9 Compactification and Supersymmetry Breaking

We have considered so far superstring theories in ten noncompact dimensions. However, our direct physical interest is in theories with four large dimensions. One way to obtain them is to make use of the Kaluza-Klein idea: consider some of the dimensions to be curled up into a compact manifold, leaving only four noncompact dimensions.

As we have seen in the case of the bosonic strings, exact solutions to the equations of motion correspond to a CFT. The classical geometric picture is only appropriate at large volume (α'-expansion). In the case of type-II string theory, vacua correspond to an $\mathcal{N} = (1,1)_2$ SCFT. In the heterotic case, vacua correspond to $\mathcal{N} = (1,0)_2$ SCFT.

We generalize the concept of compactification to four dimensions, by replacing the original flat noncompact CFT with another one, where four dimensions are still flat but the rest is described by an arbitrary unitary[1] CFT with the appropriate central charge. This type of description is more general than that of a geometrical compactification, since there are CFTs with no geometrical interpretation. In the following, we will examine both the geometric point of view and the CFT point of view, mainly via orbifold compactifications.

9.1 Narain Compactifications

The simplest possibility is the "internal compact" manifold to be a (flat) torus. This can be considered as a different background of the ten-dimensional theory, where we have given constant expectation values to the internal metric and other background fields.

Consider first the case of the heterotic string compactified to $D < 10$ dimensions. It is rather straightforward to construct the partition function of the compactified theory. There are now $D-2$ transverse noncompact coordinates, each contributing $\sqrt{\tau_2}\eta\bar{\eta}$. There is no change in the contribution of the left-moving world-sheet fermions and 16 right-moving

[1] The spectrum of dimensions of this CFT should be discrete in order to correspond to a "compactification."

compact coordinates. Finally the contribution of the $10-D$ compact coordinates is given by (4.18.40) on page 99. Putting everything together we obtain

$$Z_D^{\text{heterotic}} = \frac{\Gamma_{10-D,10-D}(G,B)\,\bar{\Gamma}_H}{\tau_2^{(D-2)/2}\eta^8\bar{\eta}^8}\,\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\,\frac{\vartheta^4[^a_b]}{\eta^4}, \tag{9.1.1}$$

where $\bar{\Gamma}_H$ stands for the partition function of either $\text{Spin}(32)/\mathbb{Z}_2$ or $E_8 \times E_8$ lattice; $G_{\alpha\beta}$, $B_{\alpha\beta}$ are the constant expectation values of the internal $(10-D)$-dimensional metric and antisymmetric tensor.

We now analyze the massless spectrum. The original ten-dimensional metric gives rise to the D-dimensional metric $G_{\mu\nu}$, $(10-D)$ U(1) gauge fields, $G_{\mu\alpha}$ and $\frac{1}{2}(10-D)(11-D)$ scalars, $G_{\alpha\beta}$. The antisymmetric tensor produces a D-dimensional antisymmetric tensor, $B_{\mu\nu}$, $(10-D)$ $U(1)$ gauge fields, $B_{\mu\alpha}$, and $\frac{1}{2}(10-D)(9-D)$ scalars, $B_{\alpha\beta}$. The ten-dimensional dilaton gives rise to another scalar. Finally the $\dim(H)$ ten-dimensional gauge fields give rise to $\dim(H)$ gauge fields, A^a_μ, and $(10-D)\cdot\dim(H)$ scalars, $A^a_\alpha \equiv Y^a_\alpha$. Similar remarks apply to the fermions.

We will consider in more detail the scalars Y^a_α coming from the ten-dimensional vectors, where a is the adjoint index and α the internal index taking values $D+1,\ldots,10$. The nonabelian field strength (8.3E) on page 215 contains a term without derivatives. This is the commutator of two gauge fields. Upon dimensional reduction this gives rise to a potential term for the (Higgs) scalars Y^a_α:

$$V_{\text{H}} \sim G^{\alpha\gamma}G^{\beta\delta}\text{Tr}[Y_\alpha,Y_\beta][Y_\gamma,Y_\delta] \sim f^a{}_{bc}f^a{}_{b'c'}\;G^{\alpha\gamma}\;G^{\beta\delta}\;Y^b_\alpha\;Y^c_\beta\;Y^{b'}_\gamma\;Y^{c'}_\delta, \tag{9.1.2}$$

where $Y_\alpha = Y^a_\alpha T^a$. This potential is minimized when the matrices Y_α are commuting. They then have arbitrary expectation values in the Cartan subalgebra. These expectations values are moduli (flat directions or continuous families of minima). We will label these values by Y^I_α, $I = 1,2,\ldots,16$. This is a normal Brout-Englert-Higgs phenomenon and it generates a mass matrix for the gauge fields

$$\left[m^2\right]^{ab} \sim G^{\alpha\beta}\,f^{ca}{}_d\,f^{cb}{}_{d'}\;Y^d_\alpha\;Y^{d'}_\beta. \tag{9.1.3}$$

This mass matrix has $\text{rank}(H)$ generic zero eigenvalues. The gauge fields belonging to the Cartan subalgebra remain massless while all the other gauge fields get a nonzero mass. Consequently, the gauge group is broken to $\text{U}(1)^{\text{rank}(H)}$. If we turn on these expectation values, the heterotic compactified partition function becomes

$$Z_D^{\text{heterotic}} = \frac{\Gamma_{10-D,26-D}(G,B,Y)}{\tau_2^{(D-2)/2}\eta^8\bar{\eta}^8}\,\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\,\frac{\vartheta^4[^a_b]}{\eta^4}, \tag{9.1.4}$$

where the structure of the $\Gamma_{10-D,26-D}$ lattice sum is described in detail in Appendix D on page 513.

The $(10-D)(26-D)$ scalar fields G, B, Y are called moduli since they can have arbitrary expectation values. Thus, the heterotic string compactified down to D dimensions provides a continuous family of vacua parametrized by the expectation values of the moduli that describe the geometry of the internal manifold (G,B) and the (flat) gauge bundle (Y).

Consider now the tree-level effective action for the bosonic massless modes in the toroidally compactified theory. It can be obtained by direct dimensional reduction of the ten-dimensional heterotic effective action, which in the string frame is given by (6.1.10) on page 146 with the addition of the gauge fields[2]

$$S_{10}^{\text{heterotic}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-\det G_{10}}e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - \frac{1}{12}\hat{H}^2 - \frac{1}{4}\text{Tr}[F^2]\right] + \mathcal{O}(\alpha'). \qquad (9.1.5)$$

The massless fields in D dimensions are obtained from those of the ten-dimensional theory by assuming that the latter do not depend on the internal coordinates x^α. Moreover we keep only the Cartan gauge fields since they are the only ones that will remain massless for generic values of the Wilson lines Y_α^I, $I = 1, 2, \ldots, 16$. So, the gauge kinetic terms become abelian $\text{Tr}[F^2] \to \sum_{I=1}^{16} F_{\mu\nu}^I F^{I,\mu\nu}$ with

$$F_{\mu\nu}^I = \partial_\mu A_\nu^I - \partial_\nu A_\mu^I. \qquad (9.1.6)$$

Also

$$\hat{H}_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2}\sum_I A_\mu^I F_{\nu\rho}^I + \text{cyclic}, \qquad (9.1.7)$$

where we have neglected the gravitational Chern-Simons contribution, since it is of higher order in α'.

There is a standard *Ansatz* to define the D-dimensional fields, such that the gauge invariance of the compactified theory is simple. This is given in Appendix E on page 516. In this way we obtain

$$S_D^{\text{heterotic}} = \int d^D x\,\sqrt{-\det G}e^{-2\phi}\left[R + 4\partial^\mu\phi\partial_\mu\phi - \frac{1}{12}\hat{H}^{\mu\nu\rho}\hat{H}_{\mu\nu\rho}\right.$$
$$\left. -\frac{1}{4}(\hat{M}^{-1})_{ij}F_{\mu\nu}^i F^{j\mu\nu} + \frac{1}{8}\text{Tr}(\partial_\mu\hat{M}\partial^\mu\hat{M}^{-1})\right], \qquad (9.1.8)$$

where $i = 1, 2, \ldots, 36 - 2D$. ϕ is the D-dimensional dilaton and

$$\hat{H}_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} - \frac{1}{2}L_{ij}A_\mu^i F_{\nu\rho}^j + \text{cyclic}, \qquad (9.1.9)$$

where L_{ij} is the invariant metric of $O(10 - D, 26 - D)$.

The moduli scalar matrix $\hat{M}$ is given in (D.4) on page 514. The action (9.1.8) has a continuous $\mathrm{O}(10 - D, 26 - D)$ symmetry. If $\Omega \in \mathrm{O}(10 - D, 26 - D)$ is a $(36 - 2D) \times (36 - 2D)$ matrix then

$$\hat{M} \to \Omega\,\hat{M}\,\Omega^T, \quad A_\mu \to \Omega \cdot A_\mu, \qquad (9.1.10)$$

leaves the effective action invariant. The presence of the massive states originating from the lattice, breaks this symmetry to the discrete infinite subgroup $\mathrm{O}(10 - D, 26 - D, \mathbb{Z})$. This is the group of T-duality symmetries. The action for the $(10 - D)(26 - D)$ scalars in (9.1.8) is the $\mathrm{O}(10 - \mathrm{D}, 26 - \mathrm{D})/(\mathrm{O}(10 - \mathrm{D}) \times \mathrm{O}(26 - \mathrm{D}))$ σ-model.

