

CHAPTER 1

WAKE UP AND SMELL THE FUNCTIONS

IT'S FRIDAY MORNING. The alarm clock next to me reads 6:55 a.m. In five minutes it'll wake me up, and I'll awake refreshed after sleeping roughly 7.5 hours. Echoing the followers of the ancient mathematician Pythagoras—whose dictum was “All is number”—I deliberately chose to sleep for 7.5 hours. But truth be told, I didn't have much of a choice. It turns out that a handful of numbers, including 7.5, rule over our lives every day. Allow me to explain.

A long time ago at a university far, far away I was walking up the stairs of my college dorm to my room. I lived on the second floor at the time, just down the hall from my friend Eric Johnson's room. EJ and I were in freshman physics together, and I often stopped by his room to discuss the class. This time, however, he wasn't there. I thought nothing of it and kept walking down the narrow hallway toward my room. Out of nowhere EJ appeared, holding a yellow Post-it note in his hand. “These numbers will change your life,” he said in a stern voice as he handed me the note. Off in the corner was a sequence of numbers:

1.5 4.5 7.5

3 6

Like Hurley from the *Lost* television series encountering his mystical sequence of numbers for the first time, my gut told me that these numbers meant something, but I didn't know what. Not knowing how to respond, I just said, “Huh?”

EJ took the note from me and pointed to the number 1.5. “One and a half hours; then another one and a half makes three,” he said. He explained that the average human sleep cycle is 90 minutes (1.5 hours) long. I started connecting the numbers in the shape of a “W.” They were all a distance of 1.5 from each other—the length of the sleep cycle. This was starting to sound like a good explanation for why some days I’d wake up “feeling like a million bucks,” while other days I was just “out of it” the entire morning. The notion that a simple sequence of numbers could affect me this much was fascinating.

In reality getting *exactly* 7.5 hours of sleep is very hard to do. What if you manage to sleep for only 7 hours, or 6.5? How awake will you feel then? We could answer these questions if we had the sleep cycle *function*. Let’s create this based on the available data.

What’s Trig Got to Do with Your Morning?

A typical sleep cycle begins with REM sleep—where dreaming generally occurs—and then progresses into non-REM sleep. Throughout the four stages of non-REM sleep our bodies repair themselves,¹ with the last two stages—stages 3 and 4—corresponding to deep sleep. As we emerge from deep sleep we climb back up the stages to REM sleep, with the full cycle lasting on average 1.5 hours. If we plotted the sleep stage S against the hours of sleep t , we’d obtain the diagram in Figure 1.1(a). The shape of this plot provides a clue as to what function we should use to describe the sleep stage. Since the graph repeats roughly every 1.5 hours, let’s approximate it by a *trigonometric function*.

To find the function, let’s begin by noting that S depends on how many hours t you’ve been sleeping. Mathematically, we say that your sleep stage S is a function of the number of hours t you’ve been asleep,ⁱ and write $S = f(t)$. We can now use what we know about sleep cycles to come up with a reasonable formula for $f(t)$.

ⁱAppendix A includes a short refresher on functions and graphs.

Since we know that our REM/non-REM stages *cycle* every 1.5 hours, this tells us that $f(t)$ is a *periodic function*—a function whose values repeat after an interval of time T called the *period*—and that the period $T = 1.5$ hours. Let’s assign the “awake” sleep stage to $S = 0$, and assign each subsequent stage to the next negative whole number; for example, sleep stage 1 will be assigned to $S = -1$, and so on. Assuming that $t = 0$ is when you fell asleep, the trigonometric function that results is ^{*1}

$$f(t) = 2 \cos \left(\frac{4\pi}{3} t \right) - 2,$$

where $\pi \approx 3.14$.

Before we go off and claim that $f(t)$ is a good mathematical model for our sleep cycle, it needs to pass a few basic tests. First, $f(t)$ should tell us that we’re awake (sleep stage 0) every 1.5 hours. Indeed, $f(1.5) = 0$ and so on for multiples of 1.5. Next, our model should reproduce the actual sleep cycle in Figure 1.1(a). Figure 1.1(b) shows the graph of $f(t)$, and as we can see it does a good job of capturing not only the awake stages but also the deep sleep times (the troughs).ⁱⁱ

In my case, though I’ve done my best to get exactly 7.5 hours of sleep, chances are I’ve missed the mark by at least a few minutes. If I’m way off I’ll wake up in stage 3 or 4 and feel groggy; so I’d like to know how close to a multiple of 1.5 hours I need to wake up so that I still feel relatively awake.

We can now answer this question with our $f(t)$ function. For example, since stage 1 sleep is still relatively light sleeping, we can ask for all of the t values for which $f(t) \geq -1$, or

$$2 \cos \left(\frac{4\pi}{3} t \right) - 2 \geq -1.$$

The quick way to find these intervals is to draw a horizontal line at sleep stage -1 on Figure 1(b). Then all of the t -values for which our graph is

ⁱⁱAs Figure 1.1(a) shows, after roughly three full sleep cycles (4.5 hours of sleep) we don’t experience the deep sleep stages again. We didn’t factor this in when designing the model, which explains why $f(t)$ doesn’t capture the shallower troughs seen in Figure 1.1(a) for $t > 5$.

