

# 1

## Introducing Graphs

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The mathematical structure known as a *graph* has the valuable feature of helping us to visualize, to analyze, to generalize a situation or problem we may encounter and, in many cases, assisting us to understand it better and possibly find a solution. Let's begin by seeing how this might happen and what these structures look like.

### FIRST, . . . FOUR PROBLEMS

We begin with four problems that have a distinct mathematical flavor. Yet any attempt to solve these problems doesn't appear to use any mathematics you may have previously encountered. However, all of the problems can be analyzed and eventually solved with the aid of a relatively new sort of mathematical object and that object is a graph. The graph we're referring to is not the kind of graph you've seen before. For example, Figure 1.1 shows the graph of the function  $y = \sin x$ . That is not the kind of graph we're referring to.

### The Problem of the Five Princes

Once upon a time, there was a kingdom ruled by a king who had five sons. It was his wish that upon his death, this kingdom should be divided into five regions, one region for each son, such that each region would have a common boundary with each of the other four regions. Can this be done?

Figure 1.2 illustrates an unsuccessful attempt to satisfy the king's wishes. Every two of the five regions, numbered 1, 2, 3, 4, 5, share some common boundary, except regions 4 and 5.

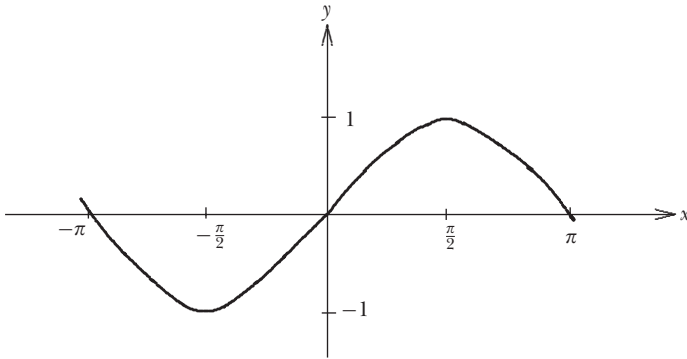


Figure 1.1. Not the sort of graph we're talking about.

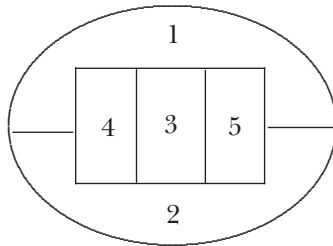


Figure 1.2. Attempting to satisfy the king's wishes.

If the kingdom can be divided into five regions in the manner desired by the king, then something else would have to be true. Place a point in each region and join two points by a line or curve if the regions containing these points have a common boundary. If  $A$  and  $B$  are two adjacent regions in the kingdom and  $C$  and  $D$  are two other adjacent regions, then it's always possible to connect each pair of points by a line in such a way that these two lines don't cross.

What we have just encountered is a graph (*our* type of graph) for the first time. A *graph*  $G$  is a collection of points (called *vertices*) and lines (called *edges*) where two vertices are joined by an edge if they are related in some way. In particular, the division of the kingdom into the five regions shown in Figure 1.2 gives rise to the graph  $G$  shown in Figure 1.3.

In order to have a solution to the king's wishes, the resulting graph must have five vertices, every two joined by an edge. Such a graph is called a *complete graph* of order 5 and expressed as  $K_5$ . Furthermore,

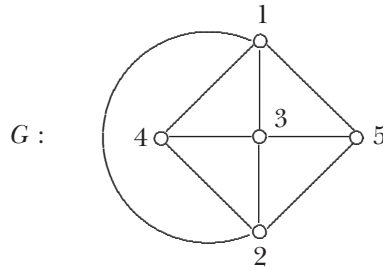


Figure 1.3. The graph representing the regions in Figure 1.2.

it must be possible to draw  $K_5$  without any of its edges crossing. Since there is no edge joining vertices 4 and 5 in Figure 1.3, the division of the kingdom into regions shown in Figure 1.2 does not represent a solution. In Chapter 10 we will visit the Problem of the Five Princes again when we will be able to give a complete solution to this problem.

### The Three Houses and Three Utilities Problem

Three houses are under construction and each house must be provided with connections to each of three utilities, namely water, electricity and natural gas. Each utility provider needs a direct line from the utility terminal to each house without passing through another provider's terminal or another house along the way. Furthermore, all three utility providers need to bury their lines at the same depth underground without any lines crossing. Can this be done?

