Chapter One

A review: The Laplacian and the d'Alembertian

1.1 THE LAPLACIAN

One of the main goals of this course is to understand well the solution of wave equation both in Euclidean space and on manifolds and then to use this knowledge to derive properties of eigenfunctions on Riemannian manifolds. This is a very classical idea. A key step in understanding properties of solutions of wave equations on manifolds will be to compute the types of distributions that include the fundamental solution of the wave operator in Minkowski space (d'Alembertian),

$$\Box = \partial^2/\partial t^2 - \Delta, \tag{1.1.1}$$

with

$$\Delta = \sum_{j=1}^{n} \partial_j^2, \quad \partial_j = \partial/\partial x_j, \tag{1.1.2}$$

being the Euclidean Laplacian on $\mathbb{R}^n$.

In the next section we shall compute fundamental solutions for $\Box$, which is central to our goal. Here, though, since it will serve for us as a good model, we shall compute the fundamental solution for Δ.

Recall that the fundamental solution of a partial differential operator $P(D) = \sum a_\alpha \partial^\alpha$ is a distribution E for which

$$P(D)E = \delta_0, \tag{1.1.3}$$

where δ_0 is the Dirac-delta distribution $\langle \delta_0, f\rangle = f(0)$, $f \in \mathcal{S}(\mathbb{R}^n)$. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of length $|\alpha| = \sum_j \alpha_j$, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, $D = \frac{1}{i}\partial/\partial x$, and $\mathcal{S}(\mathbb{R}^n)$ is the space of Schwartz class functions on $\mathbb{R}^n$, whose dual is the space of tempered distributions, $\mathcal{S}'(\mathbb{R}^n)$. If $\phi \in \mathcal{S}'(\mathbb{R}^n)$, then $\langle \phi, f\rangle$ denotes the value of ϕ acting on f. For later use, we shall also set $\alpha! = \alpha_1!\,\alpha_2!\cdots\alpha_n!$.

Using the fundamental solution, one can solve the equation

$$P(D)u = F.$$

In fact, by (1.1.3), $u = E * F$ satisfies

$$P(D)\big(E * F\big) = \big(P(D)E\big) * F = \delta_0 * F = F. \tag{1.1.4}$$

Here "$*$" denotes convolution, initially defined for say $f, g \in \mathcal{S}(\mathbb{R}^n)$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy, \tag{1.1.5}$$

and then extended to distributions by approximating them by functions. Also, you can justify the first equality in (1.1.4) by using (1.1.5).

We shall assume basic facts about distributions. Besides $\mathcal{S}'(\mathbb{R}^n)$, there is also $\mathcal{D}'(\mathbb{R}^n)$, the dual space of $C_0^\infty(\mathbb{R}^n)$ (compactly supported C^∞ functions), and $\mathcal{E}'(\mathbb{R}^n)$ (compactly supported distributions), which is the dual of the dual of $C^\infty(\mathbb{R}^n)$. In the appendix we review the basic facts that we use here and elsewhere. The reader can also refer to many texts that cover the theory of distributions, including [37], [61] and [73].

Let us now derive a fundamental solution of Δ. Thus, we seek a $E \in \mathcal{S}'$ so that

$$\Delta E = \delta_0.$$

Since both Δ and δ_0 are invariant under rotations in $\mathbb{R}^n$, it is natural to expect that E also has this property. In other words, we expect that $E(x) = f(|x|) = f(r)$, where $r = |x| = \sqrt{\sum_{j=1}^n x_j^2}$. Assuming for now that f is smooth for $r > 0$, since δ_0 is supported at the origin, we would have that

$$\begin{aligned}\Delta f(|x|) &= \sum_j \partial_j \partial_j f(|x|) = \sum_j \partial_j \big(f'(|x|)x_j/|x|\big) \\ &= \sum_j \big(f''(|x|)x_j^2/|x|^2 + f'(|x|)(1/|x| - x_j^2/|x|^3) \big) = 0, \quad x \neq 0,\end{aligned}$$

and therefore

$$f''(r) + \frac{n-1}{r} f'(r) = 0, \quad r = |x| > 0. \tag{1.1.6}$$

The last equation suggests that E should be homogeneous of degree $2-n$. Recall that $u \in \mathcal{S}'$ is homogeneous of degree σ if $\langle u, f_\lambda \rangle = \lambda^\sigma \langle u, f \rangle$, $f \in \mathcal{S}$, where $f_\lambda(x) = \lambda^{-n} f(x/\lambda)$, $\lambda > 0$ is the λ L^1-normalized dilate of f. This reasoning turns out to be correct for $n \geq 3$, but not for $n = 2$ since then $2-n = 0$ and constant functions are the ones on $\mathbb{R}_+$ which are homogeneous of degree 0, and then cannot give us our fundamental solution of the Laplacian on $\mathbb{R}^2$. But since $\ln r$ also solves (1.1.6) when $n = 2$ and r^{2-n} is a solution of the equation for $n \geq 3$, perhaps

$$f(r) = a_n r^{2-n},\ n \geq 3, \quad \text{and } f(r) = a_n \ln r,\ n = 2,$$

will work for the appropriate constants a_n.

Specifically, we claim that we can choose the a_n so that if

$$E(x) = a_n r^{2-n},\ n \geq 3, \quad \text{and } \ E(x) = a_n \ln |x|,\ n = 2, \tag{1.1.7}$$

then for $g \in \mathcal{S}$ we have

$$g(0) = \langle E, \Delta g \rangle, \quad g \in \mathcal{S}, \tag{1.1.8}$$

since $g(0) = \langle \delta_0, g \rangle = \langle \Delta E, g \rangle = \langle E, \Delta g \rangle$, for all $g \in \mathcal{S}$ if and only if $\delta_0 = \Delta E$.

