CHAPTER 1

Introduction

Slow down there, hotshot. I know you're smart—you might have always been good with numbers, you might have aced calculus—but I want you to *slow down*. Real analysis is an entirely different animal from calculus or even linear algebra. Besides the fact that it's just plain harder, the way you learn real analysis is not by memorizing formulas or algorithms and plugging things in. Rather, you need to read and reread definitions and proofs until you understand the larger concepts at work, so you can apply those concepts in your own proofs. The best way to get good at this is to take your time; read slowly, write slowly, and think carefully.

What follows is a short introduction about why I wrote this book and how you should go about reading it.

Why I Wrote This Book

Real analysis is hard. This topic is probably your introduction to proof-based mathematics, which makes it even harder. But I very much believe that anyone can learn anything, as long as it is explained clearly enough.

I struggled with my first real analysis course. I constantly felt like I was my own teacher and wished there was someone who could explain things to me in a clear, linear fashion. The fact that I struggled—and eventually pulled through—makes me an excellent candidate to be your guide. I easily recall what it was like to see this stuff for the first time. I remember what confused me, what was never really clear, and what stumped me. In this book, I hope I can preempt most of your questions by giving you the explanations I would have most liked to have seen.

My course used the textbook *Principles of Mathematical Analysis*, 3rd edition, by Walter Rudin (also known as Baby Rudin, or That Grueling Little Blue Book). It is usually considered the classic, standard real analysis text. I appreciate Rudin now—his book is well organized and concise. But I can tell you that when I used it to learn the material for the first time, it was a *slog*. It never explains anything! Rudin lists definitions without giving examples and writes polished proofs without telling you how he came up with them.

Don't get me wrong: having to figure things out for yourself can be of tremendous value. Being challenged to understand why things work—without linear steps handed to you on a silver platter—makes you a better thinker and a better learner. But I believe that as a pedagogical technique, "throwing you in the deep end" without teaching you how to swim is only good in moderation. After all, your teachers want you to learn, not drown. I think Rudin can provide all the throwing, and this book can be a lifesaver when you need it.

I wrote this book because if you are an intelligent-but-not-a-genius student (like I was), who genuinely wants to learn real analysis... you need it.

What Is Real Analysis?

Real analysis is what mathematicians would call the *rigorous* version of calculus. Being "rigorous" means that every step we take and every formula we use must be proved. If we start from a set of basic assumptions, called *axioms* or *postulates*, we can always get to where we are now by taking one justified step after another.

In calculus, you might have proved some important results, but you also took many things for granted. What exactly *are* limits, and how do you really know when an infinite sum "converges" to one number? In an introductory real analysis course, you are reintroduced to concepts you've seen before—continuity, differentiability, and so on—but this time, their foundations will be clearly laid. And when you are done, you will have basically proven that calculus *works*.

Real analysis is typically the first course in a pure math curriculum, because it introduces you to the important ideas and methodologies of pure math in the context of material you are already familiar with.

Once you are able to be rigorous with familiar ideas, you can apply that way of thinking to unfamiliar territory. At the core of real analysis is the question: "how do we expand our intuition for certain concepts—such as sums—to work in the infinite cases?" Puzzles such as infinite sums cannot be properly understood without being rigorous. Thus, you must build your hard-core proving skills to apply them to these new (not-from-high-school-calculus), more interesting problems.

How to Read This Book

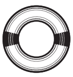
This book is not intended to be concise. Take a look at Chapter 7 as an example; I spend several pages covering what Rudin does in just two. The definitions are followed by examples in an attempt to make them less abstract. The proofs here are intended to show you not just *why* the theorem is true but also *how* you could go about proving it yourself. I try to state every fact being used in an argument, instead of omitting the more basic ones (as advanced mathematical literature would do).

If you are using Rudin, you'll find that I've purposely tried to cover all the definitions and theorems that he covers, mostly in the same order. There isn't a one-to-one mapping between this book and Rudin's (Chapter 7 math joke!); for example, the next chapter explains the basic theory of sets, whereas Rudin holds off on that until after covering real numbers. I also include a few extra pieces of information for your enrichment. But by following his structure and notations as closely as possible, you should be able to go back and forth between this book and his with ease.

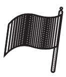
Unlike some other math books—which are meant to be glanced at, skimmed, or just referenced—you should read this one linearly. The chapters here are deliberately short and should contain the equivalent of an easily digestible one-hour lecture. Start at the beginning of a chapter and don't jump around until you make it to the end.

Now for some advice: read actively. Fill in the blanks where I tell you to. (I purposely didn't include the answers to these; the temptation to peek would just be too great.) Make notes even where I don't tell you to. Copy definitions into your notebook if you learn by repetition; draw lots of figures if you learn visually. Write any questions you may have in the margins. If, after reading a chapter twice, you still have unanswered questions, ask your study group, ask your TA, ask your professor (or ask all three; the more times you hear something, the better you'll learn it). Within each chapter, try to summarize its main ideas or methods; you'll find that almost every topic has one or two tricks that are used to do most of the proofs.

If your time is limited or you are reviewing material you've already learned, you can use the following icons to guide your skimming:



- Here begins an example or a proof that is figured out step by step.



- This is an important clarification or thing to keep in mind.

- Try this fill-in-the-blank exercise!

- This is a more complicated topic that is only mentioned briefly.

Extra resources never hurt. In fact, the more textbooks you read, the better your chances of success in learning advanced mathematics. The best strategy is to have one or two primary textbooks (for example, this one with Rudin) whose material you are committed to learning. Complement those with a library of other books from which to get extra practice and to look up an explanation if your primaries are not satisfactory. If you choose to disregard this and try to learn *all* the material in *all* the real analysis books out there... good luck to you!

This book covers most of a typical first-semester real analysis course, though it's possible your school covers more material. If this book ends before your course does, don't panic! Everything builds on what comes before it, so the most important factor for success is an understanding of the fundamentals. We will cover those fundamentals in detail, to make sure you have a solid foundation with which to swim onward (while avoiding mixed metaphors, such as this one).

For a list of some recommended books, along with my comments and criticisms, see the Bibliography.

Once you turn the page, we'll begin learning by going over some basic mathematical and logical concepts; they are critical background material for a rigorous study of real analysis. (How many times have I used the word *rigorous* so far? This many: $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} + 7$.)