Figure 1.4 shows a failed attempt to solve this problem, where the three houses are labeled A, B and C. Not only can this problem be looked at in terms of graphs, but in terms of graphs this problem is extremely similar to the Problem of the Five Princes. We can represent this situation by a graph with six vertices, three representing the three houses A, B and C and three representing the three utilities water (W), electricity (E) and natural gas (NG). Two vertices are joined by an edge when one vertex represents a house and the other represents a utility. This graph then has nine edges. This graph is denoted by  $K_{3,3}$ , indicating that there are two sets of three vertices each where each vertex in one set is joined to all vertices in the other set. To solve the Three Houses and Three Utilities

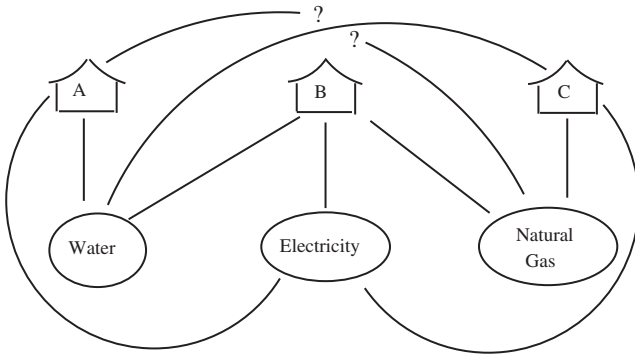


Figure 1.4. The Three Houses and Three Utilities Problem.

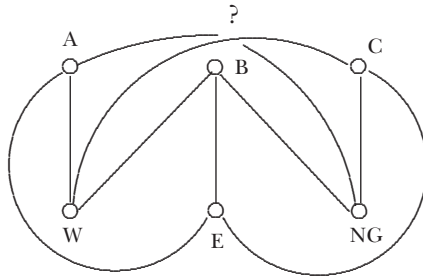


Figure 1.5. The graph representing the situation in Figure 1.4.

Problem, we need to know whether  $K_{3,3}$  can be drawn without any edges crossing. The attempted solution of the Three Houses and Three Utilities Problem in Figure 1.4 gives rise to the graph shown in Figure 1.5.

We will visit the Three Houses and Three Utilities Problem as well in Chapter 10 and explain how to solve the problem.

In our next problem a graph will be introduced whose vertices represent people. Here we assume every two people are friends or strangers.

### The Three Friends or Three Strangers Problem

What is the smallest number of people that must be present at a gathering to be certain that among them three are mutual friends or three are mutual strangers?

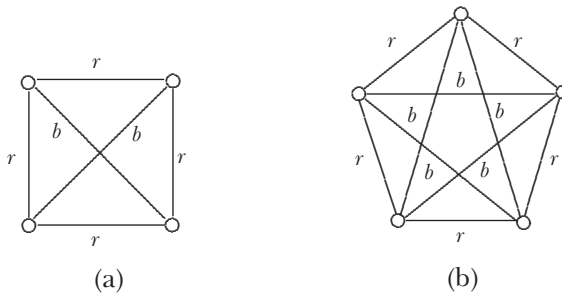


Figure 1.6. The answer to the Three Friends or Three Strangers Problem is neither four nor five.

Here too the situation can be represented by a graph, in fact by a complete graph. Suppose that four people are present at a gathering. Then we have a graph with four vertices, corresponding to the four people. We join every two vertices by an edge to indicate that these two people are friends or are strangers, resulting in the complete graph  $K_4$  with four vertices and six edges. To indicate whether two people are friends or are strangers, we color the edge red ( $r$ ) if the two people are friends and color the edge blue ( $b$ ) if the two people are strangers. Thus three mutual friends would be represented by a red triangle in our graph and three mutual strangers would be represented by a blue triangle. The situation shown in Figure 1.6a shows that with four people it is possible to avoid having three mutual friends or three mutual strangers. Likewise, when we color the complete graph  $K_5$  as in Figure 1.6b, we see that this situation can even be avoided with five people.

It turns out that the answer to the Three Friends or Three Strangers Problem is six, however. In fact, we believe that we can convince you of this, even so early in our discussion. We state this as a theorem.

**Theorem 1.1:** *The answer to the Three Friends or Three Strangers Problem is six. That is, among any six people, there must be three mutual friends or three mutual strangers.*

**Proof:** We've already seen that the answer is not five. So what we must do is consider the complete graph  $K_6$  with six vertices where each edge is colored red or blue and show that there are three vertices where all three edges joining them have the same color.

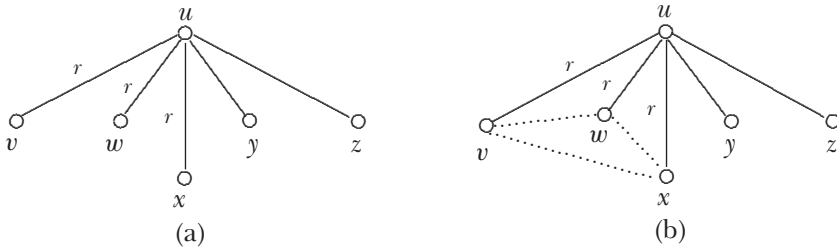


Figure 1.7. Proving Theorem 1.1.

Let's denote the vertices of  $K_6$  by  $u, v, w, x, y, z$  and look at  $u$ , say. Then there are five edges leading from  $u$  to the other five vertices. At least three of these five edges must be colored the same, say red. Suppose that three red edges lead to  $v, w$  and  $x$  as shown in Figure 1.7a. It's not important what the colors are of the edges leading  $u$  to  $y$  and  $z$ .