To verify (1.1.8), we shall need to use the divergence theorem. Note that $\nu = -x/|x|$ is the outward unit normal for a point x on the complement of the sphere of radius $\varepsilon > 0$ centered at the origin. Thus, for $n \geq 3$, if $d\sigma$ denotes the induced

Lebesgue measure on this sphere,

$$\begin{aligned}\langle E, \Delta g\rangle &= \int_{\mathbb{R}^n} \frac{a_n}{|x|^{n-2}} \Delta g(x)\, dx \\ &= \lim_{\varepsilon\to 0+} \int_{|x|\geq\varepsilon} \frac{a_n}{|x|^{n-2}} \sum_j \partial_j \big(\partial_j g(x)\big)\, dx \\ &= \lim_{\varepsilon\to 0+} \Big[-\int_{|x|\geq\varepsilon} \sum_j \partial_j \Big(\frac{a_n}{|x|^{n-2}} \Big) \partial_j g(x)\, dx \\ &\qquad - \int_{|x|=\varepsilon} \frac{a_n}{|x|^{n-2}} \sum_j \frac{x_j}{|x|} \partial_j g(x)\, d\sigma \Big] \\ &= \lim_{\varepsilon\to 0_+} \Big[\int_{|x|\geq\varepsilon} \sum \partial_j^2 \Big(\frac{a_n}{|x|^{n-2}} \Big) g(x)\, dx \\ &\qquad - \sum_j \int_{|x|=\varepsilon} \frac{a_n}{|x|^{n-2}} \frac{x_j}{|x|} \partial_j g(x)\, d\sigma \\ &\qquad + \sum_j \int_{|x|=\varepsilon} \partial_j \Big(\frac{a_n}{|x|^{n-2}} \Big) \frac{x_j}{|x|} g(x)\, d\sigma \Big].\end{aligned}$$

The first term in the right vanishes since, as noted above, $\Delta|x|^{-n+2} = 0$, when $x \neq 0$. For a given $\varepsilon > 0$, the second term is bounded by

$$\sup_x |\nabla g(x)| a_n \varepsilon^{-n+2} \int_{|x|=\varepsilon} d\sigma \leq C\varepsilon,$$

and so its limit is 0. The last term is

$$-\int_{|x|=\varepsilon} \frac{(n-2)a_n}{|x|^{n-1}} g(x)\, d\sigma = -\frac{(n-2)a_n}{\varepsilon^{n-1}} \int_{|x|=\varepsilon} g(x)\, d\sigma,$$

which, as $\varepsilon \to 0_+$, tends to

$$-(n-2)a_n g(0) \int_{|x|=1} d\sigma.$$

Therefore, if $A_n = |S^{n-1}| = \int_{|x|=1} d\sigma$ denotes the area of the unit sphere in $\mathbb{R}^n$, we have the desired identity (1.1.8) for $n \geq 3$ if

$$a_n = \frac{-1}{(n-2)A_n}, \quad n \geq 3. \tag{1.1.9}$$

We leave it as an exercise for the reader that we also have (1.1.8) for $n = 2$ if we set

$$a_2 = \frac{1}{2\pi}. \tag{1.1.10}$$

The minus signs in (1.1.9) are due to the fact that the Laplacian has a negative symbol, $-|\xi|^2$.

Let us record what we have done in the following.

Theorem 1.1.1 *If $n \geq 3$ and $E(x) = a_n|x|^{2-n}$ or $n = 2$ with $E(x) = a_n \ln|x|$, with a_n defined by* (1.1.9)–(1.1.10)*, then E is a fundamental solution of Δ, i.e., $\Delta E = \delta_0$.*

The fundamental solution for Δ, (1.1.7), that we have just computed is not unique since others are given by $E(x) + h(x)$ if $h(x)$ is a harmonic function, i.e., $\Delta h(x) = 0$ for all $x \in \mathbb{R}^n$. E, though, for $n \geq 3$ is the unique fundamental solution vanishing at infinity.

Remark 1.1.2 To help motivate computations that we shall carry out in the next chapter and throughout the text, let us see how the fundamental solution E for Δ in Theorem 1.1.1 is pulled back via a linear bijection $T : \mathbb{R}^n \to \mathbb{R}^n$. If $y = Tx$ then

$$T^t \frac{\partial}{\partial y} = \frac{\partial}{\partial x}$$

and so

$$\Delta_y = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2} = \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} g^{jk} \frac{\partial}{\partial x_k} = \Delta_g,$$

$$\text{with } g^{jk} = T^{-1}\,(T^{-1})^t. \quad (1.1.11)$$

Then $\Delta_y u(y) = F(y)$ is equivalent to $\Delta_g u(Tx) = F(Tx)$. In other words, the pullback of Δ_y via T is Δ_g. If $n \geq 3$, since $a_n|y|^{2-n} = a_n|x|_g^{2-n}$ if

$$|x|_g^2 = \sum_{j,k=1}^{n} g_{jk} x_j x_k, \quad (g_{jk}) = (g^{jk})^{-1} = T^t\,T,$$

we conclude that

$$\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} g^{jk} \frac{\partial}{\partial x_k} \big(a_n |x|_g^{2-n} \big) = \Delta_y \big(a_n |y|^{2-n} \big) = \delta_0(y)$$

$$= \delta_0(Tx) = |\text{det} T^{-1}|\, \delta_0(x) = |g|^{-\frac{1}{2}}\, \delta_0(x), \quad (1.1.12)$$

if

$$|g| = \det\,(g_{jk}).$$

Note that, by a theorem of Jacobi (Sylvester's law of inertia), every symmetric positive definite matrix g^{jk} can be written in the above form, $g^{jk} = LL^t$, where L is real and invertible. Consequently, if we take $T = L^{-1}$ and $T^tT = (g_{jk}) = (g^{jk})^{-1}$, then the above argument shows that if $n \geq 3$, then

$$a_n |g|^{1/2} \big(\sum_{j,k} g_{jk} x_j x_k\big)^{(2-n)/2}$$

is a fundamental solution for $\sum_{j,k} g^{jk} \partial_j \partial_k$.

This decomposition $g^{jk} = LL^t$ is not unique; however, it is if we require that L is a lower triangular matrix with strictly positive diagonal entries (the *Cholesky decomposition* from linear algebra). We shall make this assumption in all such factorizations to follow. Note that the matrix L in this decomposition is, roughly speaking, the matrix analog of taking the square root of a positive number.

Let us conclude this section by reviewing one more equation involving the Laplacian.

As we pointed out before, we can use E to solve Laplace's equation $\Delta u = F$. Let us briefly study one more important equation involving Δ, the Dirichlet problem for $\mathbb{R}^{1+n} = [0,\infty) \times \mathbb{R}^n$, since it will also have some relevance in our calculation of fundamental solutions for the d'Alembertian, $\Box = \partial_t^2 - \Delta$.