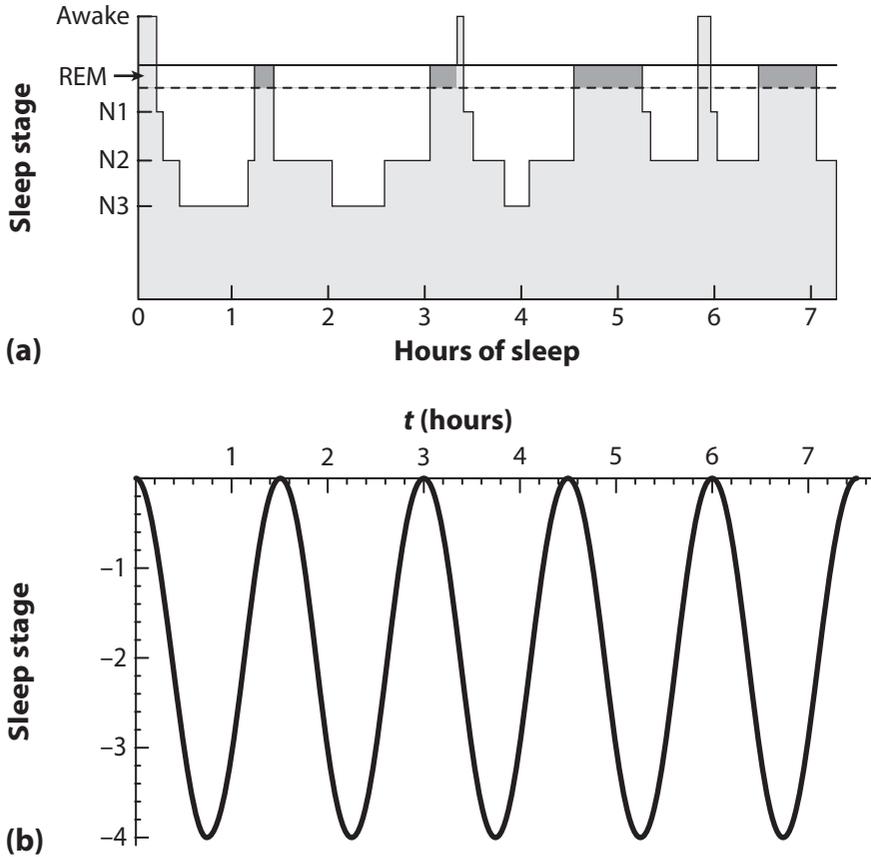


Figure 1.1. (a) A typical sleep cycle.² (b) Our trigonometric function $f(t)$.

above this line will satisfy our inequality. We could use a ruler to obtain good estimates, but we can also find the exact intervals by solving the equation $f(t) = -1$.²

$$[0, 0.25], [1.25, 1.75], [2.75, 3.25], [4.25, 4.75], [5.75, 6.25], [7.25, 7.75], \text{ etc.}$$

We can see that the endpoints of each interval are 0.25 hour—or 15 minutes—away from a multiple of 1.5. Hence, our model shows that missing the 1.5 hour target by 15 minutes on either side won't noticeably impact our morning mood.

This analysis assumed that 90 minutes represented the *average* sleep cycle length, meaning that for some of us the length is closer to 80 minutes, while for others it's closer to 100. These variations are easy to incorporate into $f(t)$: just change the period T . We could also replace the 15-minute buffer with any other amount of time. These *free parameters* can be specified for each individual, making our $f(t)$ function very customizable.

I'm barely awake and already mathematics has made it into my day. Not only has it enabled us to solve the mystery of EJ's multiples of 1.5, but it's also revealed that we all wake up with a built-in trigonometric function that sets the tone for our morning.

How a Rational Function Defeated Thomas Edison, and Why Induction Powers the World

Like most people I wake up to an alarm, but unlike most people I set *two* alarms: one on my radio alarm clock plugged into the wall and one on my iPhone. I adopted this two-alarm system back in college when a power outage made me late for a final exam. We all know that our gadgets run on electricity, so the power outage must have interrupted the flow of electricity to my alarm clock at the time. But what is "electricity," and what causes it to flow?

On a normal day my alarm clock gets its electricity in the form of *alternating current* (AC). But this wasn't always the case. In 1882 a well-known inventor—Thomas Edison—established the first electric utility company; it operated using *direct current* (DC).³ Edison's business soon expanded, and DC current began to power the world. But in 1891 Edison's dreams of a DC empire were crushed, not by corporate interests, lobbyists, or environmentalists, but instead by a most unusual suspect: a *rational function*.

The story of this rational function begins with the French physicist André-Marie Ampère. In 1820 he discovered that two wires carrying electric currents can attract or repel each other, as if they were magnets. The hunt was on to figure out how the forces of electricity and magnetism were related.

The unexpected genius who contributed most to the effort was the English physicist Michael Faraday. Faraday, who had almost no formal education or mathematical training, was able to visualize the interactions between magnets. To everyone else the fact that the “north” pole of one magnet attracted the “south” pole of another—place them close to each other and they’ll snap together—was just this, a fact. But to Faraday there was a *cause* for this. He believed that magnets had “lines of force” that emanated from their north poles and converged on their south poles. He called these lines of force a *magnetic field*.