There are three edges joining the pairs of vertices among  $v, w$  and  $x$ . If even one of these edges is red—say the edge between  $v$  and  $w$  is red—then  $u, v$  and  $w$  represent three friends at the gathering, represented by the red triangle  $uvw$ . On the other hand, if no edge joining any two of the vertices  $v, w$  and  $x$  is red, then all three of these edges are blue, implying that  $v, w$  and  $x$  are mutual strangers at the gathering, represented by the blue triangle  $vwx$ , which is shown in Figure 1.7b where the edges of the blue triangle  $vwx$  are drawn with dashed lines. ■

Although the next problem is not well known historically, it is a practical problem and shows how graphs can be used to analyze a problem that we all might encounter.

## A Job-Hunters Problem

A counselor in a high school has contacted a number of business executives she knows for the purpose of finding summer jobs for six hard-working students: Harry, Jack, Ken, Linda, Maureen, Nancy. She found six companies, each of which is willing to offer a summer position to a qualified student who is interested in the business. The six business areas are architecture, banking, construction, design, electronics, financial. The

six students apply for these positions as follows:

- Harry: architecture, banking, construction;
- Jack: design, electronics, financial;
- Ken: architecture, banking, construction, design;
- Linda: architecture, banking, construction;
- Maureen: design, electronics, financial;
- Nancy: architecture, banking, construction.

- (a) How can this situation be represented by a graph?
- (b) Can each student obtain a job for which he or she has applied?

SOLUTION:

- (a) We construct a graph  $G$  with 12 vertices, 6 of which represent the 6 students, which we denote by H, J, K, L, M, N (the first letters of their first names), and the other 6 vertices represent the 6 positions a, b, c, d, e, f, representing architecture, banking, construction, design, electronics, financial. An edge joins two vertices if one vertex represents a business and the other represents a student who applied for a position in that business area. (See Figure 1.8.)
- (b) Yes. The edges Ha, Je, Kd, Lb, Mf, Nc within the graph  $G$  show that this is possible. (See Figure 1.9.) In this situation, Ken will have a summer job in the area of design. If this business decides that they would rather hire someone other than Ken, will all six students still be able to have a summer job for which they applied? ♦

We'll see more about these kinds of "matching" problems in Chapter 7.

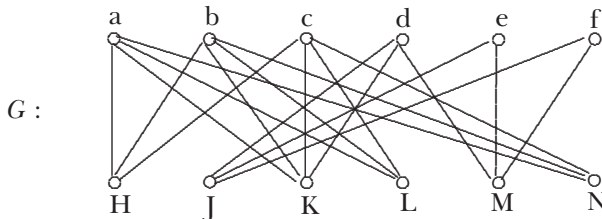


Figure 1.8. Modeling job applications by means of a graph.

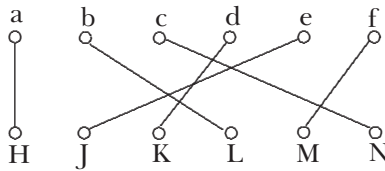


Figure 1.9. Illustrating the job situation.

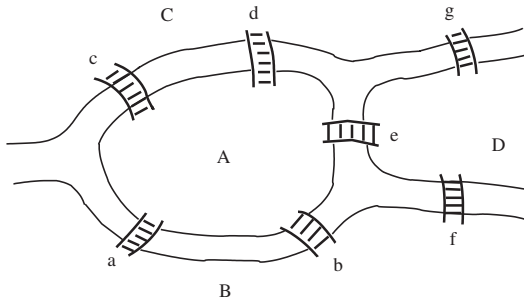


Figure 1.10. A famous problem concerning Königsberg and its seven bridges.

## NEXT, . . . FOUR FAMOUS PROBLEMS

We now look at four problems that are not only important in the history of graph theory (which we will describe later in the book) but which led to new areas within graph theory.

In 1736 the city of Königsberg was located in Prussia (in Europe). The River Pregel flowed through the city dividing it into four land areas. Seven bridges crossed the river at various locations. Figure 1.10 shows a map of Königsberg where the four land regions are A, B, C, D and the bridges are a, b, . . . , g.

### The Königsberg Bridge Problem

Is it possible to walk about Königsberg crossing each of its seven bridges exactly once?

Königsberg and this problem can be represented by a graph  $G$ —well, not exactly a graph. There are four vertices in  $G$ , one for each land region and two vertices are joined by a number of edges equal to the number of bridges joining these two land regions. What we get here is called a *multigraph* because more than one edge can join the same



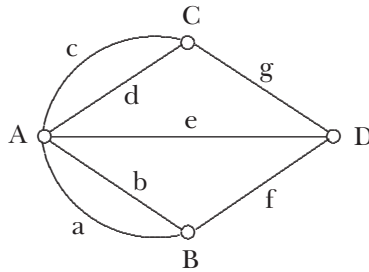


Figure 1.11. The multigraph representing the Königsberg Bridge Problem.

pair of vertices. This multigraph  $G$  is shown in Figure 1.11. In terms of this multigraph, solving the Königsberg Bridge Problem is the same as determining whether it is possible to walk about  $G$  and use each edge exactly once. Actually, there are two problems here, depending on whether we are asking whether there is a walk in Königsberg that ends where it began or whether there is a walk that ends in a land region different from the one where it began. A solution to both problems will be provided in Chapter 5.