If $(y,x) \in [0,\infty) \times \mathbb{R}^n$, the Dirichlet problem for this upper half-space is to show that for a given $f \in \mathcal{S}(\mathbb{R}^n)$ on the boundary we can find a function $u(y,x)$ that is harmonic in $\mathbb{R}^{1+n}_+$ with boundary values f, i.e., a solution of

$$\begin{cases} \big(\frac{\partial^2}{\partial y^2} + \Delta\big)u(y,x) = 0, \quad (y,x) \in (0,\infty) \times \mathbb{R}^n \\ \\ u(0,x) = f(x). \end{cases} \tag{1.1.13}$$

By using the fact that in polar coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \Delta_{S^{n-1}},$$

where $r = |x|$ and $\Delta_{S^{n-1}}$ is the induced Laplacian on S^{n-1} (see § 3.4), one easily checks that if

$$P(y,x) = \frac{y}{(y^2+|x|^2)^{(n+1)/2}},$$

then

$$\big(\frac{\partial^2}{\partial y^2} + \Delta\big)P(y,x) = 0, \quad (y,x) \in (0,\infty) \times \mathbb{R}^n.$$

Therefore, if b_n is a constant and $f \in \mathbb{S}$,

$$u(y,x) = b_n \int_{\mathbb{R}^n} P(y, x-w)\, f(w)\, dw \tag{1.1.14}$$

is harmonic in $(0,\infty) \times \mathbb{R}^n$, i.e., it satisfies the first part of (1.1.13). Since

$$P(y,x) = y^{-n}P(1, x/y) = y^{-n}(1+|x/y|^2)^{-(n+1)/2},$$

we also have that as $y \to 0_+$

$$u(y,x) = b_n \int_{\mathbb{R}^n} (1+|w|^2)^{-(n+1)/2}\, f(x-yw)\, dw \to f(x),$$

provided that the constant b_n is chosen so that

$$b_n \int_{\mathbb{R}^n} (1+|x|^2)^{-(n+1)/2}\, dx = 1,$$

in other words

$$b_n = \pi^{-(n+1)/2}\Gamma\big(\frac{n+1}{2}\big), \tag{1.1.15}$$

with

$$\Gamma(z) = \int_0^\infty e^{-s}s^{z-1}\, ds, \quad \text{Re } z > 0, \tag{1.1.16}$$

being Euler's gamma function. Thus we have shown that the Poisson integral of f, (1.1.14), with b_n given by (1.1.15) solves the Dirichlet problem for the upper half-space, (1.1.13). Note also that the constant b_n in (1.1.15) is also given by the formula

$$b_n = \frac{2}{A_{n+1}}, \tag{1.1.17}$$

where, as before, $A_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ denotes the area of the unit sphere, S^{d-1}, in $\mathbb{R}^d$.

1.2 FUNDAMENTAL SOLUTIONS OF THE D'ALEMBERTIAN

In this section we shall compute fundamental solutions for the d'Alembertian in $\mathbb{R}^{1+n}$. Thus we seek distributions E for which we have

$$\Box E = \delta_{0,0}(t,x), \quad \Box = \partial_t^2 - \Delta, \ (t,x) \in \mathbb{R}^{1+n}. \tag{1.2.1}$$

Here $\delta_{0,0}$ is the Dirac delta distribution centered at the origin in space-time.

Recall that the Lorentz transformations are the linear maps from $\mathbb{R}^{1+n}$ to itself preserving the Lorentz form

$$Q(t,x) = t^2 - |x|^2, \tag{1.2.2}$$

which is the natural quadratic form associated with $\Box$. Since both $\Box$ and $\delta_{0,0}$ are invariant under these transformations, we expect E to also enjoy this property. Thus, we expect it to be of the form $E = f(t^2 - |x|^2)$ where f is some distribution. If we plug this into our equation (1.2.1) and we use the polar coordinates formula for Δ, we can see that if E were of this form then we would have to have that

$$f''(\rho) + \frac{n+1}{2\rho} f'(\rho) = 0, \quad \text{when} \ \ \rho = t^2 - |x|^2 \neq 0. \tag{1.2.3}$$

From this we expect f to be homogeneous of degree $(1-n)/2$. If

$$H(s) = \begin{cases} 1, & s \geq 0 \\ 0, & s < 0, \end{cases} \tag{1.2.4}$$

denotes the Heaviside step function on $\mathbb{R}$, then we can write down solutions of (1.2.3) with this homogeneity. Specifically, when $n \leq 3$ the equation has the solution

$$f(\rho) = \begin{cases} c_1 H(\rho), & n = 1 \\ c_2 H(\rho)\rho^{-1/2}, & n = 2 \\ c_3 \delta(\rho), & n = 3. \end{cases}$$

Thus, we expect that for appropriate constants c_n the following are fundamental solutions for $\Box$ in spatial dimensions $n = 1, 2, 3$

$$E = \begin{cases} c_1 H(t) H(t^2 - |x|^2), & n = 1 \\ \\ c_2 H(t) H(t^2 - |x|^2)(t^2 - |x|^2)^{-1/2}, & n = 2 \\ \\ c_3 H(t) \delta(t^2 - |x|^2), & n = 3. \end{cases} \tag{1.2.5}$$

We added the factor $H(t)$ since, as we shall see below when we solve the Cauchy problem for $\Box$, it is natural to want our fundamental solution to be supported in $\mathbb{R}^{1+n}_+ = [0,\infty) \times \mathbb{R}^n$. When $n = 3$ our guess involves $\delta(t^2 - |x|^2)$, which is the Leray measure in $\mathbb{R}^{1+3}$ associated with the function $t^2 - |x|^2$ (see Theorem A.4.1 in the appendix).

Let us show that our guess is correct when $n = 2$, since this will serve as a model for arguments to follow. Thus, we wish to see that c_2 can be chosen so that whenever $F \in \mathcal{S}(\mathbb{R}^{1+2})$ we have

$$F(0,0) = c_2 \int_{\mathbb{R}^{1+2}} H(t) H(t^2 - |x|^2)(t^2 - |x|^2)^{-1/2} \left(\Box F\right)(t,x) \, dt dx. \tag{1.2.6}$$

To simplify the integration by parts arguments, we note that we can regularize our guess by extending it into the complex plane and taking limits. Specifically, instead of truncating the distribution about its singularity as we did for the fundamental solution of the Laplacian, we shall use the fact that

$$H(t^2-|x|^2)(t^2-|x|^2)^{-1/2} = \lim_{\varepsilon\to 0_+} \operatorname{Im}\, \big(\,|x|^2-(t+i\varepsilon)^2\,\big)^{-1/2},$$

due to the fact that $\lim_{\varepsilon\to 0_+}(|x|^2-(t+i\varepsilon)^2)^{-1/2}$ is real when $|x|^2>t^2$, and of positive imaginary part if (t,x) is fixed with $|x|^2<t^2$ and $\varepsilon>0$ small. Therefore, (1.2.6) is equivalent to showing that c_2 can be chosen so that we always have

$$F(0,0) = c_2 \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^2}\int_0^\infty \operatorname{Im}\, \big(\,|x|^2-(t+i\varepsilon)^2\,\big)^{-1/2} \big(\partial_t^2-\Delta\big)F(t,x)\,dtdx. \quad (1.2.7)$$

