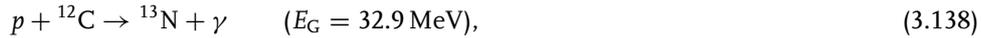
$$\tau_{pp} \sim 6 \times 10^{17} \text{ s} \sim 2 \times 10^{10} \text{ yr}, \quad (3.136)$$

as asserted above. Thus, we have shown that hydrogen fusion provides an energy source that can power the observed luminosity of the Sun over the known age of the Solar System, about 5 billion years, not only in terms of energy budget (Eq. 3.102) but also in terms of the energy generation *rate*. Furthermore, we see that the timescale to deplete the hydrogen fuel in the solar core is of order 10 billion years.

The total power density at a point in a star with a given temperature, density, and abundance will be the sum of the power densities due to all the possible nuclear reactions, each described by Eq. 3.134. Because of the exponential term in 3.134, there will be a strong preference for reactions between species with low atomic number, and hence small E_G . For example, compare the reactions



and



which have comparable nuclear cross sections S_0 (both reactions follow the same process of adding a proton to a nucleus and emitting a photon). At a typical kinetic energy of 1 keV, if the abundances of deuterium and carbon nuclei were comparable, the ratio between the rates would be

$$\frac{\mathcal{R}(p^{12}\text{C})}{\mathcal{R}(pd)} \sim \exp \left[-3 \frac{32.9^{1/3} - 0.66^{1/3}}{(4 \times 0.001)^{1/3}} \right] \sim e^{-44} \sim 10^{-20}. \quad (3.139)$$

Furthermore, the higher the Gamow energy, the more strongly will the reaction rate depend on temperature. For example, a first-order Taylor expansion of Eq. 3.134 around $T = 1.5 \times 10^7$ K, the central temperature of the Sun, shows that the $p + p \rightarrow d + e^+ + \nu_e$ rate depends on temperature approximately as T^4 , while the $p + {}^{12}\text{C} \rightarrow {}^{13}\text{N} + \gamma$ rate goes like T^{18} (see Problem 8). The steep positive temperature dependence of nuclear reactions, combined with the virial theorem, means that nuclear reactions serve as a natural “thermostat” that keeps stars stable. Suppose, for example, that the temperature inside a star rises. This will increase the rate of nuclear reactions, leading to an increase in luminosity. Due to opacity, this additional energy will not directly escape from the star, resulting in a temporary increase in total energy. Since

$$E_{\text{tot}} = \frac{1}{2} E_{\text{gr}} = -E_{\text{th}}, \quad (3.140)$$

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the gravitational energy E_{gr} will grow (i.e., become less negative), meaning the star will expand, and E_{th} will become smaller, meaning the temperature will be reduced again. This explains why main-sequence stars of very different masses have comparable core temperatures.

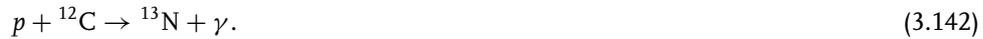
The thermostatic behavior controls also the long-term evolution of stars. Eventually, when the dominant nuclear fuel runs out, the power density ϵ will drop. The star will then contract, E_{th} will increase, and T will rise until a new nuclear reaction, involving nuclei of higher atomic number, can become effective.

A key prediction of the picture we have outlined, in which the energy of the Sun derives from the p-p chain, is that there will be a constant flux of neutrinos coming out of the Sun. As opposed to the photons, the weak interaction of the neutrinos with matter guarantees that they can escape the core of the Sun almost unobstructed.⁷ As calculated above, the thermal energy released per p-p chain completion is 26.2 MeV. The neutrino number flux on Earth should therefore be twice the solar energy flux divided by 26.2 MeV:

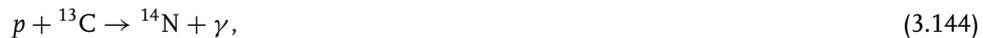
$$f_{\text{neutrino}} = \frac{2f_{\odot}}{26.2 \text{ MeV}} = \frac{2 \times 1.4 \times 10^6 \text{ erg s}^{-1} \text{ cm}^{-2}}{26.2 \times 1.6 \times 10^{-6} \text{ erg}} = 6.7 \times 10^{10} \text{ s}^{-1} \text{ cm}^{-2}. \quad (3.141)$$

This huge particle flux goes mostly unhindered through our bodies and through the entire Earth, and is extremely difficult to detect. Experiments to measure the solar neutrino flux began in the 1960s, and consistently indicated a deficit in the flux of electron neutrinos arriving from the Sun. It is now established that the **total** neutrino flux from the Sun is actually very close to the predictions of solar models. The originally observed deficit was the result of previously unknown *flavor oscillations*, in which some of the original electron neutrinos turn into other types of neutrinos en route from the Sun to the Earth.