In 1852 it was observed that in a map of England, the counties could be colored with four colors in such a way that every two counties sharing a common boundary are colored differently. This led to a much more general problem.

## The Four Color Problem

In a map consisting of regions, can the regions be colored with four or fewer colors in such a way that every two regions sharing a common boundary are colored differently?

The map in Figure 1.12 is divided into 10 regions. These regions are colored with four colors, where the colors are 1, 2, 3, 4. It turns out that the regions of this map cannot be colored with three colors so that every two regions sharing a common boundary are colored differently, however.

This example, and the Four Color Problem in general, can be looked at in terms of graphs. A point is placed in each region and, like the Problem of the Five Princes, two points are joined by a line if the regions have a common boundary. Every graph constructed in this way can be drawn without any edges crossing. Instead of coloring regions, we can color the vertices of the resulting graph so that every two vertices joined by an

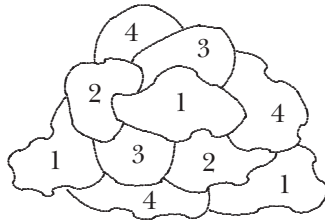


Figure 1.12. A map whose regions can be colored with four colors.

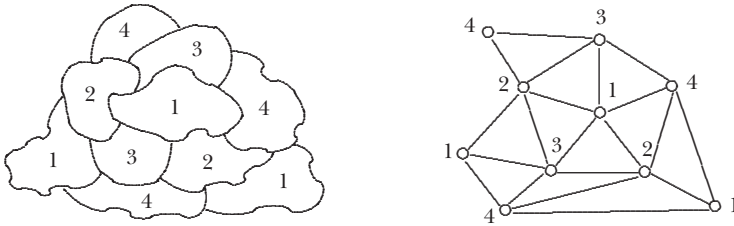


Figure 1.13. A coloring of the vertices of the graph representing the map in Figure 1.12.

edge are colored differently. This is illustrated in Figure 1.13 for the map in Figure 1.12. The Four Color Problem will be discussed in more detail in Chapter 11.

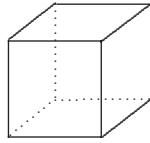
In geometry, a *polyhedron* (the plural is *polyhedra*) is a three-dimensional solid where the boundary of each face is a polygon. Figure 1.14 shows two polyhedra: the cube and the octahedron. It is common to represent the number of vertices of a polyhedron by  $V$ , the number of edges by  $E$  and the number of faces by  $F$ . These numbers for the cube and the octahedron are also given in Figure 1.14. In both cases,  $V - E + F = 2$ .

In 1750 the problem occurred as to whether  $V - E + F = 2$  was a formula for every polyhedron.

## The Polyhedron Problem

For a polyhedron with  $V$  vertices,  $E$  edges and  $F$  faces, is  $V - E + F = 2$ ?

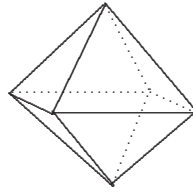
Every polyhedron can be represented by a graph whose edges do not cross. The graphs corresponding to the cube and the octahedron are shown in Figure 1.15. Here the number  $n$  of vertices of the graph is the number  $V$  of vertices of the polyhedron, the number  $m$  of edges of the



cube

$$V = 8, E = 12, F = 6$$

$$V - E + F = 8 - 12 + 6 = 2$$

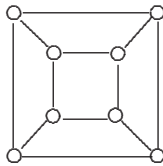


octahedron

$$V = 6, E = 12, F = 8$$

$$V - E + F = 6 - 12 + 8 = 2$$

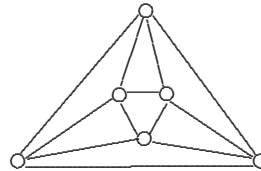
Figure 1.14. The cube and octahedron.



graph of the cube

$$n = 8, m = 12, r = 6$$

$$n - m + r = 8 - 12 + 6 = 2$$



graph of the octahedron

$$n = 6, m = 12, r = 8$$

$$n - m + r = 6 - 12 + 8 = 2$$

Figure 1.15. The graphs of the cube and octahedron.

graph is the number  $E$  of edges of the polyhedron and the number  $r$  of regions of the graph (including the outside region) is the number  $F$  of faces of the polyhedron. If it could be shown that  $n - m + r = 2$  for all such graphs, then the Polyhedron Problem would be solved. This too will be discussed in Chapter 10.

Another polyhedron is the dodecahedron, shown in Figure 1.16. For this polyhedron,  $V = 20$ ,  $E = 30$  and  $F = 12$  and, once again,  $V - E + F = 2$ . It was observed in 1856 that a round-trip can be made along edges of the dodecahedron passing through each vertex exactly once. Determining such a round-trip is known as an Around the World Problem. Consequently, a round-trip can be made along edges of the graph of the dodecahedron passing through each vertex exactly once. The graph corresponding to the dodecahedron is shown in Figure 1.17 where a trip around the world on this graph can be found by following

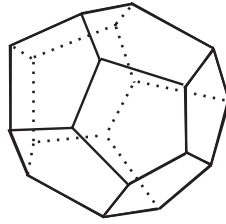


Figure 1.16. A dodecahedron.