In addition to motivating what is to follow, the advantage of (1.2.7) over (1.2.6) is that the integration by parts arguments that we require are easy for the latter. If we use the polar coordinates formula for the Laplacian it is not difficult to check that

$$(\partial_t^2-\Delta)\big(\,|x|^2-(t+i\varepsilon)^2\,\big)^{-1/2}=0, \quad \varepsilon>0, \quad (t,x)\in\mathbb{R}^{1+2},$$

and of course we also have that for $\varepsilon>0$, $\operatorname{Im}\,(|x|^2-(t+i\varepsilon)^2)^{-1/2}=0$ when $t=0$. Therefore, if we integrate by parts, we find that when $\varepsilon>0$ we have

$$\begin{aligned} c_2 &\int_{\mathbb{R}^2}\int_0^\infty \operatorname{Im}\, \big(\,|x|^2-(t+i\varepsilon)^2\,\big)^{-1/2} \big(\partial_t^2-\Delta\big)F(t,x)\,dtdx \\ &= c_2 \int_{\mathbb{R}^2} \frac{\partial}{\partial t}\Big(\operatorname{Im}\, \big(\,|x|^2-(t+i\varepsilon)^2\,\big)^{-1/2}\Big)\Big|_{t=0} F(0,x)\,dx \\ &= c_2 \int_{\mathbb{R}^2} \frac{\varepsilon}{(|x|^2+\varepsilon^2)^{3/2}}\,F(0,x)\,dx. \end{aligned}$$

The last integral involves the Poisson integral of $x\to F(0,x)$, which, as we showed in the last section, will tend to $F(0,0)$ as $\varepsilon\to 0_+$ provided that c_2 is the constant b_2 in (1.1.15).

Thus, we have verified our claim that for $n=2$, (1.2.5) provides a fundamental solution for $\Box$. Before proceeding further, we urge the reader to verify the assertion for the other two cases there, namely, $n=1$ and $n=3$, and compute c_1 and c_3.

Before we calculate fundamental solutions in other dimensions besides $n=2$, let us try to add some perspective to what we have just done. Recall that for $n\ge 3$ the fundamental solution of $-\Delta$ was a constant multiple of $|x|^{2-n}$. We can think of the latter as the pullback of $H(r)r^{(2-n)/2}$ via the map $x\to r^2=\sum x_j^2$, and the latter is the metric form for Δ. Since the space-time dimension is larger by one, ideally we would like to obtain fundamental solutions for $\Box$ by pulling back distributions like $r^{(1-n)/2}$ via the Lorentz form $Q(t,x)=t^2-|x|^2$. The problem that arises, of course, is that since Q is a semidefinite form (unlike the one for Δ), we cannot do this directly. In our calculation for $n=2$, we in effect did realize our goal of obtaining a fundamental solution of $\Box$ via a regularization argument that produced a distribution which was the limit of smooth functions involving the appropriate power of complex extensions of Q. We shall do an analogous thing when $n\ge 3$.

Let us be more specific. We first define distributions

$$W_a(t,x) = \lim_{\varepsilon\to 0_+} \operatorname{Im}\, \big(\,|x|^2-(t+i\varepsilon)^2\,\big)^a, \quad a\in -\frac{1}{2}\mathbb{N}, \quad (1.2.8)$$

by which we mean that for $F(t,x) \in \mathcal{S}(\mathbb{R}^{1+n})$,

$$\langle W_a, F\rangle = \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^{1+n}} F(t,x)\, \mathrm{Im}\,\big(\,|x|^2 - (t+i\varepsilon)^2\,\big)^a\, dtdx.$$

Note that if $|x|^2 > t^2 > 0$, $\lim_{\varepsilon\to 0_+}(|x|^2 - (t+i\varepsilon)^2)^a = (|x|^2 - t^2)^a$ is real for $a = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots$, while for $a = -\nu - 1$, $\nu = 0, 1, 2, \dots$ we leave it as an exercise for the reader to see that $W_a(t,x)$ is a constant multiple of $\mathrm{sgn}\, t\, \delta^{(\nu)}(t^2 - |x|^2)$, with the constant being equal to π when $\nu = 1$ (show this!, cf. also formula (3.2.10) in [37]). Therefore,

$$\mathrm{supp}\; W_a \subset \{\, (t,x) \in \mathbb{R}^{1+n} : |x|^2 \le t^2 \,\}, \tag{1.2.9}$$

which is the union of the forward and backward light cones through the origin. Note also that a simple integration by parts argument in the $r = |x|$ variable shows that these distributions are well-defined.

The distributions in (1.2.8) are the ones that will give us our fundamental solutions for $\Box$ in $\mathbb{R}^{1+n}$, $n \ge 2$ with $a = -(n-1)/2$. To handle variable coefficient operators and construct the Hadamard parametrix we shall also require natural extensions of the above definitions to $a \in \frac{1}{2}\mathbb{Z} \setminus -\frac{1}{2}\mathbb{N}$. Based on (1.2.9) it is natural to set

$$W_a(t,x) = H(t^2 - |x|^2)(t^2 - |x|^2)^a, \quad a \in \frac{1}{2}\mathbb{N}, \tag{1.2.10}$$

and

$$W_0(t,x) = H(t^2 - |x|^2), \tag{1.2.11}$$

with H being the Heaviside function.

Since $\Delta = \partial^2/\partial r^2 + (n-1)r^{-1}\partial/\partial r + \Delta_{S^{n-1}}$, the reader can verify that for $a \in \frac{1}{2}\mathbb{Z}$ and $t > 0$

$$\Box W_a = 2|a|(n - 1 + 2a)W_{a-1}, \quad a \ne 0,\ -(n-1)/2, \tag{1.2.12}$$

while for $a = 0$ and $t > 0$

$$\Box W_0 = 2(n-1)\pi^{-1} W_{-1}. \tag{1.2.13}$$

The limit as $\varepsilon \to 0_+$ of the other exceptional case of $a = -(n-1)/2$ ($n \ge 2$) is a candidate for a fundamental solution since it is a distribution which is invariant under the Lorentz group, and, moreover, homogeneous of degree $2 - d$, where $d = n + 1$ is the space-time dimension. What is more, we have

$$\Box\big(\,|x|^2 - (t+i\varepsilon)^2\,\big)^{-(n-1)/2} = 0, \quad \varepsilon > 0,\ (t,x) \in \mathbb{R}^{1+n}. \tag{1.2.14}$$

Indeed, we claim that if c_n is chosen appropriately, then the distribution

$$E_+ = c_n H(t) \lim_{\varepsilon\to 0_+} \mathrm{Im}\,\big(\,|x|^2 - (t+i\varepsilon)^2\,\big)^{-(n-1)/2},\ n \ge 2, \tag{1.2.15}$$

is a fundamental solution of $\Box$, i.e., $\Box E_+ = \delta_{0,0}$.