To Faraday, Ampère’s discovery hinted that magnetic fields and electric current were related. In 1831 he found out how. Faraday discovered that moving a magnet near a circuit creates an electric current in the circuit. Put another way, this *law of induction* states that a changing magnetic field produces a *voltage* in the circuit. We’re familiar with voltages produced by batteries (like the one in my iPhone), where chemical reactions release energy that results in a voltage between the positive and negative terminals of the battery. But Faraday’s discovery tells us that we don’t need the chemical reactions; just wave a magnet near a circuit and voilà, you’ll produce a voltage! This voltage will then push around the electrons in the circuit, causing a flow of electrons, or what we today call *electricity* or *electric current*.

So what does Edison have to do with all of this? Well, remember that Edison’s plants operated on DC current, the same current produced by today’s batteries. And just like these batteries operate at a fixed voltage (a 12-volt battery will never magically turn into a 15-volt battery), Edison’s DC-current plants operated at a fixed voltage. This seemed a good idea at the time, but it turned out to be an epic failure. The reason: hidden mathematics.

Suppose that Edison’s plants produce an amount V of electrical energy (i.e., voltage) and transmit the resulting electric current across a power line to a nineteenth-century home, where an appliance (perhaps a fancy new electric stove) sucks up the energy at the constant rate P_0 . The radius r and length l of the power line are related to V by

$$r(V) = k \frac{\sqrt{P_0 l}}{V},$$

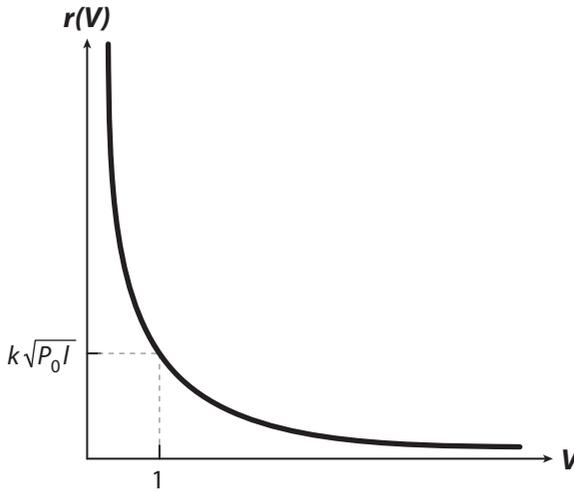


Figure 1.2. A plot of the rational function $r(V)$.

where k is a number that measures how easily the power line allows current to flow.ⁱⁱⁱ This *rational function* is the nemesis Edison never saw coming.

For starters, the easiest way to distribute electricity is through hanging power lines. And there's an inherent incentive to make these as thin (small r) as possible, otherwise they would both cost more and weigh more—a potential danger to anyone walking under them. But our rational function tells us that to carry electricity over large distances (large l) we need large voltages (large V) if we want the power line radius r to be small (Figure 1.2). And this was precisely Edison's problem; his power plants operated at the low voltage of 110 volts. The result: customers needed to live at most 2 miles from the generating plant to receive electricity. Since start-up costs to build new power plants were too high, this approach soon became uneconomical for Edison. On top of this, in 1891 an AC current was generated and transported *108 miles* at an exhibition in Germany. As they say in the sports business, Edison bet on the wrong horse.⁴

ⁱⁱⁱThis property of a material is called the *electrical resistivity*. Power lines are typically made of copper, since this metal has low electrical resistivity.

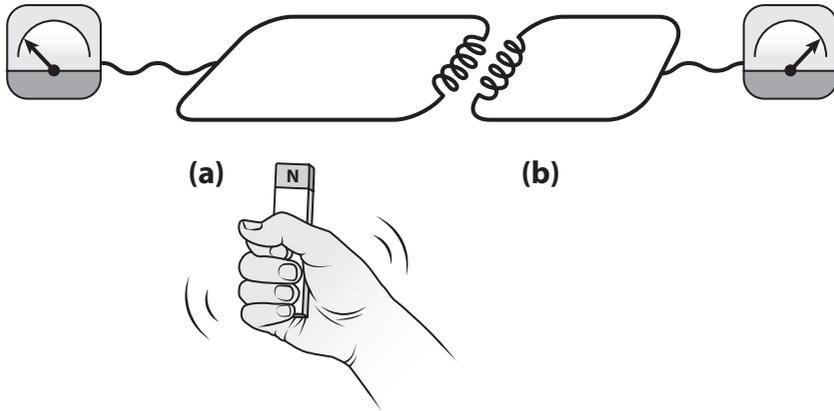


Figure 1.3. Faraday’s law of induction. (a) A changing magnetic field produces a voltage in a circuit. (b) The alternating current produced creates another changing magnetic field, producing another voltage in a nearby circuit.

But the function $r(V)$ has a split personality. Seen from a different perspective, it says that if we crank up the voltage V —by a lot—we can also increase the length l —by a bit less—and *still* reduce the wire radius r . In other words, we can transmit a very high voltage V across a very long distance l by using a very thin power line. Sounds great! But having accomplished this we’d still need a way to transform this high voltage into the low voltages that our appliances use. Unfortunately $r(V)$ doesn’t tell us how to do this. But one man already knew how: our English genius Michael Faraday.