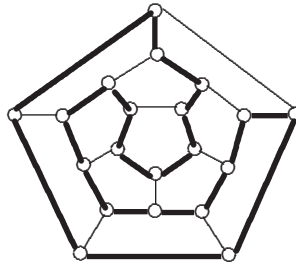


Figure 1.17. Around the world on the graph of the dodecahedron.

the edges drawn in bold. The question occurs then as to which graphs have this property.

### The Around the World Problem

Which graphs have the property that there is a round-trip along edges of the graph that passes through each vertex of the graph exactly once?

This problem will be discussed in Chapter 6.

## GRAPHS, GAMES, GALLERIES AND GRIDLOCK

The game of chess has always been considered to be a mathematical game. Perhaps then it comes as no surprise that there are puzzles and problems involving chess that have connections to graph theory. The first of these has been traced back to the year 840.

A knight is a chess piece that can move from one square to another square that is two squares forward, backward, left or right and one square

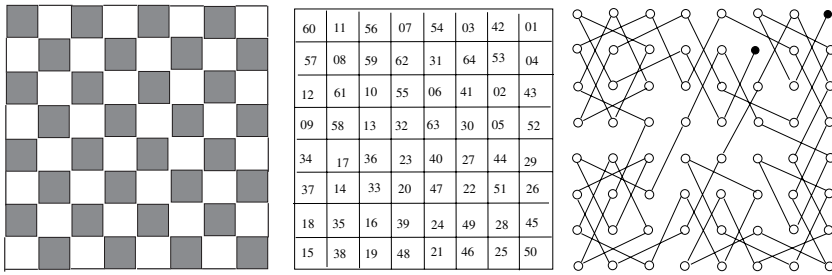


Figure 1.18. A solution of the Knight's Tour Puzzle.

perpendicular to it. A knight therefore always moves to a square whose color is different from the square where it started.

### The Knight's Tour Puzzle

Following the rules of chess, is it possible for a knight to tour an  $8 \times 8$  chessboard, visiting each square exactly once, and return to the starting square?

Figure 1.18 shows (1) a chessboard, (2) the solution of the Knight's Tour Puzzle given in 840, where the numbers on the squares indicate the order in which the squares are visited and (3) this solution given in terms of a graph, where each square of the chessboard is a vertex and where two vertices are joined by an edge if this indicates a move of the knight. This problem therefore has a great deal of similarity to the Around the World Problem mentioned in the preceding section.

The next chess problem concerns a different chess piece: the queen. The queen can move in any direction (horizontally, vertically or diagonally), any number of vacant squares. A queen is said to capture or attack a square (or a chess piece on the square) if the queen can reach that square through a single legal move. It is known that there is no way to place four queens on the squares of a chessboard so that every vacant square can be captured by a queen. The following problem asks whether this can be done with five.

### The Five Queens Puzzle

Can five queens be placed on an  $8 \times 8$  chessboard so that every vacant square can be captured by at least one of these queens?

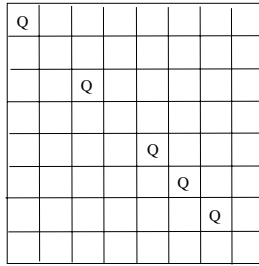


Figure 1.19. Five queens that can capture every vacant square on a chessboard.

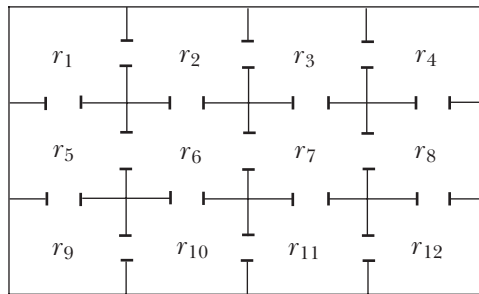


Figure 1.20. The 12 rooms in an art gallery.

The answer to this problem is yes. One possible placement of five such queens on a chessboard is shown in Figure 1.19. Once again, let  $G$  be the graph whose 64 vertices are the squares of the chessboard, where an edge joins 2 vertices if a queen can move between these two squares in a single move. The Five Queens Puzzle tells us that this graph contains 5 vertices such that each of the remaining 59 vertices is joined to at least one of these 5 vertices. We will say more about this when we discuss graph domination in Chapter 3.