We note, for completeness, that apart from the particular p-p chain described in Eqs. 3.97–3.99, which is the main nuclear reaction sequence in the Sun, other nuclear reactions occur, and produce neutrinos that are detectable on Earth (see Problem 9). In stars more massive than the Sun, hydrogen is converted to helium also via a different sequence of reactions, called the CNO cycle. In the CNO cycle, the trace amounts of carbon, nitrogen, and oxygen in the gas serve as *catalysts* in the hydrogen-to-helium burning, without any additional C, N, or O being synthesized. The main branch of the CNO cycle actually begins with reaction 3.138,



This is followed by



⁷ Typical cross sections for scattering of neutrinos on matter are of order 10^{-43} cm^2 , 10^{18} times smaller than the Thomson cross section for photons. Scaling from Eq. 3.41, the mean free path for neutrinos in the Sun is $\sim 10^{18} \text{ cm}$, 10^7 times greater than the solar radius.

and finally,



Although we noted that reaction 3.142 is slower by 20 orders of magnitude than the p-p chain's $p + d \rightarrow {}^3\text{He} + \gamma$, the $p + d$ reaction can take place only after overcoming the $p + p$ bottleneck, which has a timescale 18 orders of magnitude *longer* than $p + d$. The lack of such a bottleneck for the $p + {}^{12}\text{C}$ reaction is further compensated by this reaction's strong dependence on temperature. Although core temperature varies only weakly with stellar mass, the slightly higher core temperatures in more massive stars are enough to make the CNO cycle the dominant hydrogen-burning mechanism in main-sequence stars of mass $1.2M_{\odot}$ and higher.

3.11 Solution of the Equations of Stellar Structure

We have now derived the four differential equations and the three additional functions that, together with boundary conditions, define uniquely the equilibrium properties of a star of a given mass and composition. Along the way, we already deduced many of the observed properties of main-sequence stars. “Solving” this system of coupled equations means finding the functions $P(r)$, $T(r)$, and $\rho(r)$, which are the ones that are usually considered to describe the structure of the star. Unfortunately, there is no analytic solution to the equations, unless some unrealistic assumptions are made (see, e.g., Problems 4 and 5). Nevertheless, a numerical solution can be obtained straightforwardly, and is the most reasonable way to proceed anyway, given the complicated nature of the functions P , κ , and ϵ when all relevant processes are included. In a numerical solution, the differentials in the equations are replaced by differences. Then, an example of one possible calculation scheme is one in which the radial structure of a star is followed shell by shell, going either outward from the center or inward from the surface.

3.12 Convection

Under certain conditions, the main means of energy transport in some regions of a star is **convection**, rather than radiative transport. Convection occurs when a volume element of material that is displaced from its equilibrium position, rather than returning to the original position, continues moving in the displacement direction. For example, if the displacement is upward to a region of lower density, and after the displacement the density of the volume element is lower than that of its new surroundings, the element will continue to be buoyed upward. When convection sets in, it is very efficient at transporting heat, and becomes the dominant transport mechanism.

To see what are the conditions for the onset of convection, consider a volume element of gas at equilibrium radius r inside a star, where the temperature, pressure, and density are T , P , and ρ , respectively (see Fig. 3.10). Now let us displace the element to a radius

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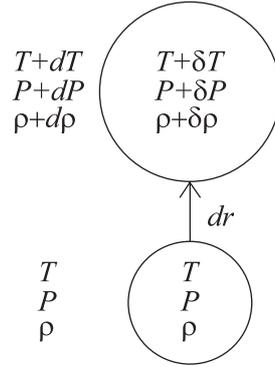


Figure 3.10 A mass element (lower circle) inside a star undergoes a small displacement dr to a higher position (upper circle), and expands adiabatically to match the new surrounding pressure $P + dP$. If, after the expansion, the density inside the element, $\rho + \delta\rho$, is larger than the surrounding density $\rho + d\rho$, the element will sink back to its former equilibrium position. If, on the other hand, the density inside the volume is lower than that of the surroundings, the mass element will be buoyed up, and convection ensues.