Faraday used what we mathematicians would call “transitive reasoning,” the deduction that if A causes B and B causes C , then A must also cause C . Specifically, since a changing magnetic field produces a current in a circuit (his law of induction), and currents flowing through circuits produce magnetic fields (Ampère’s discovery), then it should be possible to use magnetic fields to *transfer* current from one circuit to another. Here’s how he did it.

Picture Faraday—a clean-shaven tall man with his hair parted down the middle—with a magnet in his hand, waving it around a nearby circuit. Induction causes this changing magnetic field to produce a voltage V_a in one circuit (Figure 1.3(a)). The alternating current produced

would, by Ampère’s discovery, produce another changing magnetic field. The result would be *another* voltage V_b in a *nearby* circuit (Figure 1.3(b)), producing current in that circuit.

As Faraday waves the magnet around, sometimes he does so closer to the loop and sometimes farther away; sometimes he waves it fast and other times slow. In other words, the voltage V_a produced *changes*. Today, magnets are put inside objects like windmills that do the waving for us. As the blades rotate in the wind, the magnetic field produced inside the turbine also changes. In this case the changes are described by a *trigonometric function* (not by Faraday’s crazy hand-waving). This alternating voltage causes the current to alternate too, putting the “alternating” in alternating current.

Great, we can now transfer current between circuits. But we still have the voltage problem: most household plugs run at low voltages (a fact left over from Edison’s doings), yet our modern grids produce voltages as high as 765,000 volts; how do we reduce this to the standard range of 120–220 volts that most countries use?

Let’s suppose that the original circuit’s wiring has been coiled into N_a turns, and that the nearby circuit’s wiring has been coiled into N_b turns (Figure 1.4(a)). Then

$$V_b = \frac{N_b}{N_a} V_a.$$

This formula says that a high incoming voltage V_a can be “stepped down” to a low outgoing voltage V_b by using a large number of turns N_a for the incoming coiling relative to the outgoing coiling. This transfer of voltage is called *mutual induction*, and is at the heart of modern electricity transmission. In fact, if you step outside right now and look up at the power lines you’ll likely see cylindrical buckets like the one in Figure 1.4(b). These *transformers* use mutual induction to step down the high voltages produced by modern electricity plants to lower, safer voltages for household use.

The two devices that got me going on this story—the iPhone and my clock radio—honor the legacies of both Edison and Faraday. My iPhone runs on DC current from its battery, and my clock radio draws

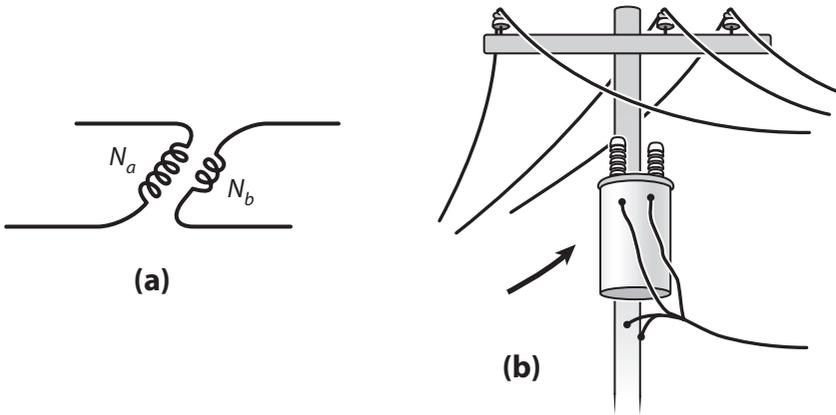


Figure 1.4. (a) Two circuits with different coiling numbers N_a and N_b . (b) A drawing of a transformer.

its power from the AC current coming through the wall plug, itself produced *dozens* of miles away at the electricity plant by an alternating voltage. And somewhere in between, Faraday’s mutual induction is at work stepping down the voltage so that we can power our devices.

But the real hero here is the rational function $r(V)$. It spelled doom for Edison, but through a different interpretation suggested that we base our electric grid on voltages *much* higher than Edison’s 110 volts. This idea of “listening” closely to mathematics to learn more about our world is a recurring theme of this book. We’ve already exposed two functions—the trigonometric $f(t)$ and the rational $r(V)$ —that follow you around everywhere you go. Let me wake up so that I can reveal even more hidden mathematics.

The Logarithms Hidden in the Air

It’s now seven in the morning and my alarm clock finally goes off. It’s set to play the radio when the alarm goes off, rather than that startling “BUZZ! BUZZ!” I can’t stand. Back when I lived in Ann Arbor I would wake up to 91.7 FM, the local National Public Radio (NPR) station.

But now that I live in Boston, 91.7 FM is pure static. What happened to the Ann Arbor station? Is my radio broken? Where's my NPR?!

The local NPR station for Boston is WBUR-FM, at 90.9 FM on the radio dial. Since I'm now far away from Ann Arbor my radio can't pick up the old 91.7 NPR station. We all intuitively know this; just drive far enough away from your home town and all your favorite radio stations will fade away. But wait a second, that's the same relationship that we saw in Figure 1.2 with the function $r(V)$. Could there be another rational function lurking somewhere in the air waves?