**Example 1.2:** *A famous art gallery contains 12 rooms  $r_1, r_2, \dots, r_{12}$  (see Figure 1.20) in which expensive paintings are on display. Every room has exits leading to neighboring rooms.*

- (a) *Represent this situation by a graph.*
- (b) *Do there exist four rooms where security guards may be placed so that every room either contains a guard or is a neighboring room of some room containing a guard?*

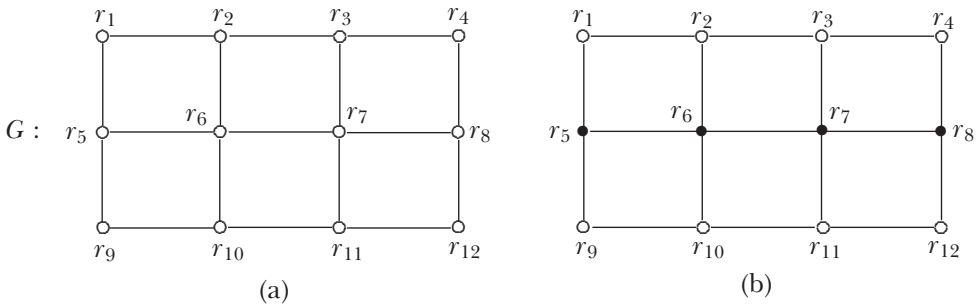


Figure 1.21. Modeling the art gallery by means of a graph.

**SOLUTION:**

- (a) Let  $G$  be a graph of order 12 where  $V = \{r_1, r_2, \dots, r_{12}\}$  and two vertices are adjacent if they represent neighboring rooms (see Figure 1.21a).
- (b) If four guards are stationed in rooms  $r_5, r_6, r_7, r_8$ , then every room either contains a guard or is a neighboring room of some room containing a guard. In the graph  $G$ , this means that every vertex is either one of the vertices  $r_5, r_6, r_7, r_8$  or is adjacent to one of these (see Figure 1.21b). This situation may suggest two other questions:
  - (1) By placing the guards in rooms  $r_5, r_6, r_7, r_8$ , the eight rooms without guards are neighboring rooms of exactly one room with a guard. It would be helpful if some of these rooms were near to more than one guard. Is it possible to place four guards in rooms in such a way that the number of rooms without guards and which are neighboring rooms of exactly one room with a guard is less than eight?
  - (2) Is it possible to place fewer than four security guards in the rooms so that every room either contains a guard or is a neighboring room of a room containing a guard?  $\blacklozenge$

**Example 1.3:** *Figure 1.22 shows an intersection of two streets where there is often heavy traffic. There are seven traffic lanes  $L_1, L_2, \dots, L_7$  where vehicles can enter the intersection of these two streets. A traffic light is located*

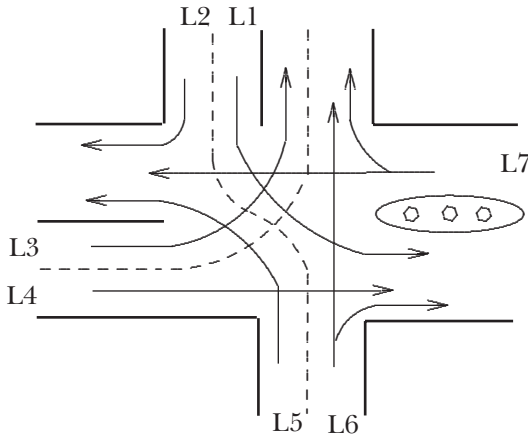


Figure 1.22. The traffic lanes at an intersection.

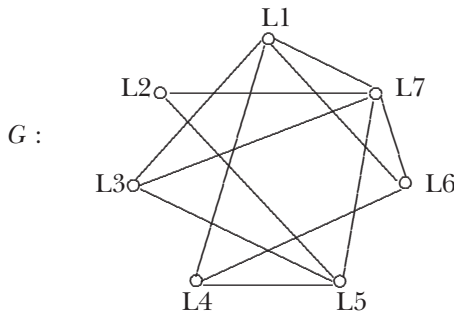


Figure 1.23. Modeling traffic lanes at an intersection by means of a graph.

at this intersection. During a certain phase of this traffic light, those cars in lanes for which the light is green may proceed safely through the intersection.

- (a) Represent this situation by a graph.
- (b) Determine whether it is possible, with four phases, for cars in all lanes to proceed safely through the intersection?

SOLUTION:

- (a) Let  $G$  be a graph with vertex set  $V = \{L1, L2, \dots, L7\}$ , where two vertices (lanes) are joined by an edge if vehicles in these two lanes cannot safely enter the intersection at the same time, as there is the possibility of an accident. (See Figure 1.23.)



- (b) Since cars in lanes L1 and L2 may proceed safely through the intersection at the same time, the traffic light can be green for both lanes at the same time. The same is true for L3 and L4, for L5 and L6 and for L7. We may represent this as  $\{L1, L2\}$ ,  $\{L3, L4\}$ ,  $\{L5, L6\}$ ,  $\{L7\}$ . This can also be accomplished as  $\{L1, L5\}$ ,  $\{L2, L6\}$ ,  $\{L3\}$ ,  $\{L4, L7\}$ . ♦

The question asked in (b) above suggests another question. Is it possible for cars in all lanes to proceed safely through the intersection where there are fewer than four phases for the traffic light? Questions of this type will be studied and answered in Chapter 11.

There are occasions when we might choose to give the edges an “orientation” to indicate a direction or perhaps a preference relation. Orienting all the edges of the complete graph  $K_n$  results in a “tournament” with  $n$  players where the orientation of an edge indicates the winner of a match played between two players. In Chapter 9, we’ll discover the following amazing result:

For any tournament with  $n$  players, there is always a way to number the players in such a way that Player 1 beat Player 2, Player 2 beat Player 3, Player 3 beat Player 4 and so up to Player  $n - 1$  beat Player  $n$ .