Thus, we need to verify that c_n, $n \ge 2$, can be chosen so that whenever $F(t,x) \in \mathcal{S}(\mathbb{R}^{1+n})$, we have

$$F(0,0) = \lim_{\varepsilon\to 0_+} c_n \int_0^\infty \int_{\mathbb{R}^n} \mathrm{Im}\,\big(\,|x|^2 - (t+i\varepsilon)^2\,\big)^{-(n-1)/2}\, \Box F(t,x)\, dtdx, \tag{1.2.16}$$

assuming that $n > 1$. We just did this for $n = 2$ and the argument for higher dimensions is practically the same. Indeed, if we use (1.2.14) and the fact that

$$\operatorname{Im}\,\big(\,|x|^2 - (t+i\varepsilon)^2\,\big)^{-(n-1)/2}\Big|_{t=0} = 0,$$

we can integrate by parts as we just did in the 2-dimensional case to see that the right side of (1.2.16) equals

$$c_n \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^n} F(0,x)\frac{\partial}{\partial t}\operatorname{Im}\,\big(\,|x|^2 - (t+i\varepsilon)^2\,\big)^{-(n-1)/2}\Big|_{t=0}\,dx$$
$$= (n-1)c_n \lim_{\varepsilon\to 0_+} \int_{\mathbb{R}^n} F(0,x)\frac{\varepsilon}{(\varepsilon^2+|x|^2)^{(n+1)/2}}\,dx.$$

For a given $\varepsilon > 0$, the last integral is our friend, the Poisson integral of $x \to F(0,x)$ that arose in the Dirichlet problem for the Laplacian. So if for $n \geq 2$ we choose c_n here to be

$$c_n = b_n/(n-1),$$

where b_n is as in (1.1.15) and (1.1.17), i.e.,

$$c_n = \frac{1}{2}\pi^{-(n+1)/2}\Gamma\Big(\frac{n-1}{2}\Big) = \frac{2}{(n-1)A_{n+1}}, \tag{1.2.17}$$

then we conclude that (1.2.16) must be valid, i.e., E_+ is a fundamental solution for $\Box$.

For $n = 1$ this calculation does not work, but based on what we just did we find that

$$E_+ = \frac{1}{2}H(t)H(t^2 - |x|^2), \quad n = 1,$$

satisfies $\Box E = \delta_{0,0}$ in $\mathbb{R}\times\mathbb{R}$.

We also notice that

$$E_+ = \frac{1}{2\pi}H(t)H(t^2-|x|^2)(t^2-|x|^2)^{-1/2}, \quad n = 2,$$

and in three spatial-dimensions we have the Kirchhoff formula

$$E_+ = \frac{1}{2\pi}H(t)\delta(t^2-|x|^2) = \frac{1}{4\pi t}H(t)\delta(t-|x|), \quad n = 3.$$

Therefore, when $n \leq 3$, E_+ is a measure supported in the forward light cone. When $n \geq 4$ it is a more singular and complicated distribution than this. On the other hand, $n = 2k+1$ is odd with $k \geq 1$, then E_+ involves $(k-1)$ derivatives of the δ-function on $\mathbb{R}$ pulled back via the Lorentz form (1.2.2), which is equivalent to the familiar spherical means formulae for solutions of the wave equation in odd-dimensions (see [62], formula (1.6)).

By construction, the fundamental solution E_+ is supported in the forward light cone, $\{(t,x) \in [0,\infty)\times\mathbb{R}^n : |x| \leq t\}$. Let us now argue that it is the only such fundamental solution. Any other would have to be of the form $E_+ + u$ where u is a distribution satisfying $\Box u = 0$ and supp $u \subset [0,\infty)\times\mathbb{R}^n$. But then $u = \delta_{0,0} * u = (\Box E_+) * u = E_+ * \Box u = 0$.[1] So not only is E_+ the only such fundamental solution,

[1] Note that these calculations are justified by the discussion at the end of §A.5 of the appendix.

it actually is the only one supported in $\mathbb{R}^{1+n}_+$. Because of this, it is often called the advanced fundamental solution.

If E_- is the reflection of E_+ about the origin in $\mathbb{R}^{1+n}$, i.e.,

$$\langle E_-, F\rangle = \langle E_+, F_-\rangle, \quad F \in \mathcal{S},$$

where $F_-(t,x) = F(-t,-x)$, then clearly E_- is a fundamental solution, which is called the retarded fundamental solution since it is supported in the backward light cone. Another fundamental solution then is given by

$$E = \frac{1}{2}(E_+ + E_-),$$

which is sometimes called the Feynman-Wheeler fundamental solution that makes use of both the advanced and retarded fundamental solutions, E_+ and E_-.

Let us summarize what we have just done in the following.

Theorem 1.2.1 *The distribution E_+ on $\mathbb{R}^{1+n}$ defined by (1.2.15) and (1.2.19) is a fundamental solution, supported in the forward light cone, and is the unique fundamental solution supported in $[0,\infty)\times\mathbb{R}^n$. Its reflection through the origin, E_- is the unique fundamental solution supported in the backward light cone through the origin, $\{(t,x)\in(-\infty,0]\times\mathbb{R}^n : t^2-|x|^2\ge 0\}$, and $E=\frac{1}{2}(E_++E_-)$ is also a fundamental solution of $\Box$.*

The construction of E_+ using W_a, $a=-(n-1)/2$, will turn out to be very useful when we replace Δ by a variable coefficient operator later. Let us now see, though, that if we use the Fourier transform, we can write down a simple and natural formula for E_+.