Let's get back to WBUR to figure this out. The station's "effective radiative power"—a measure of its signal strength—is 40,000 watts.⁵ You should recognize the unit here from your experience with light bulbs; just as a 100-watt bulb left on for one hour would *consume* 100 watt-hours of energy, WBUR's station *emits* 40,000 watt-hours of energy in one hour. That's the equivalent of $40,000/100 = 400$ light bulbs worth of energy every hour! But where does that energy go?

Picture a light bulb placed on the floor in the middle of a dark room. Turn it on and the light it emits will light up everything in the room. The bulb radiates its energy, partly in the form of light, evenly throughout the space in the room. Similarly, WBUR's antenna radiates its energy outward in the form of *radio waves*.

Now, just as you'd perceive the bulb's light to be brighter the closer you are to it, the radio signals coming from WBUR's antenna come through clearer when you're closer to the antenna. We can measure this by calculating the *intensity* $J(r)$ of the signal at a distance r from the antenna:

$$J(r) = \frac{\text{radiated power}}{\text{surface area}} = \frac{40,000}{4\pi r^2} = \frac{10,000}{\pi r^2}, \quad (1)$$

where I've assumed that the energy is radiated spherically outward. Aha! Here's the *rational* function we had predicted. Let's see if we can "listen" to it and learn something about how radios work.

The $J(r)$ formula tells us that the intensity of the signal decreases as the distance r from the antenna increases. This explains what happened in my move from Ann Arbor to Boston: it's not that the Ann Arbor

NPR station doesn't reach me anymore, but that its signal intensity is too weak to be picked up by my radio. On the other hand, at my current distance from WBUR's antenna my radio has no problem picking up the station.

While I lie there still in a haze, I pick up a few headlines from the voice coming out of the radio; something about the economy and later about politics. Nothing too exciting so I just stay in bed, listening. There's always a danger I'll fall back asleep (think "second alarm"); to thwart this I decide to boot up my brain by asking a simple question: what am I listening to?

Certainly the answer is WBUR at 90.9 FM. But that's a radio wave, and we humans can't *hear* a radio wave; the ear's frequency range is from 20 to 20,000 hertz,⁶ and WBUR's signal is broadcast at 90.9 megahertz.^{iv} Ergo, it's not the radio wave I hear. What I hear is the *sound* waves coming from my radio. And somehow that little gadget manages to convert a radio wave—which I can't hear—into a sound wave, which I can. But how?

Part of the answer is hidden in the fact that WBUR transmits at 90.9 megahertz. All sounds have a frequency associated with them; for example, the 49th key—called A4—on an 88-key piano has a frequency of 440 hertz. And we know (either from Appendix A or from your general knowledge) that phenomena with frequencies can be represented as oscillating functions, just like our sleep cycle functions. But then, what's oscillating in this case? *Something* has to move back and forth between the radio and my ear. And the only possibility is air, so the answer must be related to changes in *air pressure*.

In a nutshell, *sound is a pressure wave*. This is easy to confirm: hold your palm very close to your mouth and try to speak without any air hitting your palm. Good luck, because without the movement of air molecules there's no pressure wave. Now hold your hand somewhat close to your ear and fan it ferociously back and forth. You should hear a periodic sound as your arm oscillates: that sound is the pressure wave.

^{iv} 1 megahertz (MHz) is 1×10^6 hertz. Hertz (Hz) is the unit of frequency (see Appendix A for a quick refresher).

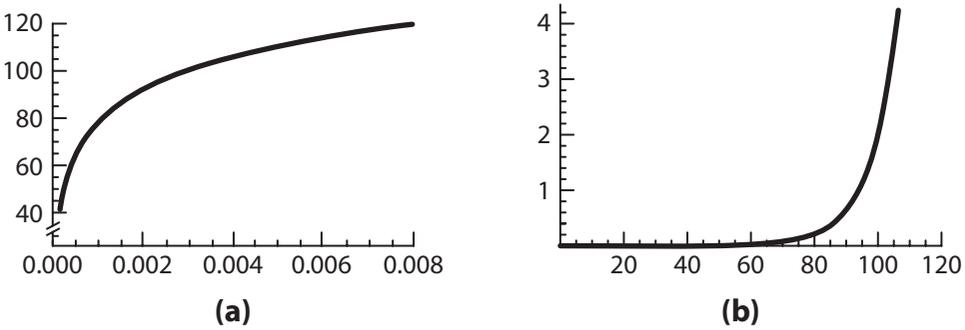


Figure 1.5. (a) A plot of the function $L(p)$. (b) A plot of the function $p(L)$.