Figure 1.24a shows the complete graph  $K_5$  where the labels  $u, v, w, x, y$  indicate five players where every two will participate in some sports match. Figure 1.24b shows the outcome of these 10 matches. By referring to the players  $u, v, w, x, y$  as 2, 3, 5, 1, 4, respectively, we see that Player 1 beat Player 2, Player 2 beat Player 3 and so on, as shown in Figure 1.24c.

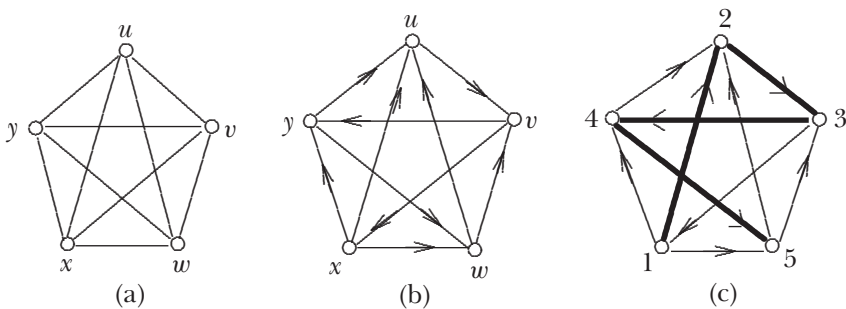


Figure 1.24. The outcome of a five-player tournament.

## THE ARRIVAL OF GRAPH THEORY

While the games, puzzles, problems and results we've mentioned were not initially part of graph theory as there was not yet an area of mathematics called graph theory, all this changed in 1891 when the first purely theoretical article dealing with graphs as mathematical objects was written by the Danish mathematician Julius Petersen (1839–1910). That he called these objects by the name “graph” was perhaps the deciding factor in that name being used from that time on. While we have used this term a number of times, we have yet to give a formal definition of a graph and describe some of the basic terminology associated with graphs. It is now time to do this. Although there are some variations on how mathematicians define these terms, the definitions we are about to present are among the most common.

As we have noted and the reader will have already experienced, a graph can be represented by a diagram, like the one in Figure 1.25. Associated with a graph, there are two sets, a vertex set  $V$  and an edge set  $E$ . For instance, the graph  $H$  in Figure 1.25 has  $V = \{u, v, w, x, y, z\}$  and  $E = \{uv, ux, uz, vw, vx, vz, xz\}$ . In every graph, these sets are finite so they cannot have an infinite number of points. Also, each element of  $E$  consists of two different elements of  $V$ , where the order does not matter. (For instance, the edge  $uv$  is the same as the edge  $vu$ .) Mathematicians refer to the elements of  $E$  as 2-element subsets of  $V$ .

The formal definition of a graph adopted by mathematicians goes as follows. A *graph*  $G$  is a finite nonempty set  $V$  of objects called *vertices* (the singular is *vertex*) together with a set  $E$  consisting of 2-element subsets of  $V$ . Each element of  $E$  is called an *edge* of  $G$ . The sets  $V$  and  $E$  are called the *vertex set* and *edge set* of  $G$ . In fact,  $G$  is often written as  $G = (V, E)$ . Sometimes the vertex set and edge set of a graph  $G$  are expressed as  $V(G)$

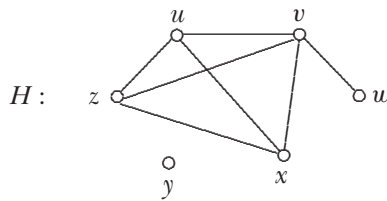


Figure 1.25. A graph.

and  $E(G)$ , respectively, to emphasize that the graph  $G$  is involved.

The number of vertices in a graph  $G$  is called the *order* of  $G$  and the number of edges in  $G$  is its *size*.

The order and size of a graph are typically denoted by  $n$  and  $m$ , respectively. The order of the graph  $H$  of Figure 1.25 is  $n = 6$  and its size is  $m = 7$ . We often represent a graph  $G$  by means of a diagram (and refer to the diagram as the graph), where each vertex is indicated by a small circle (and called a vertex) and an edge  $ab$  is indicated by placing a straight-line segment or a curve between the vertices  $a$  and  $b$ . Although the edges  $ux$  and  $vz$  intersect in the diagram of the graph  $H$  of Figure 1.25, the point of intersection is *not* a vertex of  $H$ .

Since  $e = uv$  is an edge of the graph  $H$  of Figure 1.25, the vertices  $u$  and  $v$  are said to be *adjacent* and  $e$  is said to *join* the vertices  $u$  and  $v$ . The vertices  $u$  and  $w$  are *nonadjacent vertices*. Since  $u$  and  $v$  are adjacent vertices, they are *neighbors* of each other. Because  $e = uv$  is an edge of  $H$ , the vertex  $u$  and the edge  $e$  are *incident*, as are  $v$  and  $e$ . Since  $uv$  and  $vw$  are incident with the same vertex  $v$ , they are *adjacent edges*. The edges  $ux$  and  $vz$  are not adjacent.