[2] We have rescaled the gauge fields $\ell_s^4 A_\mu/\sqrt{2} \to A_\mu$ so that now they are dimensionless; see (H.42) in appendix H.5 on page 526.

In the Einstein frame (using (6.1.11)), the action becomes

$$S_D^{\text{heterotic}} = \int d^Dx \sqrt{-\det G_E}\left[R - \frac{4}{D-2}\partial^\mu\phi\partial_\mu\phi - \frac{e^{-8\phi/(D-2)}}{12}\hat{H}^{\mu\nu\rho}\hat{H}_{\mu\nu\rho} - \frac{e^{-4\phi/(D-2)}}{4}(\hat{M}^{-1})_{ij}F^i_{\mu\nu}F^{j\mu\nu} + \frac{1}{8}\text{Tr}(\partial_\mu\hat{M}\partial^\mu\hat{M}^{-1})\right]. \tag{9.1.11}$$

In section 4.18.6 on page 100 we have described the effect of gauge symmetry enhancement. This applies to the toroidal compactifications of the heterotic string. Whenever new currents appear on the nonsupersymmetric side, new massless gauge bosons appear in the effective theory. This is very much like the bosonic string. There is a difference, however, here: currents that appear on the supersymmetric side do not generate new massless gauge bosons. The reason is that the massless states on the supersymmetric sides come from the fermionic oscillators $\psi^i_{-1/2}$ which are not affected by changing the torus moduli. Therefore, there is no symmetry enhancement coming from the supersymmetric sector. The abelian gauge bosons originating from the supersymmetric side (i.e., $\psi^I_{-1/2}\bar{a}^\mu_{-1}|p\rangle$) are graviphotons.[3]

Whenever the lattice contains as a sublattice, the root lattice of a Lie algebra $\mathfrak{g}$, the gauge group contains G as a gauge group.[4] Moving away from that point is equivalent to the Brout-Englert-Higgs breaking (sometimes partially) of the G symmetry.

We will now pay special attention to the $D = 4$ compactifications. Here, the ten-dimensional gravitino produces four four-dimensional Majorana gravitini. Consequently, the four-dimensional compactified theory has $\mathcal{N} = 4_4$ local SUSY. The relevant massless $\mathcal{N} = 4_4$ supermultiplets are the supergravity multiplet and the vector multiplet. The supergravity multiplet contains the metric, six vectors (the graviphotons), a scalar and an antisymmetric tensor, as well as four Majorana gravitini and four Majorana spin-$\frac{1}{2}$ fermions. The vector multiplet contains a vector, four Majorana spin $\frac{1}{2}$ fermions and six scalars. In total we have, apart from the supergravity multiplet, 22 vector multiplets.

In $D = 4$, the antisymmetric tensor is equivalent (on shell) via a duality transformation, to a pseudoscalar a, the “axion.” The relation (in the Einstein frame) is

$$e^{-4\phi}\hat{H}_{\mu\nu\rho} = E_{\mu\nu\rho}{}^{\sigma}\nabla_\sigma a \tag{9.1.12}$$

with the E tensor defined as in (B.12) on page 506. This relation is such that the $B_{\mu\nu}$ equations of motion $\nabla^\mu(e^{-4\phi}\hat{H}_{\mu\nu\rho}) = 0$ are automatically solved by substituting (9.1.12). The Bianchi identity for $\hat{H}$ from (9.1.9) is

$$E^{\mu\nu\rho\sigma}\,\partial_\mu\hat{H}_{\nu\rho\sigma} = -L_{ij}F^i_{\mu\nu}\tilde{F}^{j,\mu\nu}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}E^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{9.1.13}$$

Using (9.1.12), it becomes the equation of motion for the axion:

$$\nabla^\mu\,(e^{4\phi}\nabla_\mu a) = -\frac{1}{4}L_{ij}F^i_{\mu\nu}\tilde{F}^{j,\mu\nu}. \tag{9.1.14}$$

[3] It is known that making some of the graviphotons part of a nonabelian symmetry is equivalent to gauging the associated supergravity. Gauged supergravities are very interesting and useful but they rarely have flat supersymmetric vacua. They correspond typically to compactifications with fluxes.

[4] Only simply laced (A-D-E) algebras with rank at most $26 - D$ can appear.

This equation can be obtained from the "dual" action

$$\tilde{S}^{\text{heterotic}}_{D=4} = \int d^4x\sqrt{-\det g_E}\Big[R - 2\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}e^{4\phi}\partial^\mu a\partial_\mu a$$
$$-\frac{1}{4}e^{-2\phi}(\hat{M}^{-1})_{ij}F^i_{\mu\nu}F^{j,\mu\nu} + \frac{1}{4}aL_{ij}F^i_{\mu\nu}\tilde{F}^{j,\mu\nu} + \frac{1}{8}\mathrm{Tr}(\partial_\mu\hat{M}\partial^\mu\hat{M}^{-1})\Big]. \tag{9.1.15}$$

We define the complex axion-dilaton S field

$$S = S_1 + iS_2 = a + ie^{-2\phi}, \tag{9.1.16}$$

and write the action as

$$\tilde{S}^{\text{heterotic}}_{D=4} = \int d^4x\sqrt{-\det g_E}\Big[R - \frac{1}{2}\frac{\partial^\mu S\partial_\mu\bar{S}}{S_2^2} - \frac{1}{4}S_2(\hat{M}^{-1})_{ij}F^i_{\mu\nu}F^{j,\mu\nu}$$
$$+\frac{1}{4}S_1L_{ij}F^i_{\mu\nu}\tilde{F}^{j,\mu\nu} + \frac{1}{8}\mathrm{Tr}(\partial_\mu\hat{M}\partial^\mu\hat{M}^{-1})\Big]. \tag{9.1.17}$$

From the definition (9.1.16), S_2 is the string loop expansion parameter (heterotic string coupling constant). The scalar field S takes values in the upper half plane $\mathcal{H}_2$ =SL(2,$\mathbb{R}$)/U(1). The scalars $\hat{M}$ parametrize the coset space O(6,22)/O(6)×O(22). As we will see later on, the four-dimensional heterotic string has a nonperturbative SL(2,$\mathbb{Z}$) action on S by fractional linear transformations. It entails electric-magnetic duality transformations on the abelian gauge fields as described in appendix G on page 522.

We will briefly describe here the toroidal compactification of type-II string theory to four dimensions. As discussed in section 7.7.1 on page 174, upon toroidal compactification the IIA and IIB theories are equivalent. Consequently, we need only examine the compactification of the type-IIA theory.

We compactify on a six-torus to four dimensions. The two Majorana-Weyl gravitini and fermions give rise to eight $D = 4$ Majorana gravitini and 48 spin-$\frac{1}{2}$ Majorana fermions. Therefore, the $D = 4$ theory has maximal $\mathcal{N} = 8_4$ supersymmetry. The ten-dimensional metric produces the four-dimensional metric, six U(1) vectors, and 21 scalars. The antisymmetric tensor produces (after four-dimensional dualization), six U(1) vectors and 16 scalars. The dilaton gives an extra scalar. The R-R U(1) gauge field gives one gauge field and six scalars. The R-R three-form gives a three-form (no physical degrees of freedom in four dimensions) 15 vectors and 26 scalars. All the degrees of freedom form the $\mathcal{N} = 8_4$ supergravity multiplet that contains the graviton, 28 vectors, 70 scalars, eight gravitini, and 56 fermions. We will see more on the symmetries of this theory in chapter 11. We note that there is no perturbative gauge symmetry enhancement in type-II string theory.

9.2 World-sheet versus Space-time Supersymmetry

There is an interesting relation between world-sheet and space-time supersymmetry. To uncover it, we consider first the case of the heterotic string compactified to $D = 4$. The four dimensions are described by a flat Minkowski space.

An N-extended supersymmetry algebra in four dimensions is generated by N Weyl supercharges Q^I_a and their Hermitian conjugates $\bar{Q}^I_{\dot{\alpha}}$ satisfying the algebra

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ},$$
$$\{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}^{IJ},$$
$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = \delta^{IJ}\, \sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \tag{9.2.1}$$

where Z^{IJ} is the antisymmetric matrix of central charges.