Like your arm, a radio pulses its speakers back and forth to produce the pressure waves that our ears detect as sound. And just like your arm, the more violently the speakers vibrate the louder the sound that’s created. Mathematically, if we denote by p the sound pressure of a pressure wave, then the “sound level” $L(p)$ of that sound is given by the *logarithmic* function (Figure 1.5(a))

$$L(p) = 20 \log_{10}(50,000p) \text{ decibels.}$$

Let’s examine the familiar *decibel* (dB) units. As a reference, the water coming out of a showerhead makes a sound of about 80 decibels, and a jet engine at about 100 feet makes a sound of 140 decibels. From these numbers you can see why long-term exposure to sounds at levels as low as 90 decibels has the potential to cause hearing loss.⁷ We’re all more used to the decibel scale than to measuring pressure waves, so let’s invert the $L(p)$ equation. We arrive at the exponential function^{*3} (Figure 1.5(b))

$$p(L) = \frac{1}{50,000} 10^{L/20}.$$

The $p(L)$ equation tells us that, for example, a sound level of $L = 0$ decibels gives a pressure of $p(0) = 1/50,000 = 20 \times 10^{-6}$ pascals, the unit of pressure. This sound level and pressure combination roughly correspond to the sound a mosquito would make as it flaps its wings roughly 10 feet away from you,⁸ hence the small pressure number.

Now that I've gotten myself up and about figuring out this pressure thing, a nagging thought has developed in my head. Just a few minutes ago I was somewhere along my sleep cycle—modeled by trigonometric function $f(t)$ —and then my radio turned on, thanks to our rational function $r(V)$ and WBUR's antenna intensity function $J(r)$. The NPR reporter's voice then created a pressure wave that I interpreted as sound via the $L(p)$ function (we actually *hear* logarithmic functions; how cool is that?). There's so much going on. Is there any order to this chaos? Does my morning consist of chance encounters with different functions, or are they all related somehow? A hierarchy or a unifying principle would be nice.

The Frequency of Trig Functions

My new quest gives me something to think about while I pick my clothes out. On the other end of the bedroom is a small closet that my wife Zoraida and I cram our clothes into. I'm shuffling clothes around looking for something to wear after I shower. In the background, a soft sound begins to steadily increase in intensity; Zoraida is snoring. I figure I'll wake her up (we've got to get to work soon) by turning on the TV; she likes waking up to the morning shows. Naturally, I reach for another one of our modern gadgets: the remote control.

With the control in hand, I push the "channel up" button, looking for something she'd like. The remote sends out *infrared light waves* at frequencies of about 36,000 hertz. Although I can't see these signals—they are outside our frequency range of vision—the pulses of 1's and 0's that are emitted instruct the TV to change to the next channel. I find one of those morning shows and put the volume just loud enough to eventually wake her up.

Now that I've picked out a pair of khakis and a shirt, I get back to thinking about this "unifying principle" business on my way to the shower. The hallway's dark; it's a cloudy day outside. I'm hoping that since it's July the rain will quickly be followed by sunshine. This triggers a memory of a conversation I had back in high school with my friend Blake about light. We were talking about how the colors we see are

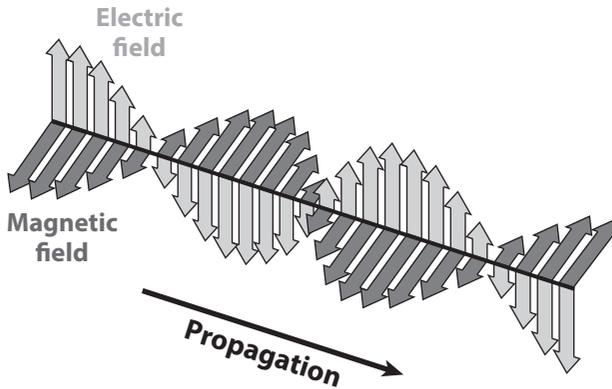


Figure 1.6. An electromagnetic wave. The electric and magnetic fields it carries oscillate perpendicular to each other as the wave propagates. Image from <http://www.molphys.leidenuniv.nl/monos/smo/index.html?basics/light.htm>.

described by different frequencies of light. For example, red light has a frequency range of about 430 to 480 terahertz.^{v.9} Blake was wondering if aliens would see red light—light in the frequency range of 430 to 480 terahertz—as actually “red.” This was in biology class, so we spent some time talking about what our eyes think “red” is.

Midway through my recollection I’m interrupted by a simple, clearly articulated word: frequency. And then it clicks. The AC current, the radio waves, the infrared waves, and sunlight, they all have a frequency associated with them. Here’s the unifying principle I’ve been looking for! Because they are characterized by a frequency, these are all oscillating functions—*trigonometric* functions.

This *mathematical* unifying principle also has a *physical* analogue. All of these waves—with the exception of AC current, which we’ll discuss shortly—are all particular types of *electromagnetic waves* (EM waves for short). As the name suggests, an electromagnetic wave carries along with it an electric *and* a magnetic field.^{vi} These fields oscillate perpendicular to each other as the wave propagates, and each can be represented by trigonometric functions (Figure 1.6).

^vOne terahertz (THz) is 1×10^{12} hertz.

^{vi}An electric field is the analogue of a magnetic field, where positive and negative charges play the roles of the north and south poles of a magnet.