$r + dr$, where the parameters of the surroundings are $T + dT$, $P + dP$, and $\rho + d\rho$. Since the gas in the star obeys

$$\rho \propto \frac{P}{T}, \quad (3.148)$$

taking the logarithmic derivative gives

$$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T}. \quad (3.149)$$

To simplify the problem, we will assume that, at its new location, the volume element expands **adiabatically** (i.e., without exchanging heat with its new surroundings, so $dQ = 0$, and therefore the entropy, defined as $dS = dQ/T$, remains constant). The element expands until its pressure matches the surrounding pressure, and reaches new parameters $T + \delta T$, $\rho + \delta\rho$, and $P + \delta P = P + dP$, where we have identified the small changes inside the element with a “ δ ” rather than a “ d .” Since we approximate the expansion of the element to be adiabatic, it obeys an equation of state

$$P \propto \rho^\gamma, \quad (3.150)$$

where the *adiabatic index*, γ , is the usual ratio of heat capacities at constant pressure and constant volume. Taking again the logarithmic derivative, we obtain

$$\frac{\delta\rho}{\rho} = \frac{1}{\gamma} \frac{\delta P}{P}. \quad (3.151)$$

The element will continue to float up, rather than falling back to equilibrium, if after the expansion its density is lower than that of the surroundings, i.e.,

$$\rho + \delta\rho < \rho + d\rho, \quad (3.152)$$

or simply

$$\delta\rho < d\rho, \quad (3.153)$$

(recall that both $\delta\rho$ and $d\rho$ are negative), or dividing both sides by ρ ,

$$\frac{\delta\rho}{\rho} < \frac{d\rho}{\rho}. \quad (3.154)$$

Substituting from Eqs. 3.149 and 3.151, the condition for convection becomes

$$\frac{1}{\gamma} \frac{\delta P}{P} < \frac{dP}{P} - \frac{dT}{T}. \quad (3.155)$$

Recalling that $\delta P = dP$, this becomes

$$\frac{dT}{T} < \frac{\gamma - 1}{\gamma} \frac{dP}{P}, \quad (3.156)$$

or upon division by dr ,

$$\frac{dT}{dr} < \frac{\gamma - 1}{\gamma} \frac{T}{P} \frac{dP}{dr}. \quad (3.157)$$

Since the radial temperature and pressure gradients are both negative, the condition for convection is that the temperature profile must fall fast enough with increasing radius, i.e., convection sets in when

$$\left| \frac{dT}{dr} \right| > \frac{\gamma - 1}{\gamma} \frac{T}{P} \left| \frac{dP}{dr} \right|. \quad (3.158)$$

For a nonrelativistic gas without internal degrees of freedom (e.g., ionized hydrogen), $\gamma = 5/3$. As the number of internal degrees of freedom increases, γ becomes smaller, making convection possible even if $|dT/dr|$ is small. This can occur when the gas is made of atoms, which can be excited or ionized, or of molecules that have rotational and vibrational degrees of freedom and can be dissociated. Convection therefore occurs in some cool regions of stars, where atoms and molecules exist. This applies to the outer layers of intermediate-mass main-sequence stars and red giants, and to large ranges in radius in low-mass stars. Another range of applicability of convection is in the cores of massive stars. The Sun is convective in the outer 28% of its radius.