Recall that the Fourier transform is an isometry on $\mathcal{S}(\mathbb{R}^n)$ given by

$$\hat f(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx, \quad f\in\mathcal{S}(\mathbb{R}^n), \tag{1.2.18}$$

whose inverse is given by Fourier's inversion formula

$$f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat f(\xi)\,d\xi, \quad f\in\mathcal{S}(\mathbb{R}^n). \tag{1.2.19}$$

Note that if $f,g\in\mathcal{S}$ then

$$\int_{\mathbb{R}^n} \hat f\, g\,dx = \int f\,\hat g\,dx.$$

This allows us to define the Fourier transform of $u\in\mathcal{S}'(\mathbb{R}^n)$. To do so, we just define $\hat u$ by requiring that

$$\langle \hat u, f\rangle = \langle u, \hat f\rangle.$$

Having recalled these definitions, we claim that we have the following natural formula for E_+:

$$E_+(t,x) = H(t)\times(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{\sin t|\xi|}{|\xi|}\,d\xi = E_+(t), \tag{1.2.20}$$

where $E_+(t)\in\mathcal{S}'(\mathbb{R}^n)$ denotes the distribution, whose Fourier transform is $\sin(t|\xi|)/|\xi|$, when $t>0$ and 0 if $t\le 0$, i.e.,

$$\langle E_+(t), f\rangle = (2\pi)^{-n}H(t)\int_{\mathbb{R}^n}\frac{\sin t|\xi|}{|\xi|}\hat f(\xi)\,d\xi.$$

If (1.2.20) is valid it shows that E_+ is a continuous function of $t > 0$ with values in $\mathcal{E}'(\mathbb{R}^n)$, but this follows from the earlier formula as well.

Since the second term in (1.2.20) is a distribution supported in $[0, \infty) \times \mathbb{R}^n$, in view of Theorem 1.2.1, we would prove the formula if we could show that for all $F \in \mathcal{S}(\mathbb{R}^{1+n})$ we have

$$F(0,0) = (2\pi)^{-n} \int_0^\infty \int_{\mathbb{R}^n} \frac{\sin t|\xi|}{|\xi|} (\Box F)\hat{}\,(t,\xi)\, d\xi dt, \tag{1.2.21}$$

where here we are taking the $\mathbb{R}^n$ Fourier transform so that

$$(\Box F)\hat{}\,(t,x) = (\partial_t^2 + |\xi|^2)\hat{F}(t,\xi).$$

Since

$$(\partial_t^2 + |\xi|^2)\frac{\sin t|\xi|}{|\xi|} = 0, \quad \text{and} \quad \sin(t|\xi|) = 0, \quad \text{when} \quad t = 0,$$

if we integrate by parts we conclude that the right side of (1.2.21) equals

$$(2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \frac{\sin t|\xi|}{|\xi|} \Big|_{t=0} \hat{F}(0,\xi)\, d\xi = (2\pi)^{-n} \int \hat{F}(0,\xi)\, d\xi = F(0,0),$$

as desired, which proves (1.2.20). We also remark that if E is as in Theorem 1.2.1, then this also gives us that

$$2E(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin|t\xi|}{|\xi|}\, d\xi. \tag{1.2.22}$$

In addition to showing that

$$\langle E_+(t), h\rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{\sin t|\xi|}{|\xi|} \hat{h}(\xi)\, d\xi, \quad h \in \mathcal{S}(\mathbb{R}^n),$$

has a limit as $t \to 0_+$, which is 0, (1.2.20) also shows that $E'_+(t) = \partial_t E_+(t)$, which is given by

$$\langle E'_+(t), h\rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} \cos(t|\xi|)\, \hat{h}(\xi)\, d\xi,$$

has a limit in $\mathcal{S}'$, which is δ_0, the Dirac distribution for $\mathbb{R}^n$.

Using these facts we can solve the Cauchy problem:

$$\begin{cases} \Box u(t,x) = F(t,x), & t > 0 \\ u(0,x) = f(x), \ \partial_t u(0,x) = g(x). \end{cases} \tag{1.2.23}$$

Corollary 1.2.2 *Suppose that $f, g \in C^\infty(\mathbb{R}^n)$ and that $F \in C^\infty([0,\infty) \times \mathbb{R}^n)$ the Cauchy problem* (1.2.23) *has a unique solution $u \in C^\infty([0,\infty) \times \mathbb{R}^n)$ given by*

$$u(t,\,\cdot\,) = E'_+(t) * f + E_+(t) * g + \int_0^t E_+(t-s) * F(s,\,\cdot\,)\, ds. \tag{1.2.24}$$

If we assume that $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $F \in C^\infty(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$, we also have

$$u(t,x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos(t|\xi|)\hat{f}(\xi)\, d\xi + (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|} \hat{g}(\xi)\, d\xi$$
$$+ (2\pi)^{-n} \int_0^t \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin(t-s)|\xi|}{|\xi|} \hat{F}(s,\xi)\, d\xi. \tag{1.2.25}$$

We leave it up to the reader to show that (1.2.25) follows from (1.2.24) and the above Fourier transform calculations. By §A.5 of the appendix, if we set $v(t,\,\cdot\,) = E'_+(t) * f + E_+(t) * g$ is in $C^\infty([0,\infty) \times \mathbb{R}^n)$, then $v \in C^\infty([0,\infty) \times \mathbb{R}^n)$, since, for each fixed t, $E_+(t) \in \mathcal{E}'(\mathbb{R}^n)$ and $t \to E_+(t)$ is a smooth function of t with values in $\mathcal{E}'(\mathbb{R}^n)$. Since $\Box E_+(t) = 0$ and $\Box E'_+(t) = 0$ for $t > 0$ and $E_+(+0) = \lim_{t\to 0_+} E_+(t) = 0$ and $E'(+0) = \lim_{t\to 0_+} E'_+(t) = \delta_0$, it is clear that v solves (1.2.23) with vanishing forcing term F. So what remains is to show that $w(t,\,\cdot\,) = \int_0^t E_+(t-s) * F(s,\,\cdot\,)\,ds$ satisfies (1.2.23) with vanishing Cauchy data, i.e., $f, g = 0$, since the same considerations that showed that v was smooth imply the same for w. But using once again that $E_+(+0) = 0$ and $E'(+0) = \delta_0$ we have

$$\partial_t \int_0^t E_+(t-s) * F(s,\,\cdot\,)\,ds = \int_0^t E'_+(t-s) * F(s,\,\cdot\,)\,ds,$$

and

$$\partial_t^2 \int_0^t E_+(t-s) * F(s,\,\cdot\,)\,ds = \int_0^t E''_+(t-s) * F(s,\,\cdot\,)\,ds + \delta_0 * F(t,\,\cdot\,).$$

The first of these formulas tells us that $\partial_t w$ vanishes when $t = 0$ and clearly so does w. The second formula tells us that $\Box w = F$ when $t > 0$, since $\Box E_+ = 0$ when $t > 0$. This establishes the claim about w and finishes the proof of the existence results in the corollary.