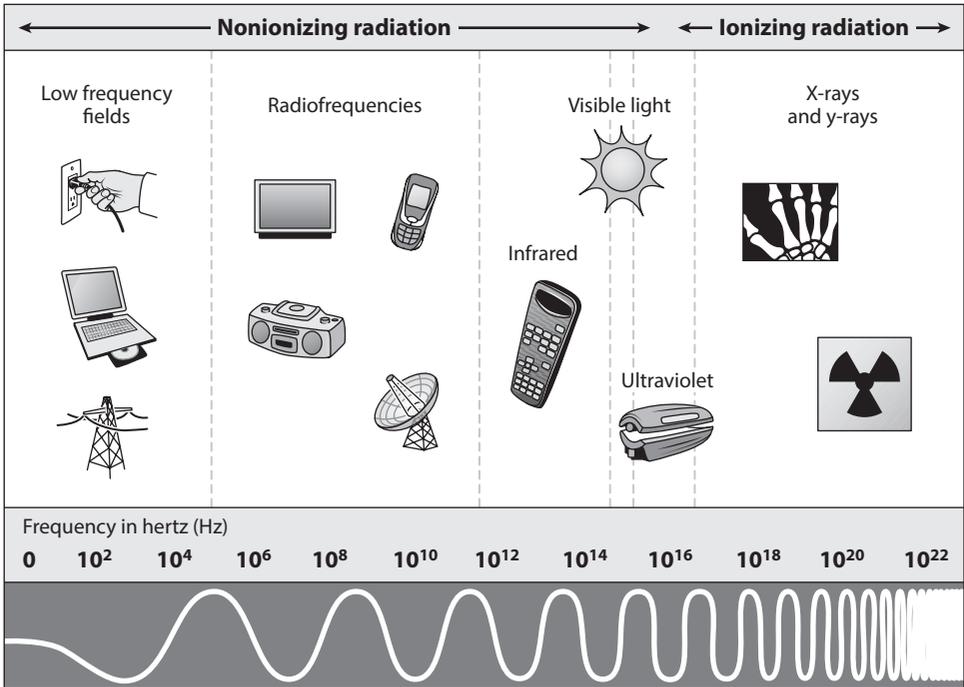


Figure 1.7. The electromagnetic wave spectrum. Image from <http://www.hermes-program.gr/en/emr.aspx>.

One of the greatest discoveries of the nineteenth century—due to our induction discoverer Michael Faraday—was that light itself is an EM wave. This explains why light has a *frequency* associated with it. Thus, infrared light, radio waves, and any other radiation that has a frequency associated with it is an EM wave (Figure 1.7). Alternating current, although not an electromagnetic wave itself, emits electromagnetic waves as it travels down a wire. An electromagnetic wave, along with its mathematical representation as a trigonometric function, is the unifying concept I was looking for.

When I turn on the light in my bathroom, I pause for a second to marvel at all the EM waves around me. The light the bulb produces? An EM wave. The sunlight coming through the window? Another EM wave. The radio waves transmitting NPR to the bedroom radio? Yep,

just another electromagnetic wave. So, not only can we *hear* logarithms (recall our function $L(p)$); now you know that we can *see* trigonometric functions (light). Who knew that trigonometric functions occurred so frequently throughout the day? (Pun intended.)

Galileo's Parabolic Thinking

I turn the tub's faucet on and switch on the showerhead; the water is freezing! It'll take a minute or so for it to heat up. No problem, I'll just brush my teeth while I wait. While I brush up and down, left and right (don't worry, I won't mention the trigonometric function here; oops, I just did!), I continue looking at the water stream, as if that'll make it heat up faster.

Inspired by Faraday's ability to see magnetic fields, I start trying to see the "gravitational field" and its effect on the stream. I know the field exists, since the water doesn't shoot out in a straight line, even though it comes out of the showerhead with a high velocity; instead it looks like it's "attracted" to the floor. Of course, there's no magnetism going on here, it's just gravity, but that's the *physics*. What about the *math*? The man who figured this out, Galileo Galilei, was referred to by none other than Einstein himself as the "father of modern science." He invented the telescope, and later used it to decisively confirm that the Earth revolves around the Sun and not the other way around. In addition, Galileo is also well known for his experiments with falling objects. The most famous of these is the Leaning Tower of Pisa experiment. Vincenzo Viviani, Galileo's pupil, described the experiment in a biography of Galileo. He wrote that Galileo had dropped balls of different masses from the tower to test the conjecture that they would reach the ground at the same time, regardless of their mass.^{vii,10} Galileo, in his earlier writings, had proposed that a falling object would fall with a uniform (constant) acceleration. By using this simple proposition, he had also demonstrated mathematically that the distance the object traveled

^{vii}This popular story might actually be a legend, but Vincenzo is no longer around to set the record straight.