Once convection sets in, it mixes material at different radii and thus works toward equilibrating temperatures, i.e., lowering the absolute value of the temperature gradient $|dT/dr|$. Therefore, convection can be implemented into stellar structure computations by testing, at every radius, if the convection condition has been met. If it has, convection will bring the temperature gradient back to its critical value, and therefore the radiative energy transport equation (3.51) can be replaced by Eq. 3.158, but with an equality sign:

$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma} \frac{T(r)}{P(r)} \frac{dP}{dr}. \quad (3.159)$
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Problems

1. In Eqs. 3.23–3.28, we saw that, for a star composed of a classical, nonrelativistic, ideal gas, $E_{\text{total}} = E_{\text{th}}^{\text{tot}} + E_{\text{gr}} = -E_{\text{th}}^{\text{tot}}$, and therefore the star is bound. Repeat the derivation, but for a classical, **relativistic** gas of particles. Recall (Eq. 3.75) that the equation of state of a relativistic gas is $P = \frac{1}{3}E_{\text{th}}/V$. Show that, in this case, $E_{\text{gr}} = -E_{\text{th}}^{\text{tot}}$, and therefore $E_{\text{total}} = E_{\text{th}}^{\text{tot}} + E_{\text{gr}} = 0$, i.e., the star is marginally bound. As a result, stars dominated by radiation pressure are unstable.

2. The pressure inside a normal star is given by (Eq. 3.76)

$$P = P_g + P_{\text{rad}} = \frac{\rho k T}{\bar{m}} + \frac{1}{3}aT^4.$$

Using parameters appropriate to the Sun, show that throughout the Sun, including the core, where the internal temperature is about 10^7 K, the kinetic pressure dominates.

3. Because of the destabilizing influence of radiation pressure (see Problem 1), the most massive stars that can form are those in which the radiation pressure and the nonrelativistic kinetic pressure are approximately equal. Estimate the mass of the most massive stars, as follows.

- a. Assume that the gravitational binding energy of a star of mass M and radius R is $|E_{\text{gr}}| \sim GM^2/R$. Use the virial theorem (Eq. 3.22),

$$\bar{P} = -\frac{1}{3} \frac{E_{\text{gr}}}{V},$$

to show that

$$P \sim \left(\frac{4\pi}{3^4}\right)^{1/3} GM^{2/3} \rho^{4/3},$$

where ρ is the typical density.

- b. Show that if the radiation pressure, $P_{\text{rad}} = \frac{1}{3}aT^4$, equals the kinetic pressure, then the total pressure is

$$P = 2 \left(\frac{3}{a}\right)^{1/3} \left(\frac{k\rho}{\bar{m}}\right)^{4/3}.$$

- c. Equate the expressions for the pressure in (a) and (b), to obtain an expression for the maximal mass of a star. Find its value, in solar masses, assuming a fully ionized hydrogen composition.

Answer: $M = 110M_{\odot}$.

4. Consider a hypothetical star of radius R , with density ρ that is constant, i.e., independent of radius. The star is composed of a classical, nonrelativistic, ideal gas of fully ionized hydrogen.

- a. Solve the equations of stellar structure for the pressure profile, $P(r)$, with the boundary condition $P(R) = 0$.
Answer: $P(r) = (2\pi/3)G\rho^2(R^2 - r^2)$.
- b. Find the temperature profile, $T(r)$.
- c. Assume that the nuclear energy production rate depends on temperature as $\epsilon \sim T^4$. (This is the approximate dependence of the rate for the p-p chain at the temperature in the core of the Sun.) At what radius does ϵ decrease to 0.1 of its central value, and what fraction of the star's volume is included within this radius?

5. Suppose a star of total mass M and radius R has a density profile $\rho = \rho_c(1 - r/R)$, where ρ_c is the central density.

- a. Find $M(r)$.
- b. Express the total mass M in terms of R and ρ_c .
- c. Solve for the pressure profile, $P(r)$, with the boundary condition $P(R) = 0$.

Answer:

$$P(r) = \pi G\rho_c^2 R^2 \left[\frac{5}{36} - \frac{2}{3}\left(\frac{r}{R}\right)^2 + \frac{7}{9}\left(\frac{r}{R}\right)^3 - \frac{1}{4}\left(\frac{r}{R}\right)^4 \right].$$

6. Consider a star of mass $M = 10M_\odot$, composed entirely of fully ionized ^{12}C . Its core temperature is $T_c = 6 \times 10^8$ K (compared to $T_{c,\odot} = 1.5 \times 10^7$ K for the Sun).

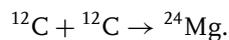
- a. What is the mean particle mass \bar{m} , in units of m_{H} ?

Answer: 12/7.

- b. Use the classical ideal gas law, the dimensional relation between mass, density, and radius, and the virial theorem to find the scaling of the stellar radius r_* with total mass M , mean particle mass \bar{m} , and core temperature T_c . Using the values of these parameters for the Sun, derive the radius of the star.