We need to prove the uniqueness results. The reader can verify that it just follows from the argument that showed that E_+ was the unique fundamental solution of $\Box$ supported in $[0,\infty) \times \mathbb{R}^n$. On the other hand, let us present an argument based on energy estimates which shows that under the stronger hypotheses in (1.2.25) there is uniqueness, since this will also serve as a model for material to follow.

The energy inequality for $\Box$ says that, if, say, $u \in C^2$ vanishes for large x, then

$$\|u'(t,\,\cdot\,)\|_{L^2(\mathbb{R}^n)} \le \|u'(0,\,\cdot\,)\|_{L^2(\mathbb{R}^n)} + \int_0^t \|\Box u(s,\,\cdot\,)\|_{L^2(\mathbb{R}^n)}\,ds, \tag{1.2.26}$$

where

$$u' = (\partial_t u, \nabla_x u)$$

denotes the space-time gradient of u. Clearly (1.2.26) implies that there is a unique solution to the equation (1.2.23) with the given data (f, g) and forcing term F. For if u_1 and u_2 were two such solutions then their difference would be a solution of the equation $\Box(u_1 - u_2) = 0$ with zero Cauchy data, and hence be identically zero in $\mathbb{R}^{1+n}$ by (1.2.26).

To prove (1.2.26) we use the identity

$$2\partial_t u \Box u = \partial_t |u'|^2 - 2\sum_{j=1}^n \partial_j \big(\partial_t u \partial_j u\big), \tag{1.2.27}$$

and integrate by parts to get

$$\begin{aligned}\partial_t \|u'(t,\,\cdot\,)\|_{L^2}^2 = \int \partial_t |u'|^2\,dx &= \int \partial_t |u'|^2\,dx - 2\sum_{j=1}^n \int \partial_j \big(\partial_t u \partial_j u\big)\,dx \\ &= 2\int \partial_t u \Box u\,dx \le 2\|u'(t,\,\cdot\,)\|_{L^2} \|\Box u(t,\,\cdot\,)\|_{L^2}.\end{aligned}$$

Since this implies

$$\partial_t \|u'(t,\,\cdot\,)\|_{L^2} \le \|\Box u(t,\,\cdot\,)\|_{L^2},$$

we obtain (1.2.26) via integration from 0 to t.

Remark 1.2.3 As noted in Theorem 1.2.1, E_+ is supported in the forward light cone. Thus, its reflection about $\{t=0\}$, E_-, is a fundamental solution for the d'Alembertian supported in the backward light cone, $\{(t,x)\in\mathbb{R}^{1+n} : t\le 0,\, t^2-|x|^2\ge 0\}$, and $E=\frac{1}{2}(E_+ + E_-)$ is another fundamental solution but with support in the full light cone $\{(t,x)\in\mathbb{R}^{1+n} : t^2-|x|^2\ge 0\}$. By (1.2.22) we have

$$E(t,x)=\frac{1}{2}\,(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,\frac{\sin|t\xi|}{|\xi|}\,d\xi.$$

With this in mind, it is not difficult to check that another natural fundamental solution to the d'Alembertian in $\mathbb{R}^{1+n}$, $n\ge 2$, is given by the formula

$$E_F(t,x)=\frac{1}{2}\,(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\left(\frac{\sin|t\xi|}{|\xi|}-i\frac{\cos t|\xi|}{|\xi|}\right)d\xi \\ =\frac{1}{2i}\,(2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{i|t\xi|}\,\frac{d\xi}{|\xi|}. \qquad (1.2.28)$$

We leave the verification of this assertion as a good exercise for the reader. This distribution, E_F, is called the *Feynman fundamental solution for the d'Alembertian.* Unlike E_+, E_- or E its support is not contained in the light cone. Instead $\operatorname{supp} E_F = \mathbb{R}^{1+n}$, which is a physical curiosity since, as noted above, the d'Alembertian has propagation speed one (see also Theorem 2.4.2 below).

Another good exercise is to verify that if $n\ge 2$ then by (1.2.15) and (1.2.17)

$$E_F(t,x)=\lim_{\varepsilon\to 0_+}\frac{1}{i(n-1)A_{n+1}}\left(|x|^2-(|t|+i\varepsilon)^2\right)^{-(n-1)/2}, \qquad (1.2.29)$$

with, as above, A_d denoting the area of the unit sphere in $\mathbb{R}^d$. (See (6.2.1) in [37] for another proof of this fact.) Using (1.2.29) one finds that the difference between E_F and E has support equal to $\{(t,x)\in\mathbb{R}^{1+n} : |x|\ge|t|\}$ (the exterior of the full light cone) when n is even, while for odd spatial dimensions the support of E_F-E is all of Minkowski space.

Let us conclude this section by using Corollary 1.2.2 to make some calculations regarding the distributions $W_a(t,x)$ defined in (1.2.8), (1.2.10) and (1.2.12). As we saw above if we set

$$E_0=\frac{1}{2}\pi^{-(n+1)/2}\Gamma\Big(\frac{n-1}{2}\Big)\,H(t)W_{-(n-1)/2}(t,x),$$

then $\Box E_0=\delta_{0,0}$, and since, by (1.2.12) and (1.2.13), $\Box W_{-(n-1)/2+\nu}$ is a nonzero multiple of $W_{-(n-1)/2+\nu-1}$, we conclude that for a given $n\ge 2$ we can inductively choose nonzero constants α_ν so that if we set

$$E_\nu=\alpha_\nu H(t)W_{-(n-1)/2+\nu}, \quad \nu=1,2,3,\dots,$$

then we have $\Box E_\nu=\nu E_{\nu-1}, \quad \nu=1,2,3,\dots.$

Computing the constants α_ν is a bit tedious. Let us derive another formula that will be easier to use. We first note that if we set for $\nu = 0, 1, 2, \dots$

$$E_\nu(t,x) = \lim_{\varepsilon \to 0_+} \nu!\,(2\pi)^{-n-1} \iint_{\mathbb{R}^{1+n}} e^{ix\cdot\xi + it\tau} (|\xi|^2 - (\tau - i\varepsilon)^2)^{-\nu-1}\, d\xi d\tau \quad (1.2.30)$$

in the sense of distributions, then supp $E_\nu \subset [0,\infty) \times \mathbb{R}^n = \mathbb{R}^{1+n}_+$, by the Paley-Weiner theorem (see §7.3-7.4 of [37] or Theorem 19.2 of [53]).[2] Since for $F \in \mathcal{S}(\mathbb{R}^{1+n})$, we have

$$\langle E_0, \Box F\rangle = \lim_{\varepsilon \to 0_+} (2\pi)^{-n-1} \iint \frac{(|\xi|^2 - \tau^2)\hat{F}(\xi,\tau)}{|\xi|^2 - (\tau - i\varepsilon)^2}\, d\xi d\tau$$
$$= (2\pi)^{-n-1} \iint \hat{F}(\xi,\tau)\, d\xi d\tau = F(0,0),$$

where now $\hat{F}$ denotes the space-time Fourier transform, we conclude that E_0 defined by (1.2.30) must be a fundamental solution for $\Box$. Since it is supported in $\mathbb{R}^{1+n}_+$, it must be the advanced fundamental solution E_+ constructed before. Note that our original construction of E_+ involved extending the real quadratic form $|x|^2 - t^2$ into the complex plane, while the one given by (1.2.30) for $\nu = 0$ involves a similar construction, but this time on the Fourier transform side.