As we have seen in section 7.5 on page 168, the space-time supersymmetry charges can be constructed from the massless fermion vertex at zero momentum. In our case we have

$$Q^I_\alpha = \frac{1}{2\pi i}\oint dz\, e^{-\phi/2} S_\alpha \Sigma^I, \quad \bar{Q}^I_{\dot{\alpha}} = \frac{1}{2\pi i}\oint dz\, e^{-\phi/2} C_{\dot{\alpha}} \bar{\Sigma}^I, \tag{9.2.2}$$

where S, C are the spinor and conjugate spinor of O(4) and $\Sigma^I, \bar{\Sigma}^I$ are operators in the R sector of the internal CFT with conformal weight $\frac{3}{8}$. We will also need

$$: e^{q_1\phi(z)} :: e^{q_2\phi(w)} \doteqdot (z-w)^{-q_1 q_2} : e^{(q_1+q_2)\phi(w)} : + \cdots, \tag{9.2.3}$$

$$S_\alpha(z) C_{\dot{\alpha}}(w) = \frac{1}{\sqrt{2}} \sigma^\mu_{\alpha\dot{\alpha}}\, \psi^\mu(w) + \mathcal{O}(z-w), \tag{9.2.4}$$

$$S_\alpha(z) S_\beta(w) = \frac{\epsilon_{\alpha\beta}}{\sqrt{z-w}} + \mathcal{O}(\sqrt{z-w}),$$
$$C_{\dot{\alpha}}(z) C_{\dot{\beta}}(w) = \frac{\epsilon_{\dot{\alpha}\dot{\beta}}}{\sqrt{z-w}} + \mathcal{O}(\sqrt{z-w}). \tag{9.2.5}$$

Imposing the anticommutation relations (9.2.1) we find that the internal operators must satisfy the following OPEs:

$$\Sigma^I(z)\bar{\Sigma}^J(w) = \frac{\delta^{IJ}}{(z-w)^{3/4}} + (z-w)^{1/4} J^{IJ}(w) + \cdots, \tag{9.2.6}$$

$$\Sigma^I(z)\Sigma^J(w) = (z-w)^{-1/4}\Psi^{IJ}(w) + \cdots,$$
$$\bar{\Sigma}^I(z)\bar{\Sigma}^J(w) = (z-w)^{-1/4}\bar{\Psi}^{IJ}(w) + \cdots, \tag{9.2.7}$$

where J^{IJ} are some weight-1 operators of the internal CFT and $\Psi^{IJ}, \bar{\Psi}^{IJ}$ have weight 1/2. The central charges are given by $Z^{IJ} = \oint \Psi^{IJ}$. The R fields $\Sigma, \bar{\Sigma}$ have square root branch cuts with respect to the internal supercurrent

$$G^{\rm int}(z)\Sigma^I(w) \sim (z-w)^{-1/2}, \quad G^{\rm int}(z)\bar{\Sigma}^I(w) \sim (z-w)^{-1/2}. \tag{9.2.8}$$

BRST invariance of the fermion vertex implies that the OPE $(e^{-\phi/2} S_\alpha \Sigma^I)(e^\phi G)$ has a single pole term. This in turn implies that there are no more singular terms in (9.2.8).

Consider an extra scalar X with two-point function

$$\langle X(z)X(w)\rangle = -\log(z-w). \tag{9.2.9}$$

Construct the dimension-$\frac{1}{2}$ operators

$$\lambda^I(z) = \Sigma^I(z) e^{iX/2}, \quad \bar{\lambda}^I(z) = \bar{\Sigma}^I(z) e^{-iX/2}. \tag{9.2.10}$$

Using (9.2.6) and (9.2.7) we can verify the following OPEs:

$$\lambda^I(z)\bar{\lambda}^J(w) = \frac{\delta^{IJ}}{z-w} + \hat{J}^{IJ} + \mathcal{O}(z-w), \quad \hat{J}^{IJ} = J^{IJ} + \tfrac{i}{2}\delta^{IJ}\partial X, \tag{9.2.11}$$

$$\lambda^I(z)\lambda^J(w) = e^{iX}\Psi^{IJ} + \mathcal{O}(z-w), \quad \bar{\lambda}^I(z)\bar{\lambda}^J(w) = e^{-iX}\bar{\Psi}^{IJ} + \mathcal{O}(z-w). \tag{9.2.12}$$

Thus, $\lambda^I, \bar{\lambda}^I$ are N complex free fermions and they generate an $O(2N)_1$ current algebra. Moreover, this immediately shows that $\Psi^{IJ} = -\Psi^{JI}$. Thus, the fields Ψ^{IJ} belong to the coset $O(2N)_1/U(1)$. It is not difficult to show that as current algebras, $O(2N)_1 \sim U(1)\times SU(N)_1$. The U(1) is precisely the one generated by ∂X.

We may now compute the OPE of the Cartan currents $\hat{J}^{II}$,

$$\hat{J}^{II}(z)\hat{J}^{JJ}(w) = \frac{\delta^{IJ}}{(z-w)^2} + \text{regular}. \tag{9.2.13}$$

Using (9.2.11) we finally obtain

$$J^{II}(z)J^{JJ}(w) = \frac{\delta^{IJ} - 1/4}{(z-w)^2} + \text{regular}. \tag{9.2.14}$$

9.2.1 $\mathcal{N} = 1_4$ space-time supersymmetry

In this case, there is a single field Σ and a single current that we will call J

$$J = 2J^{11}, \quad J(z)J(w) = \frac{3}{(z-w)^2} + \text{regular}, \tag{9.2.15}$$

and no Ψ operator because of the antisymmetry. From (9.2.6) we compute the three-point function to find

$$\langle J(z_1)\Sigma(z_2)\bar{\Sigma}(z_3)\rangle = \frac{3}{2}\frac{z_{23}^{1/4}}{z_{12}z_{13}}. \tag{9.2.16}$$

We learn that $\Sigma, \bar{\Sigma}$ are affine primaries with U(1) charges 3/2 and −3/2 respectively. Bosonize the U(1) current and separate the charge degrees of freedom

$$J = i\sqrt{3}\partial\Phi, \quad \Sigma = e^{i\sqrt{3}\Phi/2}W^+, \quad \bar{\Sigma} = e^{-i\sqrt{3}\Phi/2}W^-, \quad \langle\Phi(z)\Phi(w)\rangle = -\log(z-w), \tag{9.2.17}$$

where $W^\pm$ do not depend on Φ. If we write the internal Virasoro operator as $T^{\text{int}} = \hat{T} + T_\Phi$ with $T_\Phi = -(\partial\Phi)^2/2$, then $\hat{T}$ and T_Φ commute. The fact that the dimension of the Σ fields is equal to the U(1) charge squared over 2 implies that $W^\pm$ have dimension zero and thus must be proportional to the identity. Consequently $\Sigma, \bar{\Sigma}$ are pure vertex operators of the field Φ.

Now consider the internal supercurrent and expand it in operators with well-defined U(1) charge

$$G^{\text{int}} = \sum_{q\geq 0} e^{iq\Phi}T^{(q)} + e^{-iq\Phi}T^{(-q)}, \tag{9.2.18}$$

where the operators $T^{(\pm q)}$ do not depend on Φ. Then, (9.2.8) implies that q in (9.2.18) can only take the value $q = 1/\sqrt{3}$. We can write $G^{\text{int}} = G^+ + G^-$ with

$$J(z)G^{\pm}(w) = \pm\frac{G^{\pm}(w)}{(z-w)} + \cdots . \tag{9.2.19}$$

Finally the $\mathcal{N} = (1,0)_2$ superconformal algebra satisfied by G^{int} implies that, separately, $G^{\pm}$ are Virasoro primaries with weight 3/2. Moreover the fact that G^{int} satisfies (4.13.8) on page 78 implies that J, $G^{\pm}$, T^{int} satisfy the $\mathcal{N} = (2,0)_2$ superconformal algebra (4.13.16)–(4.13.21) with $c = 9$. The reverse argument is obvious: if the internal CFT has $\mathcal{N} = (2,0)_2$ invariance, then one can use the (chiral) operators of charge $\pm 3/2$ to construct the space-time supersymmetry charges. In section 4.13.2 on page 79 we have shown, using the spectral flow, that such R operators are always in the spectrum since they are the images of the NS ground state.

We now describe how the massless spectrum emerges from the general properties of the internal $\mathcal{N} = (2,0)_2$ superconformal algebra. As discussed in section 4.13.2, in the NS sector of the internal $\mathcal{N} = (2,0)_2$ CFT, there are two relevant ground states, the vacuum $|0\rangle$ and the chiral ground states $|\Delta, q\rangle = |1/2, \pm 1\rangle$. We have also the four-dimensional left-moving world-sheet fermion oscillators ψ_r^{μ} and the four-dimensional right-moving bosonic oscillators $\bar{\alpha}_n^{\mu}$. In the right-moving sector of the internal CFT, we have, apart from the vacuum state, a collection of $\bar{\Delta} = 1$ states. Combining the internal ground states, we obtain

$$|\Delta, q; \bar{\Delta}\rangle : \quad |0,0;0\rangle, \quad |0,0;1\rangle^I, \quad |1/2,\pm 1;1\rangle^i, \tag{9.2.20}$$

where the indices $I = 1, 2, \ldots, M$, $i = 1, 2 \ldots, \bar{M}$ count the various such states. The physical massless bosonic states are

- $\psi^{\mu}_{-1/2}\bar{\alpha}^{\nu}_{-1}|0,0;0\rangle$, which provide the graviton, antisymmetric tensor, and dilaton,
- $\psi^{\mu}_{-1/2}|0,0;1\rangle^I$, which provide the massless vectors of the gauge group with dimension M,
- $|1/2,\pm 1;1\rangle^i$, which provide $\bar{M}$ complex scalars.