Answer: $0.70r_\odot$.

- c. If the luminosity of the star is $L = 10^7 L_\odot$, what is the effective surface temperature?
- d. Suppose the star produces energy via the reaction



The atomic weight of ^{12}C is 12, and that of ^{24}Mg is 23.985. (The atomic weight of a nucleus is defined as the ratio of its mass to 1/12 the mass of a ^{12}C nucleus.) What fraction of the star's mass can be converted into thermal energy?

Answer: 6.3×10^{-4} .

- e. How much time does it take for the star to use up 10% of its carbon?

Answer: 950 yr.

7. We saw that the nuclear reaction rate in a star depends on

$$\langle \sigma v \rangle \propto \int_0^\infty f(E) dE,$$

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where

$$f(E) \equiv e^{-E/kT} e^{-\sqrt{E_G/E}},$$

and E_G is the Gamow energy (Eq. 3.131).

a. By taking the derivative of $f(E)$ and equating to zero, show that $f(E)$ has a maximum at

$$E_0 = \left(\frac{kT}{2}\right)^{2/3} E_G^{1/3}.$$

b. Perform a Taylor expansion, to second order, of $f(E)$ around E_0 , to approximate $f(E)$ with a Gaussian. Show that the width parameter (i.e., the “ σ ”) of the Gaussian is

$$\Delta = \frac{2^{1/6}}{3^{1/2}} E_G^{1/6} (kT)^{5/6}.$$

Hint: Take the logarithm of $f(E)$, before Taylor expanding, and then exponentiate again the Taylor expansion.

c. Show that

$$\int_0^\infty f(E) dE = \sqrt{2\pi} f(E_0) \Delta.$$

8. Show that the dependence on temperature of the nuclear power density (Eq. 3.134) at a temperature T near T_0 can be approximated as a power law, $\epsilon \propto T^\beta$, where

$$\beta = \left(\frac{E_G}{4kT_0}\right)^{1/3} - \frac{2}{3}.$$

Evaluate β at $T_0 = 1.5 \times 10^7$ K, for the reactions $p + p \rightarrow d + e^+ + \nu_e$ and $p + {}^{12}\text{C} \rightarrow {}^{13}\text{N} + \gamma$.

Hint: From Eq. 3.134, find $\ln \epsilon$, and calculate $d(\ln \epsilon)/d(\ln T)|_{T_0}$. This is the first-order coefficient in a Taylor expansion of $\ln \epsilon$ as a function of $\ln T$ (a pure power-law relation between ϵ and T would obey $\ln \epsilon = \text{const.} + \beta \ln T$).

9. We saw (Eq. 3.141) that, on Earth, the number flux of solar neutrinos from the p-p chain is

$$\begin{aligned} f_{\text{neutrino}} &= \frac{2f_\odot}{26.2 \text{ MeV}} = \frac{2 \times 1.4 \times 10^6 \text{ erg s}^{-1} \text{ cm}^{-2}}{26.2 \times 1.6 \times 10^{-6} \text{ erg}} \\ &= 6.7 \times 10^{10} \text{ s}^{-1} \text{ cm}^{-2}. \end{aligned}$$

Other nuclear reactions in the Sun supplement this neutrino flux with a small additional flux of higher-energy neutrinos. A neutrino detector in Japan, named SuperKamiokande, consists of a tank of 50 kton of water, surrounded by photomultiplier tubes. The tubes detect the flash of **Cerenkov radiation** emitted by a recoiling electron when a high-energy neutrino scatters on it.

- a. How many electrons are there in the water of the detector?
- b. Calculate the detection rate for neutrino scattering, in events per day, if 10^{-6} of the solar neutrinos have a high enough energy to be detected by this experiment, and each electron poses a scattering cross section $\sigma = 10^{-43} \text{ cm}^2$.

Hint: Consider the density of neutrino targets “seen” by an individual electron, with a relative velocity of c between the neutrinos and the electron, to obtain the rate at which one electron interacts with the incoming neutrinos, and multiply by the total number of electrons, from (a), to obtain the rate in the entire detector.

Answers: 1.6×10^{34} electrons; 9 events per day.