Let us collect more facts about these distributions which will prove useful in the sequel.

Proposition 1.2.4 *Let E_ν, $\nu = 0, 1, 2, \dots$ be the distributions defined in* (1.2.30)*. Then*

$$\textit{supp}\ E_\nu \subset \{(t,x) \in [0,\infty) \times \mathbb{R}^n : |x| \le t\}, \quad \nu = 0, 1, 2, 3, \dots, \quad (1.2.31)$$

$$\Box E_0 = \delta_{0,0}, \quad (1.2.32)$$

and

$$\Box E_\nu = \nu E_{\nu-1}, \quad -2\partial E_\nu/\partial x = x E_{\nu-1},$$
$$2\partial E_\nu/dt = t E_{\nu-1}, \quad \nu = 1, 2, 3, \dots. \quad (1.2.33)$$

Furthermore,

$$E_\nu(+0) = \lim_{t \to 0_+} E_\nu(t, \cdot\,) = 0, \quad \nu = 0, 1, 2, 3, \dots,$$

and

$$\partial_t^k E_\nu(+0) = 0 \ \textit{for} \ \ k \le 2\nu.$$

To verify this, we note that we have already proven the assertions for $\nu = 0$. The Paley-Weiner theorem yields that supp $E_\nu \subset \mathbb{R}^{1+n}_+$. Since (1.2.33) is easy to check directly from the definitions, we see from Corollary 1.2.2 that we have the stronger statement (1.2.31). By the discussion preceding the Proposition, E_ν must be a nonzero multiple of $W_{-(n-1)/2+\nu}$ for every $\nu = 0, 1, 2, \dots$, which implies the last part of the proposition since it is easy to check the assertions for these distributions.

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[2] In the case of $\nu = 0$, this can also be verified directly using the formula $\lim_{\varepsilon\to 0+}(|\xi|^2 - (\tau - i\varepsilon)^2)^{-1} = (|\xi|^2 - \tau^2 + i(\text{sgn}\ t)0)^{-1}$, which leads to the expected formula $\lim_{\varepsilon \to 0_+}(2\pi)^{-1}\int e^{it\tau}(|\xi|^2 - (\tau - i\varepsilon)^2)^{-1}\, d\tau = H(t)\sin t|\xi|/|\xi|$. The asserted support properties for E_ν, $\nu = 1, 2, 3, \dots$, follow from this and the fact that $2\partial E_\nu/\partial t = tE_{\nu-1}$, $\nu = 1, 2, 3, \dots$.

Remark 1.2.5 If we use (1.2.20), we can derive another formula for these distributions. Indeed, by using Corollary 1.2.2 we conclude that for $\nu = 1, 2, 3, \dots$

$$\begin{aligned} E_\nu(t,x) &= \nu(2\pi)^{-n} \int_0^t \int e^{ix\cdot\xi} \frac{\sin(t-s)|\xi|}{|\xi|} \hat{E}_{\nu-1}(s,\xi)\, d\xi ds \\ &= \nu!\,(2\pi)^{-n} \int_{0\le s_1\le s_2\le\cdots\le s_\nu\le t} \int e^{ix\cdot\xi} \frac{\sin(t-s_\nu)|\xi|}{|\xi|} \frac{\sin(s_\nu - s_{\nu-1})|\xi|}{|\xi|} \cdots \\ &\qquad \times \frac{\sin(s_2-s_1)|\xi|}{|\xi|} \frac{\sin s_1|\xi|}{|\xi|}\, d\xi ds_1 \dots ds_\nu. \end{aligned} \tag{1.2.34}$$

For instance, when $\nu = 1$, we have

$$E_1(t,x) = \frac{H(t)}{2} \times (2\pi)^{-n} \int e^{ix\cdot\xi} \left(\frac{\sin t|\xi|}{|\xi|} - t\cos t|\xi| \right) \frac{d\xi}{|\xi|^2}. \tag{1.2.35}$$

The reader can check that the factor inside the parenthesis vanishes to second order at $\xi = 0$, which cancels out the singularity coming from the factor $1/|\xi|^2$. One can also prove (1.2.35) by using the last identity in (1.2.33).

By an induction argument using either (1.2.34) or (1.2.33), one can show that E_ν, $\nu = 1, 2, \dots$, is a finite linear combination of Fourier integrals of the form

$$H(t)t^j\,(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sin t|\xi|\; |\xi|^{-\nu-1-k}\, d\xi$$

and

$$H(t)t^j\,(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cos t|\xi|\; |\xi|^{-\nu-1-k}\, d\xi$$

with j and k being nonnegative integers satisfying $j+k=\nu$. We can ignore the contributions from frequencies $|\xi| \le 1$ in our applications, since for any $\nu = 0, 1, 2, \dots$, the difference between E_ν and the inverse Fourier transform of $\xi \to \hat{E}_\nu(t,\xi)\mathbf{1}_{|\xi|\ge 1}$ will be a smooth function of t and x. Here $\mathbf{1}_{|\xi|\ge 1}$ denotes the characteristic function of $\{\xi \in \mathbb{R}^n : |\xi| \ge 1\}$. Based on this, if we use Euler's formula, we conclude that for any $\nu = 0, 1, 2, 3, \dots$, modulo smooth functions of $(t,x) \in \mathbb{R}^n_+$, E_ν is a finite linear combination of terms of the form

$$\begin{aligned} &H(t)\,t^j\,(2\pi)^{-n} \int_{|\xi|\ge 1} e^{ix\cdot\xi \pm it|\xi|} |\xi|^{-\nu-1-k}\, d\xi, \\ &\text{for some } \; j,k = 0,1,2,\dots, \text{ with } j+k=\nu. \end{aligned} \tag{1.2.36}$$