Taking into account also the fermions from the R sector, we can organize the massless spectrum in multiplets of $\mathcal{N} = 1_4$ supersymmetry. Using the results of appendix D, we obtain the $\mathcal{N} = 1_4$ supergravity multiplet, one tensor multiplet (equivalent under a duality transformation to a chiral multiplet), M vector multiplets, and $\bar{M}$ chiral multiplets.

9.2.2 $\mathcal{N} = 2_4$ space-time supersymmetry

In this case there are two fields $\Sigma^{1,2}$ and four currents J^{IJ}. Define $J^s = J^{11} + J^{22}$, $J^3 = (J^{11} - J^{22})/2$ in order to diagonalize (9.2.14):

$$J^s(z)J^s(w) = \frac{1}{(z-w)^2} + \cdots ,$$
$$J^3(z)J^3(w) = \frac{1/2}{(z-w)^2} + \cdots ,$$
$$J^s(z)J^3(w) = \cdots . \tag{9.2.21}$$

As before we compute, using (9.2.6), (9.2.7) the three-point functions $\langle J\Sigma\Sigma\rangle$. From these we learn that under (J^s, J^3), Σ^1 has charges $(1/2, 1/2)$, Σ_2 has $(1/2, -1/2)$, $\bar{\Sigma}^1$ $(-1/2, -1/2)$, and $\bar{\Sigma}^2$ $(-1/2, 1/2)$. Moreover, their charges saturate their conformal weights so that if we bosonize the currents then the fields Σ, $\bar{\Sigma}$ are pure vertex operators

$$J^s = i\partial\phi, \quad J^3 = \frac{i}{\sqrt{2}}\partial\chi, \tag{9.2.22}$$

$$\Sigma^1 = \exp\left[\frac{i}{2}\phi + \frac{i}{\sqrt{2}}\chi\right], \quad \Sigma^2 = \exp\left[\frac{i}{2}\phi - \frac{i}{\sqrt{2}}\chi\right], \tag{9.2.23}$$

$$\bar{\Sigma}^1 = \exp\left[-\frac{i}{2}\phi - \frac{i}{\sqrt{2}}\chi\right], \quad \bar{\Sigma}^2 = \exp\left[-\frac{i}{2}\phi + \frac{i}{\sqrt{2}}\chi\right]. \tag{9.2.24}$$

Using these in (9.2.6) we obtain that $J^{12} = \exp[i\sqrt{2}\chi]$ and $J^{21} = \exp[-i\sqrt{2}\chi]$. Thus, J^3, J^{12}, J^{21} form the current algebra $SU(2)_1$. Moreover, $\Psi^{12} = \exp[i\phi]$, $\bar{\Psi}^{12} = \exp[-i\phi]$.

We again consider the internal supercurrent and expand it in charge eigenstates. Using (9.2.5) we can verify that the charges that can appear are $(\pm 1, 0)$ and $(0, \pm 1/2)$. We can split

$$G^{\text{int}} = G_{(2)} + G_{(4)}, \quad G_{(2)} = G^+_{(2)} + G^-_{(2)},$$
$$G_{(4)} = G^+_{(4)} + G^-_{(4)}, \tag{9.2.25}$$

where $G^{\pm}_{(2)}$ have charges $(\pm 1, 0)$ and $G^{\pm}_{(4)}$ have charges $(0, \pm 1/2)$. This is attested by the following OPEs:

$$J^s(z) G^{\pm}_{(2)}(w) = \pm \frac{G^{\pm}_{(2)}(w)}{z-w} + \cdots, \quad J^3(z) G^{\pm}_{(4)}(w) = \pm \frac{1}{2}\frac{G^{\pm}_{(4)}(w)}{z-w} + \cdots, \tag{9.2.26}$$

$$J^s(z) G^{\pm}_{(4)}(w) = \text{finite}, \quad J^3(z) G^{\pm}_{(2)}(w) = \text{finite}, \quad G^{\pm}_{(2)} = e^{\pm i\phi} Z^{\pm}. \tag{9.2.27}$$

$Z^{\pm}$ are dimension-1 operators. They can be written in terms of scalars as $Z^{\pm} = i\partial X^{\pm}$. The vertex operators $e^{\pm i\phi}$ are those of a complex free fermion. Thus, the part of the internal theory corresponding to $G^{(2)}$ is a free two-dimensional CFT with $c = 3$. Finally it can be shown that the SU(2) algebra acting on $G^{\pm}_{(4)}$ supercurrents generates two more supercurrents that form the $\mathcal{N} = (4, 0)_2$ superconformal algebra (4.13.29)–(4.13.31) on page 81 with $c = 6$.

Since there is a complex free fermion $\psi = e^{i\phi}$ in the $c = 3$ internal CFT we can construct two massless vector boson states $\psi_{-1/2}\bar{a}^{\mu}_{-1}|p\rangle$ and $\bar{\psi}_{-1/2}\bar{a}^{\mu}_{-1}|p\rangle$. One of them is the graviphoton belonging to the $\mathcal{N} = 2_4$ supergravity multiplet while the other is the vector belonging to the vector-tensor multiplet (to which the dilaton and $B_{\mu\nu}$ also belong). The vectors of massless vector multiplets correspond to states of the form $\psi^{\mu}_{-1/2}\bar{J}^a_{-1}|p\rangle$, where $\bar{J}^a$ is a right-moving affine current. The associated massless complex scalar of the vector multiplet corresponds to the state $\psi_{-1/2}\bar{J}^a_{-1}|p\rangle$. Massless hypermultiplet bosons arise from the $\mathcal{N} = (4, 0)_2$ internal CFT. As already described in section 4.13.3 on page 81, an $\mathcal{N} = (4, 0)_2$ superconformal CFT with $c = 6$ always contains states with $\Delta = \frac{1}{2}$ that transform as two conjugate doublets of the $SU(2)_1$ current algebra. Combining them with a right-moving operator with $\bar{\Delta} = 1$ gives the four massless scalars of a hypermultiplet.

In the maximal case, namely $\mathcal{N} = 4_4$ space-time supersymmetry, the internal CFT must be free (toroidal). You are invited to show this in exercise 9.4 on page 287. The six graviphotons participating in the $\mathcal{N} = 4_4$ supergravity multiplet are states of the form

$\bar{a}^{\mu}_{-1}\psi^{I}_{-1/2}|p\rangle$ where $I = 1, \ldots, 6$ and the ψ^I are the fermionic partners of the six left-moving currents of the toroidal CFT mentioned above.

In our previous discussion, there are no constraints due to space-time SUSY on the right-moving side of the heterotic string.

To summarize, in the $D = 4$ heterotic string, the internal CFT has at least $\mathcal{N} = (1, 0)_2$ invariance. If it has $\mathcal{N} = (2, 0)_2$ then we have $\mathcal{N} = 1_4$ space-time SUSY. If we have a $(c = 3)$ $\mathcal{N} = (2, 0)_2 \oplus (c = 6)$ $\mathcal{N} = (4, 0)_2$ CFT then we have $\mathcal{N} = 2_4$ in space-time. Finally, if we have six free left-moving coordinates then we have $\mathcal{N} = 4_4$ in four-dimensional space-time.

In the type-II theory, the situation is similar, but here the supersymmetries can come from either the right-moving and/or the left-moving side. For example, $\mathcal{N} = 1_4$ space-time supersymmetry needs a $\mathcal{N} = (2, 1)_2$ or $\mathcal{N} = (1, 2)_2$ world-sheet SUSY. For $\mathcal{N} = 2_4$ space-time supersymmetry there are two possibilities. Either we must have $\mathcal{N} = (2, 2)_2$, in which one supersymmetry comes from the right-moving sector and the other from the left-moving sector, or $(c = 3)$ $\mathcal{N} = (2, 1)_2 \oplus (c = 6)$ $\mathcal{N} = (4, 1)_2$ CFT in which both space-time supersymmetries come from one side.

9.3 Orbifold Reduction of Supersymmetry

We are interested in vacua with a four-dimensional flat space-time times some compact internal manifold. In the most general case, such vacua are given by the tensor product of a four-dimensional noncompact flat CFT and an internal (compact) CFT. A CFT with appropriate central charge and world-sheet symmetries is an exact solution of the (tree-level) string equations of motion to all orders in α'. In the heterotic case, this internal CFT must have $\mathcal{N} = (1, 0)_2$ invariance and $(c, \bar{c}) = (9, 22)$. In the type-II case it must have $\mathcal{N} = (1, 1)_2$ superconformal invariance and $(c, \bar{c}) = (9, 9)$. If the CFT has a large volume limit, then an α'-expansion is possible and we can recover the leading σ-model (geometrical) results.

In this section we will consider orbifold CFTs which will provide compactification spaces that reduce the maximal supersymmetry in four dimensions. The advantage of orbifolds is that they are exactly soluble CFTs and yet they have the essential characteristics of nontrivial curved compactifications. In the next few sections we will give examples of orbifolds with $\mathcal{N} = 2_4$ and $\mathcal{N} = 1_4$ supersymmetry. We will focus first on the heterotic string.