When a graph  $G$  is given in terms of a diagram and we want to refer to the vertex set of  $G$  or discuss particular vertices in  $G$ , it is useful to assign each vertex of  $G$  a label. In this case,  $G$  is called a *labeled graph*. On the other hand, if there is no particular advantage to labeling the vertices of  $G$ , then  $G$  is an *unlabeled graph*. The graph  $F$  of Figure 1.26 is labeled, while the graph  $J$  of Figure 1.26 is unlabeled.

A graph with exactly one vertex is a *trivial graph*. Thus a *nontrivial graph* has order at least 2. A graph with no edges is an *empty graph*. A *nonempty graph* therefore has at least one edge.

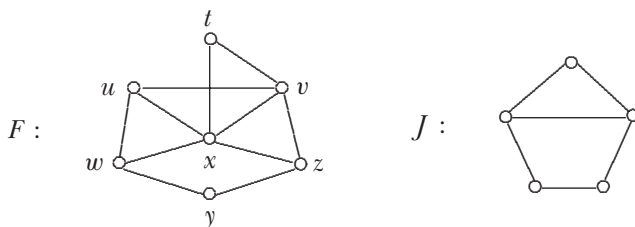


Figure 1.26. A labeled graph and an unlabeled graph.

## THE FIRST THEOREM OF GRAPH THEORY

The number of edges incident with a vertex  $v$  in a graph  $G$  is called the *degree* of  $v$  and denoted by  $\deg_G v$ . When the graph  $G$  under consideration is understood, we write the degree of  $v$  more simply as  $\deg v$ . If  $v$  is a vertex in a graph  $G$  with  $n$  vertices, then  $v$  can have at most  $n - 1$  neighbors; that is, if  $v$  is a vertex in a graph  $G$  of order  $n$ , then  $0 \leq \deg v \leq n - 1$ . In the graph  $H$  of Figure 1.25,

$$\deg y = 0, \deg w = 1, \deg u = 3, \deg x = 3, \deg z = 3, \deg v = 4.$$

A vertex of degree 0 is an *isolated vertex*, while a vertex of degree 1 is called an *end-vertex*. The vertex  $y$  is therefore an isolated vertex and  $w$  is an end-vertex. The *minimum degree* of a vertex of  $G$  is denoted by  $\delta(G)$  or simply  $\delta$  while the *maximum degree* is denoted by  $\Delta(G)$  or  $\Delta$ , where  $\delta$  and  $\Delta$  are the lowercase and uppercase Greek letter delta, respectively. For the graph  $H$  of Figure 1.25,  $\delta = 0$  and  $\Delta = 4$ . Notice that when the degrees of the vertices of any graph  $G$  are added, each edge of  $G$  is counted twice. This gives us the following result, namely a theorem that is sometimes called the *First Theorem of Graph Theory* since it is felt that if anyone were to study graph theory on his or her own, this would likely be the first result he or she would discover.

In any graph, the sum of the degrees of the vertices is twice of the number of edges.

More formally, and reinforcing your vocabulary, we state this fact as follows.

**Theorem 1.4:** *Let  $G$  be a graph of order  $n$  and size  $m$  with vertices  $v_1, v_2, \dots, v_n$ . Then*

$$\deg v_1 + \deg v_2 + \dots + \deg v_n = 2m.$$

So how many edges can a graph of order  $n$  have? For a graph to have maximum size, it would have to be complete, where every vertex is adjacent to all other vertices and so every vertex would have degree  $n - 1$ . Applying Theorem 1.4 then tells us that

$$(n - 1) + (n - 1) + \dots + (n - 1) = 2m$$

and therefore  $m = n(n - 1)/2$ . Another way that we may answer this question is to observe that for an edge  $e$ , there are  $n$  choices for one vertex in  $e$  and  $n - 1$  choices for the other vertex, or  $n(n - 1)$  choices in all. But since  $vu$  is the same as  $uv$  for every edge  $uv$ , the number  $n(n - 1)$  counts each edge twice and so  $m = n(n - 1)/2$ .

Some people have referred to Theorem 1.4 as the *Handshaking Lemma*. (A lemma is a mathematical result, usually not of primary interest but useful in proving some theorem of greater interest.) Suppose that there was a gathering of people, some pairs of whom shook hands with each other, followed by an inquiry as to how many hands each person shook. If all of these numbers were added, we would arrive at an even number, namely twice the total number of handshakes that took place.

A vertex is called *even* or *odd* according to whether its degree is even or odd. The graph  $H$  in Figure 1.25 then has two even vertices and four odd vertices. That the graph  $H$  has an even number of odd vertices is a consequence of Theorem 1.4.

**Corollary 1.5:** *Every graph has an even number of odd vertices.*

**Proof:** According to Theorem 1.4, when we add the degrees of the even vertices and the degrees of the odd vertices of any graph, the result is always an even number. Thus the sum of the degrees of all odd vertices is even, implying that the graph must have an even number of odd vertices. ■

Degrees of vertices will be discussed in considerable detail in Chapter 2.