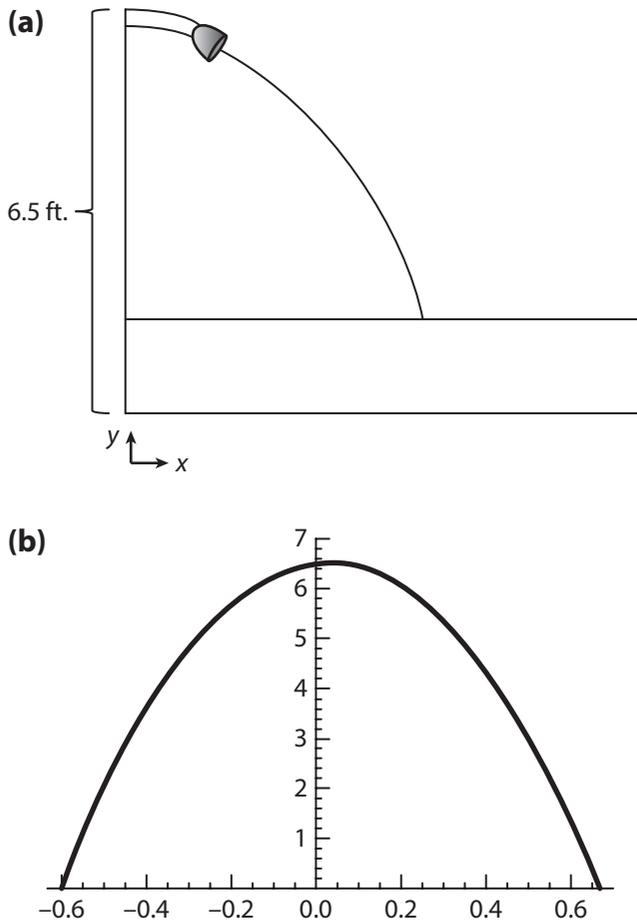


Figure 1.8. A schematic of my shower (a) along with the graph of the parabolic function $y(x) = 6.5 + x - 16x^2$ (b).

would be proportional to the square of the amount of time the object was in motion.¹¹

To fully appreciate this result, let's consider what it means in the context of the water coming out of my shower. Figure 1.8(a) shows a profile of my shower. We can define a *coordinate system* whose origin is on the ground, directly underneath my showerhead. Let's call the horizontal direction x and the vertical direction y , and suppose that the water is coming out of the showerhead with a constant speed of v_x

in the x -direction and v_y in the y -direction. Since gravity acts only in the vertical direction, *there is no acceleration in the horizontal direction* (as the joke goes, “sometimes gravity gets me down,” but never “left,” “right,” or “up”). We can now use the familiar formula *distance = rate \times time* to determine the *horizontal* distance $x(t)$ traveled by a water molecule:

$$x(t) = v_x t,$$

where we’ll measure t in seconds since the water molecule left the showerhead.

What about the vertical (y) direction motion? Each water molecule coming out of the showerhead is being pulled down by gravity, which Galileo says accelerates objects at a constant rate; let’s denote this by $-g$, where the negative sign is there to remind us that this acceleration is downward. Using this, along with the fact that our water molecule’s initial speed is v_y at time $t = 0$ and what we will call $v(t)$ at time $t > 0$, we then find*⁴ that our water molecule’s vertical speed $v(t)$ is the *linear* function

$$v(t) = v_y - gt,$$

its initial speed plus the contribution from gravity. It was also known in Galileo’s time that the distance traveled by objects whose speed varies linearly with time is given by

$$y(t) = y_0 + v_{\text{avg}} t, \quad \text{where} \quad v_{\text{avg}} = \frac{1}{2} (v_{\text{initial}} + v_{\text{final}}),$$

where y_0 is the initial position of the object. For our water molecule, since its vertical position is 6.5 feet above the ground when it comes out of the showerhead, we know that $y_0 = 6.5$. Moreover, since its initial vertical speed was v_y and its final vertical speed was $v(t) = v_y - gt$, then its average speed is

$$v_{\text{avg}} = v_y - \frac{1}{2}gt, \quad \text{so that} \quad y(t) = 6.5 + v_y t - \frac{1}{2}gt^2.$$

Unlike the $x(t)$ formula, the water molecule's vertical position is a *polynomial* function of t ; more specifically, it's a *quadratic* function.

We can put these two formulas together by solving the $x(t)$ equation for t and substituting the result into the $y(t)$ formula. We arrive at*⁵

$$y(x) = 6.5 + \frac{v_y}{v_x}x - \frac{g}{2v_x^2}x^2.$$

Since v_x , v_y , and g are numbers, this formula can be put in the form $y = 6.5 + Bx - Ax^2$, which is the equation for a *parabola* (Figure 1.8(b)). And since the coefficient of x^2 is negative, this parabola opens *downward*. Therefore, the mathematics is telling us that the water coming out of my shower bends toward the ground. And that's exactly what happens!

This formula, in my opinion, is one of the greatest achievements of medieval science. It applies not just to the water coming out of my showerhead, but also to a football, a Frisbee, or any other object thrown in the air. It tells us that *all objects moving on Earth follow parabolic trajectories*. To medieval scientists working at a time when religion was the predominant way to understand the world, results like these were seen as glimpses into the mind of God. They inspired future scientists to continue applying mathematics to our world in the hopes of achieving equally profound insights.

We'll spend the next chapter talking about one such scientist—Isaac Newton—who followed in Galileo's footsteps and made equally revolutionary advances for his time. For now, I hope this chapter has convinced you that functions are not abstract mathematical constructs. Instead, as Galileo and Faraday showed us, they can be seen, heard, and felt all around us every day. The journey that got us here started with the Pythagoreans' belief that "All is number," but this chapter suggests the more current Pythagorean-like dictum: "All are functions."