We have already seen in section 9.1 that the toroidal compactification of the heterotic string down to four dimensions, gives a theory with $\mathcal{N} = 4_4$ supersymmetry. We have to find orbifold symmetries under which some of the four four-dimensional gravitini are not invariant. They will be projected out of the spectrum and we will be left with a theory that has less supersymmetry. To find such symmetries we have to look carefully at the vertex operators of the gravitini first. We will work in the light-cone gauge and it will be convenient to bosonize the eight transverse left-moving fermions ψ^i into four left-moving scalars. Pick a complex basis for the fermions

$$\Psi^0 = \frac{1}{\sqrt{2}}(\psi^3 + i\psi^4), \quad \Psi^1 = \frac{1}{\sqrt{2}}(\psi^5 + i\psi^6), \tag{9.3.1}$$

$$\Psi^2 = \frac{1}{\sqrt{2}}(\psi^7 + i\psi^8), \quad \Psi^3 = \frac{1}{\sqrt{2}}(\psi^9 + i\psi^{10}), \tag{9.3.2}$$

and similarly for $\bar{\Psi}^I$. They satisfy

$$\langle \Psi^I(z)\bar{\Psi}^J(w)\rangle = \frac{\delta^{IJ}}{z-w}, \quad \langle \Psi^I(z)\Psi^J(w)\rangle = \langle \bar{\Psi}^I(z)\bar{\Psi}^J(w)\rangle = 0. \tag{9.3.3}$$

The four Cartan currents of the left-moving $O(8)_1$ current algebra $J^I = \Psi^I\bar{\Psi}^I$ can be written in terms of four free bosons as

$$J^I(z) = i\partial_z\phi^I(z), \quad \langle \phi^I(z)\phi^J(w)\rangle = -\delta^{IJ}\log(z-w). \tag{9.3.4}$$

In terms of the bosons

$$\Psi^I =: e^{i\phi^I} :, \quad \bar{\Psi}^I =: e^{-i\phi^I} :. \tag{9.3.5}$$

The spinor primary states are given by

$$V(\epsilon_I) =: \exp\left[\frac{i}{2}\sum_{I=0}^{3}\epsilon_I\,\phi^I\right]:, \tag{9.3.6}$$

with $\epsilon_I = \pm 1$. This operator has $2^4 = 16$ components and contains both the S and the C O(8) spinors.

The fermionic system has an O(8) global symmetry (the zero-mode part of the $O(8)_1$ current algebra). Its $U(1)^4$ abelian subgroup acts as

$$\Psi^I \to e^{2\pi i\theta^I}\Psi^I, \quad \bar{\Psi}^I \to e^{-2\pi i\theta^I}\bar{\Psi}^I. \tag{9.3.7}$$

This acts equivalently on the bosons as

$$\phi^I \to \phi^I + 2\pi\,\theta^I. \tag{9.3.8}$$

A $\mathbb{Z}_2$ subgroup of the $U(1)^4$ symmetry, namely, $\theta^I = 1/2$ for all I, is the $(-1)^{F_R}$ symmetry. Under this transformation, the fermions are odd. The spinor vertex operator transforms with a phase $\exp[i\pi(\sum_I \epsilon^I)/2]$. Therefore,

- $\sum_I \epsilon^I = 4k$, $k \in \mathbb{Z}$ corresponds to the spinor S,
- $\sum_I \epsilon^I = 4k+2$, $k \in \mathbb{Z}$ corresponds to the conjugate spinor C.

The standard GSO projection picks one of the two spinors, let us say the S. Consider the massless physical vertex operators given by

$$V^{\pm,\epsilon} = \bar{\partial}X^{\pm}\,V_S(\epsilon)e^{ip\cdot X}, \quad X^{\pm} = \frac{1}{\sqrt{2}}(X^3 \pm iX^4). \tag{9.3.9}$$

The boson ϕ^0 was constructed from the $D=4$ light-cone space-time fermions and thus carries four-dimensional helicity. The $X^{\pm}$ bosons also carry four-dimensional helicity ± 1. The subset of the vertex operators in (9.3.9) that corresponds to the gravitini are $\bar{\partial}X^+V(\epsilon^0=1)$, with helicity 3/2, and $\bar{\partial}X^-V(\epsilon^0=-1)$, with helicity $-3/2$. Taking also into account the GSO projection we find four helicity ($\pm 3/2$) states, as we expect in an $\mathcal{N}=4_4$ theory.

Consider the maximal subgroup $O(2)\times O(6)\subset O(8)$ where the O(2) corresponds to the four-dimensional helicity. The O(6) symmetry is an internal symmetry from the four-dimensional point of view. It is the so-called R-symmetry of $\mathcal{N}=4_4$ supersymmetry, since the supercharges transform as the four-dimensional spinor of O(6). O(6) is an automorphism of the $\mathcal{N} = 4_4$ supersymmetry algebra. Since the supercharges are used to generate the states of an $\mathcal{N} = 4_4$ supermultiplet, the various states inside the multiplet have well-defined transformation properties under the O(6) R-symmetry. Here are some useful examples.

The $\mathcal{N} = 4_4$ SUGRA multiplet. It contains the graviton (singlet of O(6)) four Majorana gravitini (spinor of O(6)), six graviphotons (vector of O(6)), four Majorana fermions (conjugate spinor of O(6)), and two scalars (singlets).

The massless spin-3/2 multiplet. It contains a gravitino (singlet), four vectors (spinor), seven Majorana fermions (vector plus singlet), and eight scalars (spinor + conjugate spinor).

The massless vector multiplet. It contains a vector (singlet), four Majorana fermions (spinor), and six scalars (vector).

To break the $\mathcal{N} = 4_4$ symmetry, it is enough to break the O(6) R-symmetry.

We now search for symmetries of the CFT that will reduce, after orbifolding, the supersymmetry. In order to preserve Lorentz invariance, the symmetry should not act on the four-dimensional supercoordinates X^μ, ψ^μ. The rest are symmetries acting on the internal left-moving fermions and a simple class are the discrete subgroups of the $U(1)^3$ subgroup of O(6) acting on the fermions. There are also symmetries acting on the bosonic $(6, 22)$ compact CFT. An important constraint on such symmetries is that they leave the internal supercurrent

$$G^{\rm int} = \sum_{i=5}^{10} \psi^i \, \partial X^i \tag{9.3.10}$$

invariant. The reason is that $G^{\rm int}$ along with $G^{D=4}$ (which is invariant since we are not acting on the $D = 4$ part) define the constraints responsible for the absence of ghosts. Messing them up will jeopardize the unitarity of the orbifold theory.

The generic symmetries of the internal toroidal theory are translations and SO(6) rotations of the (6,6) part as well as gauge transformations of the (0,16) part. Thus a generic orbifold group will be a combination of them all. Translations and gauge transformations do not affect the massless gravitini. Under the SO(6) rotations the gravitini transform as a four-dimensional spinor. We must therefore study the transformation of the spinor under an SO(6) rotation. Any rotation can be conjugated to the Cartan subalgebra, so it will be a combination of three O(2) rotations in the three planes of T^6. Let, θ^1 be the angle of rotation in the 5-6 plane, θ^2 in the 7-8 plane, and θ^3 in the 9-10 plane. Then, the respective fermions transform as in (9.3.7) and the transformation of the spinors can be obtained from (9.3.6). If we like to preserve a single gravitino, lets say the one corresponding to $(++++)$, $(----)$, then the condition on the rotation angles is

$$\theta^1 + \theta^2 + \theta^3 = 0 \pmod{2\pi}. \tag{9.3.11}$$

The original four-dimensional SO(6) spinor decomposes as $4 \to (1 + 3)$ under the group $G \subset SO(6)$. Therefore, G can be at most SU(3). The final result is that orbifold rotations inside an SU(3) subgroup of SO(6) preserve at least $\mathcal{N} = 1_4$ space-time supersymmetry.

9.4 A Heterotic Orbifold with $\mathcal{N}=2_4$ Supersymmetry

We will describe here a simple example of a $\mathbb{Z}_2$ orbifold that will produce $\mathcal{N}=2_4$ supersymmetry.

Consider the toroidal compactification of the heterotic string. Set the Wilson lines to zero and pick appropriately the internal six-torus G, B so that the (6, 22) lattice factorizes as $(2,2)\otimes(4,4)\otimes(0,16)$. This lattice has a symmetry that changes the sign of all the (4,4) bosonic coordinates. To keep the internal supercurrent invariant we must also change the sign of the fermions ψ^i, $i=7,8,9,10$. This corresponds to shifting the associated bosons

$$\phi^2 \to \phi^2+\pi, \quad \phi^3 \to \phi^3-\pi. \tag{9.4.1}$$

Under this transformation, two of the gravitini vertex operators are invariant while the other two transform with a minus sign. This is exactly what we need. It turns out, however, that this simple orbifold action does not give a modular-invariant partition function.

We must make a further action somewhere else. What remains is the (0, 16) part. Consider the case in which it corresponds to the $E_8\times E_8$ lattice. As we have mentioned already, $E_8 \ni [\mathbf{248}] \to [\mathbf{120}]\oplus[\mathbf{128}] \in O(16)$. Decomposing further with respect to the $SU(2)\times SU(2)\times O(12)$ subgroup of O(16), we obtain:

$$[\mathbf{120}] \to [\mathbf{3},\mathbf{1},\mathbf{1}]\oplus[\mathbf{1},\mathbf{3},\mathbf{1}]\oplus[\mathbf{1},\mathbf{1},\mathbf{66}]\oplus[\mathbf{2},\mathbf{1},\mathbf{12}]\oplus[\mathbf{1},\mathbf{2},\mathbf{12}], \tag{9.4.2}$$

$$[\mathbf{128}] \to [\mathbf{2},\mathbf{1},\mathbf{32}]\oplus[\mathbf{1},\bar{\mathbf{2}},\mathbf{32}]. \tag{9.4.3}$$

We choose the $\mathbb{Z}_2$ action on E_8 to take the spinors (the [**2**]'s) of the two SU(2) subgroups to minus themselves, but keep the conjugate spinors (the [$\bar{\mathbf{2}}$]'s) invariant. This projection keeps the $[\mathbf{3},\mathbf{1},\mathbf{1}],[\mathbf{1},\mathbf{3},\mathbf{1}],[\mathbf{1},\mathbf{1},\mathbf{66}],[\mathbf{1},\bar{\mathbf{2}},\mathbf{32}]$ representations that combine to form the group $SU(2)\times E_7$. This can be seen by decomposing the adjoint of E_8 under its $SU(2)\times E_7$ subgroup.

$$E_8 \ni [\mathbf{248}] \to [\mathbf{1},\mathbf{133}]\oplus[\mathbf{3},\mathbf{1}]\oplus[\mathbf{2},\mathbf{56}] \in SU(2)\times E_7, \tag{9.4.4}$$

where in this basis the above transformation corresponds to $[\mathbf{3}]\to[\mathbf{3}]$ and $[\mathbf{2}]\to-[\mathbf{2}]$. The reason why we considered a more complicated way in terms of orthogonal groups is that, in this language, the construction of the orbifold blocks is straightforward.

We will now construct the various orbifold blocks. The left-moving fermions contribute

$$\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\,\frac{\vartheta^2[^a_b]\vartheta[^{a+h}_{b+g}]\vartheta[^{a-h}_{b-g}]}{\eta^4}. \tag{9.4.5}$$

The bosonic (4,4) blocks can be constructed in a similar fashion to (4.21.10) on page 110. We obtain

$$Z_{(4,4)}[^0_0]=\frac{\Gamma_{4,4}}{\eta^4\bar\eta^4}, \quad Z_{(4,4)}[^h_g]=2^4\,\frac{\eta^2\bar\eta^2}{\vartheta^2[^{1-h}_{1-g}]\bar\vartheta^2[^{1-h}_{1-g}]}, \quad (h,g)\neq(0,0). \tag{9.4.6}$$

The blocks of the E_8 factor in which our projection acts are given by

$$\frac{1}{2}\sum_{\gamma,\delta=0}^{1}\frac{\bar\vartheta[^{\gamma+h}_{\delta+g}]\bar\vartheta[^{\gamma-h}_{\delta-g}]\bar\vartheta^6[^{\gamma}_{\delta}]}{\bar\eta^8}. \tag{9.4.7}$$

Finally there is a (2,2) toroidal and an E_8 part that are not touched by the projection. Putting all things together we obtain the heterotic partition function of the $\mathbb{Z}_2$ orbifold

$$Z^{\text{heterotic}}_{\text{N=2}} = \frac{1}{2}\sum_{h,g=0}^{1}\frac{\Gamma_{2,2}\,\bar\Gamma_{E_8}Z_{(4,4)}[^h_g]}{\tau_2\eta^4\bar\eta^{12}}\;\frac{1}{2}\sum_{\gamma,\delta=0}^{1}\frac{\bar\vartheta[^{\gamma+h}_{\delta+g}]\bar\vartheta[^{\gamma-h}_{\delta-g}]\bar\vartheta^6[^{\gamma}_{\delta}]}{\bar\eta^8}$$
$$\times\frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\,\frac{\vartheta^2[^a_b]\vartheta[^{a+h}_{b+g}]\vartheta[^{a-h}_{b-g}]}{\eta^4}. \tag{9.4.8}$$

This partition function is modular invariant. The massless spectrum is as follows: from the untwisted sector ($h = 0$) we obtain the graviton, an antisymmetric tensor, vectors in the adjoint of $G = U(1)^4 \times SU(2) \times E_7 \times E_8$, a complex scalar in the adjoint of the gauge group G, 16 more neutral scalars as well as scalars transforming as four copies of the $[\mathbf{2}, \mathbf{56}]$ representation of $SU(2) \times E_7$. From the twisted sector ($h = 1$), we obtain scalars only, transforming as 32 copies of the $[\mathbf{1}, \mathbf{56}]$ and 128 copies of the $[\mathbf{2}, \mathbf{1}]$.

As mentioned before, this theory has $\mathcal{N} = 2_4$ local supersymmetry. The associated R-symmetry is SU(2), which rotates the two supercharges. We will describe the relevant massless representations and their transformation properties under the R-symmetry.

The SUGRA multiplet contains the graviton (singlet), two Majorana gravitini (doublet), and a vector (singlet).

The vector multiplet contains a vector (singlet), two Majorana fermions (doublet), and a complex scalar (singlet).

The vector-tensor multiplet contains a vector (singlet), two Majorana fermions (doublet), a real scalar (singlet), and an antisymmetric tensor (singlet).

The hypermultiplet contains two Majorana fermions (singlets) and four scalars (two doublets).

The vector-tensor multiplet and the vector multiplet are related by a duality transformation of the two-form.

We can now arrange the massless states into $\mathcal{N} = 2_4$ multiplets. We have the SUGRA multiplet, a vector-tensor multiplet (containing the dilaton), a vector multiplet in the adjoint of $U(1)^2 \times SU(2) \times E_7 \times E_8$; the rest are hypermultiplets transforming under $SU(2) \times E_7$ as $4[\mathbf{1}, \mathbf{1}] + [\mathbf{2}, \mathbf{56}] + 8[\mathbf{1}, \mathbf{56}] + 32[\mathbf{2}, \mathbf{1}]$.

We will investigate further the origin of the SU(2) R-symmetry. Consider the four real left-moving fermions $\psi^{7,\ldots,10}$. Although they transform with a minus sign under the orbifold action, their $O(4) \sim SU(2) \times SU(2)$ currents, being bilinear in the fermions, are invariant. Relabel the four real fermions as ψ^0 and ψ^a, $a = 1, 2, 3$. Then, the $SU(2)_1 \times SU(2)_1$ current algebra is generated by

$$J^a = -\frac{i}{2}\left[\psi^0\psi^a + \frac{1}{2}\epsilon^{abc}\psi^b\psi^c\right], \quad \tilde J^a = -\frac{i}{2}\left[\psi^0\psi^a - \frac{1}{2}\epsilon^{abc}\psi^b\psi^c\right]. \tag{9.4.9}$$

Although both SU(2)'s are invariant in the untwisted sector, the situation in the twisted sector is different. The O(4) spinor ground state decomposes as $[\mathbf{4}] \to [\mathbf{2},\mathbf{1}] + [\mathbf{1},\mathbf{2}]$ under $SU(2) \times SU(2)$. The orbifold projection acts trivially on the spinor of the first SU(2) and with a minus sign on the spinor of the second. The orbifold projection breaks the second SU(2) invariance. The remaining $SU(2)_1$ invariance becomes the R-symmetry of the $\mathcal{N} = 2_4$ theory. Moreover, the only operators at the massless level that transform non-trivially under the SU(2) are the linear combinations

$$V^{\pm}_{\alpha\beta} = \pm i(\delta_{\alpha\beta}\psi^0 \pm i\sigma^a_{\alpha\beta}\ \psi^a), \tag{9.4.10}$$

which transform as the $[\mathbf{2}]$ and $[\bar{\mathbf{2}}]$, respectively, as well as the $[\mathbf{2}]$ spinor in the R-sector. We obtain

$$V^{+}_{\alpha\gamma}(z)V^{+}_{\gamma\beta}(w) = V^{-}_{\alpha\gamma}(z)V^{-}_{\gamma\beta}(w) = \frac{\delta_{\alpha\beta}}{z-w} - 2\sigma^a_{\alpha\beta}(J^a(w) - \tilde{J}^a(w)) + \mathcal{O}(z-w), \tag{9.4.11}$$

$$V^{+}_{\alpha\gamma}(z)V^{-}_{\gamma\beta}(w) = \frac{3\delta_{\alpha\beta}}{z-w} + 4\sigma^a_{\alpha\beta}\tilde{J}^a(w) + \mathcal{O}(z-w), \tag{9.4.12}$$

$$V^{-}_{\alpha\gamma}(z)V^{+}_{\gamma\beta}(w) = \frac{3\delta_{\alpha\beta}}{z-w} - 4\sigma^a_{\alpha\beta}\ J^a(w) + \mathcal{O}(z-w), \tag{9.4.13}$$

where a summation over γ is implied.

This $SU(2)_1$ current algebra combines with four operators of conformal weight 3/2 to make the $\mathcal{N} = (4,0)_2$ superconformal algebra in any theory with $\mathcal{N} = 2_4$ space-time supersymmetry. This agrees with the general discussion of section 9.2.

In an $\mathcal{N} = 2_4$ theory, the complex scalars that are partners of the gauge bosons, belonging to the Cartan of the gauge group, are moduli (they have no potential). If they acquire generic expectation values, they break the gauge group down to the Cartan. All charged hypermultiplets acquire masses in such a case.

A generalization of the above orbifold, where all Higgs expectation values are turned on, corresponds to splitting the original (6,22) lattice to (4,4)⊕(2,18). We perform a $\mathbb{Z}_2$ reversal in the (4, 4) part, which will break $\mathcal{N} = 4_4 \to \mathcal{N} = 2_4$. In the (2,18) lattice we can only perform a $\mathbb{Z}_2$ translation (otherwise the supersymmetry will be broken further). We perform a translation by $\epsilon/2$, where $\epsilon \in L_{2,18}$. Then the partition function is

$$Z^{\text{heterotic}}_{\text{N=2}} = \frac{1}{2}\sum_{h,g=0}^{1} \frac{\Gamma_{2,18}(\epsilon)\left[{h \atop g}\right] Z_{(4,4)}\left[{h \atop g}\right]}{\tau_2\eta^4\bar{\eta}^{20}}\ \frac{1}{2}\sum_{a,b=0}^{1}(-1)^{a+b+ab}\ \frac{\vartheta^2\left[{a \atop b}\right]\vartheta\left[{a+h \atop b+g}\right]\vartheta\left[{a-h \atop b-g}\right]}{\eta^4}; \tag{9.4.14}$$

the shifted lattice sum $\Gamma_{2,18}(\epsilon)\left[{h \atop g}\right]$ is described in Appendix B.

The theory depends on the 2×18 moduli of $\Gamma_{2,18}(\epsilon)\left[{h \atop g}\right]$ and the 16 moduli in $Z_{4,4}\left[{0 \atop 0}\right]$. There are, apart from the vector-tensor multiplet, another 18 massless vector multiplets. The 2×18 moduli are the scalars of these vector multiplets. There are also four neutral hypermultiplets whose scalars are the untwisted (4,4) orbifold moduli. At special submanifolds of the vector multiplet moduli space, extra massless vector multiplets and/or hypermultiplets can appear. We have seen such a symmetry enhancement already at the level of the CFT.

The local structure of the vector moduli space is $O(2,18)/O(2)\times O(18)$. From the real moduli, $G_{\alpha\beta}, B_{\alpha\beta}, Y^I_\alpha$ we can construct the 18 complex moduli $T=T_1+iT_2$, $U=U_1+iU_2$, $W^I=W^I_1+iW^I_2$ as follows:

$$G=\frac{T_2-\frac{W^I_2W^I_2}{2U_2}}{U_2}\begin{pmatrix}1 & U_1\\ U_1 & |U|^2\end{pmatrix},$$

$$B=\left(T_1-\frac{W^I_1W^I_2}{2U_2}\right)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \tag{9.4.15}$$

and $W^I=-Y^I_2+UY^I_1$. There is also one more complex scalar, the S field with $\mathrm{Im}S=S_2=e^{-\phi}$, whose real part is the axion a, which comes from dualizing the antisymmetric tensor. The tree-level prepotential and Kähler potential are[5]

$$\mathcal{F}=S(TU-\tfrac{1}{2}W^IW^I),\quad K=-\log(S_2)-\log\left[U_2T_2-\tfrac{1}{2}W^I_2W^I_2\right]. \tag{9.4.16}$$

The hypermultiplets belong to the quaternionic manifold $O(4,4)/O(4)\times O(4)$. $\mathcal{N}=2_4$ supersymmetry does not permit neutral couplings between vector and hypermultiplets at the two-derivative level. The dilaton belongs to a vector multiplet. Therefore, the hypermultiplet moduli space does not receive perturbative or nonperturbative corrections.

In this class of $\mathcal{N}=2_4$ vacua, we will consider the helicity supertrace B_2 which traces the presence of $\mathcal{N}=2_4$ (short) BPS multiplets.[6] The computation is straightforward, using the definitions of appendix J on page 537 and is the subject of exercise 9.8 on page 288. We find

$$\tau_2\,B_2=\tau_2\,\langle\lambda^2\rangle=\Gamma_{2,18}[^0_1]\frac{\bar\vartheta_3^2\bar\vartheta_4^2}{\bar\eta^{24}}-\Gamma_{2,18}[^1_0]\frac{\bar\vartheta_2^2\bar\vartheta_3^2}{\bar\eta^{24}}-\Gamma_{2,18}[^1_1]\frac{\bar\vartheta_2^2\bar\vartheta_4^2}{\bar\eta^{24}}=\frac{\Gamma_{2,18}[^0_0]+\Gamma_{2,18}[^0_1]}{2}\bar F_1$$
$$-\frac{\Gamma_{2,18}[^0_0]-\Gamma_{2,18}[^0_1]}{2}\bar F_1-\frac{\Gamma_{2,18}[^1_0]+\Gamma_{2,18}[^1_0]}{2}\bar F_+-\frac{\Gamma_{2,18}[^1_0]-\Gamma_{2,18}[^1_0]}{2}\bar F_- \tag{9.4.17}$$

with

$$\bar F_1=\frac{\bar\vartheta_3^2\bar\vartheta_4^2}{\bar\eta^{24}},\quad \bar F_\pm=\frac{\bar\vartheta_2^2(\bar\vartheta_3^2\pm\bar\vartheta_4^2)}{\bar\eta^{24}}. \tag{9.4.18}$$

For all $\mathcal{N}=2_4$ heterotic vacua, B_2 transforms as

$$\tau\to\tau+1:\quad B_2\to B_2,\quad \tau\to-\frac{1}{\tau}:\quad B_2\to\tau^2\,B_2. \tag{9.4.19}$$

All functions $\bar F_i$ have positive Fourier coefficients and have the expansions

$$F_1=\frac{1}{q}+\sum_{n=0}^{\infty}d_1(n)q^n=\frac{1}{q}+16+156q+\mathcal{O}(q^2), \tag{9.4.20}$$

$$F_+=\frac{8}{q^{3/4}}+q^{1/4}\sum_{n=0}^{\infty}d_+(n)q^n=\frac{8}{q^{3/4}}+8q^{1/4}(30+481q+\mathcal{O}(q^2)), \tag{9.4.21}$$

$$F_-=\frac{32}{q^{1/4}}+q^{3/4}\sum_{n=0}^{\infty}d_-(n)q^n=\frac{32}{q^{1/4}}+32q^{3/4}(26+375q+\mathcal{O}(q^2)). \tag{9.4.22}$$

[5] The definitions of the prepotential and Kähler potential may be found in appendix I.2 on page 535. The expressions in (9.4.16) can be obtained from these definitions and the kinetic terms of the moduli scalars from (E.22 on page 518).
[6] You will find the definition of helicity supertraces and their relation to BPS multiplicities in appendix J on page 537.

The lattice sums $\frac{1}{2}(\Gamma_{2,18}[^h_0] \pm \Gamma_{2,18}[^h_1])$ also have positive multiplicities. An overall plus sign corresponds to vectorlike multiplets,[7] while a minus sign corresponds to hypermultiplets. A vectorlike multiplet contributes 1 to the supertrace, and a hypermultiplet −1.

The contribution of the generic massless multiplets is given by the constant coefficient of F_1; it agrees with what we expected: $16 = 20 - 4$ since we have the supergravity multiplet and 19 vector multiplets contributing 20 and 4 hypermultiplets contributing −4.

We will analyze the BPS mass formulas associated with (9.4.17). We will use the notation for the shift vector $\epsilon = (\vec{\epsilon}_L; \vec{\epsilon}_R, \vec{\zeta})$, where ϵ_L, ϵ_R are two-dimensional integer vectors and ζ is a vector in the Spin(32)/$\mathbb{Z}_2$ lattice. We also have the modular-invariance constraint $\epsilon^2/2 = \vec{\epsilon}_L \cdot \vec{\epsilon}_R - \vec{\zeta}^2/2 = 1 \pmod 4$.

Using the results of Appendix D, we can write the BPS mass formulas associated with the lattice sums above. In the untwisted sector ($h = 0$), the mass formula is

$$M^2 = \frac{|-m_1 U + m_2 + Tn_1 + (TU - \frac{1}{2}\vec{W}^2)n_2 + \vec{W} \cdot \vec{Q}|^2}{4S_2\left(T_2 U_2 - \frac{1}{2}\mathrm{Im}\vec{W}^2\right)}, \tag{9.4.23}$$

where $\vec{W}$ is the 16-dimensional complex vector of Wilson lines. When the integer

$$\rho = \vec{m} \cdot \vec{\epsilon}_R + \vec{n} \cdot \epsilon_L - \vec{Q} \cdot \vec{\zeta} \tag{9.4.24}$$

is even, these states are vectorlike multiplets with multiplicity function $d_1(s)$ of (9.4.20) and

$$s = \vec{m} \cdot \vec{n} - \frac{1}{2}\vec{Q} \cdot \vec{Q}; \tag{9.4.25}$$

when ρ is odd, these states are hyperlike multiplets with multiplicities $d_1(s)$.

In the twisted sector ($h = 1$), the mass formula is

$$\begin{aligned} M^2 = &\big|(m_1 + \tfrac{1}{2}\epsilon_L^1)U - (m_2 + \tfrac{1}{2}\epsilon_L^2) - T(n_1 + \tfrac{1}{2}\epsilon_R^1) \\ &- (TU - \tfrac{1}{2}\vec{W}^2)(n_2 + \tfrac{1}{2}\epsilon_R^2) \\ &- \vec{W} \cdot (\vec{Q} + \tfrac{1}{2}\vec{\zeta})\big|^2 / 4S_2\left(T_2 U_2 - \tfrac{1}{2}\mathrm{Im}\vec{W}^2\right). \end{aligned} \tag{9.4.26}$$

The states with ρ even are vector-multiplet-like with multiplicities $d_+(s')$, with

$$s' = \left(\vec{m} + \frac{\vec{\epsilon}_L}{2}\right) \cdot \left(\vec{n} + \frac{\vec{\epsilon}_R}{2}\right) - \frac{1}{2}\left(\vec{Q} + \frac{\vec{\zeta}}{2}\right) \cdot \left(\vec{Q} + \frac{\vec{\zeta}}{2}\right), \tag{9.4.27}$$

while the states with ρ odd are hypermultiplets with multiplicities $d_-(s')$.

9.5 Spontaneous Supersymmetry Breaking

We have seen in the previous section that we can break the maximal supersymmetry by the orbifolding procedure. The extra gravitini are projected out of the spectrum. This type of orbifold breaking of supersymmetry we will call explicit breaking.

It turns out that there is an important difference between freely acting and non-freely-acting orbifolds with respect to the restoration of the broken supersymmetry. The example

[7] Vectorlike multiplets are, the vector multiplets, vector tensor multiplets, and the supergravity multiplet.

of the previous section (explicit breaking) corresponded to a non-freely-acting orbifold action.

To make the difference transparent, consider the $\mathbb{Z}_2$ twist on T^4 described before, under which two of the gravitini transform with a minus sign and are thus projected out. Consider also performing at the same time a $\mathbb{Z}_2$ translation (by a half period) in one direction of the extra (2,2) torus. Take the two cycles to be orthogonal, with radii R, R', and do an $X \to X + \pi$ shift on the first cycle. The oscillator modes are invariant but the vertex operator states $|m, n\rangle$ transform with a phase $(-1)^m$.

This is a freely-acting orbifold, since the action on the circle is free. Although the states of the two gravitini, $\bar{a}^{\mu}_{-1}|S^I_a\rangle$ $I = 1, 2$ transform with a minus sign under the twist, the states $\bar{a}^{\mu}_{-1}|S^I_a\rangle \otimes |m = 1, n\rangle$ are invariant! They have the space-time quantum numbers of two gravitini, but they are no longer massless. In fact, in the absence of the state $|m = 1, n\rangle$ they would be massless, but now we have an extra contribution to the mass coming from that state:

$$m_L^2 = \frac{1}{4}\left(\frac{1}{R} + \frac{nR}{\ell_s^2}\right)^2, \quad m_R^2 = \frac{1}{4}\left(\frac{1}{R} - \frac{nR}{\ell_s^2}\right)^2. \tag{9.5.1}$$

The matching condition $m_L = m_R$ implies $n = 0$, so that the mass of these states is $m^2 = 1/4R^2$. These are massive (KK) gravitini and in this theory, the $\mathcal{N} = 4_4$ supersymmetry is broken spontaneously to $\mathcal{N} = 2_4$. In field theory language, the effective field theory is a gauged version of $\mathcal{N} = 4_4$ supergravity where the supersymmetry is spontaneously broken to $\mathcal{N} = 2_4$ at the minimum of the potential.

We will note here some important differences between explicit and spontaneous breaking of supersymmetry.

- In spontaneously broken supersymmetric vacua, the behavior at high energies is softer than the case of explicit breaking. If supersymmetry is spontaneously broken, there are still broken Ward identities that govern the short distance properties of the theory. In such theories, there is a characteristic energy scale, namely the gravitino mass $m_{3/2}$ above which supersymmetry is effectively restored. A scattering experiment at energies $E \gg m_{3/2}$ will reveal supersymmetric physics. This has important implications on effects such as the running of low-energy couplings. We will return to this later, towards the end of section 10.5 on page 309.

- There is also a technical difference. As we already argued, in the case of the freely acting orbifolds, the states coming from the twisted sector have moduli-dependent masses that are generically nonzero (although they can become zero at special points of the moduli space). This is unlike non-freely-acting orbifolds, where the twisted sector masses are independent of the original moduli and one obtains generically massless states from the twisted sector.

- In vacua with spontaneously broken supersymmetry, the supersymmetry-breaking scale $m_{3/2}$ is an expectation value since it depends on compactification radii. If at least one supersymmetry is left unbroken, then the radii are moduli with arbitrary expectation values. In particular, there are corners of the moduli space where $m_{3/2} \to 0$, and physics becomes supersymmetric at all scales. These points are an infinite distance away using the natural metric of the moduli scalars.

In our simple example above, $m_{3/2} \sim 1/R \to 0$ when $R \to \infty$. At this point, an extra dimension of space-time becomes non-compact and supersymmetry is restored in five dimensions. This behavior is generic in all vacua where the free action originates from translations.

If however there is no leftover supersymmetry, then generically there is a potential for the radii. In such a case $m_{3/2}$ is dynamically determined.

Consider the class of $\mathcal{N} = 2_4$ orbifold vacua we described in (9.4.14). If the (2,18) translation vector ϵ lies within the (0,16) part of the lattice, then the breaking of $\mathcal{N} = 4_4 \to \mathcal{N} = 2_4$ is "explicit." When, however, $(\vec{\epsilon}_L, \vec{\epsilon}_R) \neq (\vec{0}, \vec{0})$ then the breaking is spontaneous.

In the general case, there is no global identification of the massive gravitini inside the moduli space due to surviving duality symmetries. To illustrate this in the previous simple example, consider instead the $(-1)^{m+n}$ translation action. In this case there are two candidate states with the quantum numbers of the gravitini: $\bar{a}^{\mu}_{-1}|S^I_a\rangle \otimes |m = 1, n = 0\rangle$ with mass $m_{3/2} \sim 1/R$, and $\bar{a}^{\mu}_{-1}|S^I_a\rangle \otimes |m = 0, n = 1\rangle$ with mass $\tilde{m}_{3/2} \sim R$. In the region of large R, the first set of states behaves like light gravitini, while in the region of small R it is the second set that is light.

Freely acting orbifolds breaking supersymmetry are stringy versions of Scherk-Schwarz compactifications.

9.6 A Heterotic $\mathcal{N} = 1_4$ Orbifold and Chirality in Four Dimensions

So far, we have used orbifold techniques to remove two of the four gravitini, ending up with $\mathcal{N} = 2_4$ supersymmetry. We will carry this procedure one step further in order to reduce the supersymmetry to $\mathcal{N} = 1_4$.

For phenomenological purposes, $\mathcal{N} = 1_4$ supersymmetry is optimal, since it is the only supersymmetric case that admits chiral representations in four dimensions. Although the very low-energy world is not supersymmetric, we seem to need some supersymmetry beyond Standard-Model energies to explain the gauge hierarchy.

Consider splitting the (6,22) lattice in the $\mathcal{N} = 4_4$ heterotic string as

$$(6, 22) = \oplus_{i=1}^{3} (2, 2)_i \oplus (0, 16). \tag{9.6.1}$$

Label the coordinates of each two-torus as $X_i^{\pm}$, $i = 1, 2, 3$. Consider the following $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolding action: The element g_1 of the first $\mathbb{Z}_2$ acts with a minus sign on the coordinates of the first and second two-torus, the element g_2 of the second $\mathbb{Z}_2$ acts with a minus sign on the coordinates of the first and third torus, and g_1g_2 acts with a minus sign on the coordinates of the second and third torus. Only one of the four four-dimensional gravitini survives this orbifold action. You are invited to verify this in exercise 9.9 on page 288.

To ensure modular invariance we also have to act on the gauge sector. We will consider the $E_8 \times E_8$ string, with the E_8's fermionically realized. We will split the 16 real fermions realizing the first E_8 into groups of 10+2+2+2. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection will act in a similar way in the three groups of two fermions each, while the other ten will be invariant.

The partition function for this $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is:

$$Z^{N=1}_{\mathbb{Z}_2\times\mathbb{Z}_2} = \frac{1}{\tau_2\,\eta^2\bar\eta^2}\,\frac{1}{4}\sum_{h_1,g_1=0,h_2,g_2=0}^{1}\frac{1}{2}\sum_{\alpha,\beta=0}^{1}(-)^{\alpha+\beta+\alpha\beta}$$
$$\times\frac{\vartheta[^{\alpha}_{\beta}]}{\eta}\frac{\vartheta[^{\alpha+h_1}_{\beta+g_1}]}{\eta}\frac{\vartheta[^{\alpha+h_2}_{\beta+g_2}]}{\eta}\frac{\vartheta[^{\alpha-h_1-h_2}_{\beta-g_1-g_2}]}{\eta}\,\frac{\bar\Gamma_8}{\bar\eta^8}\,Z^1_{2,2}[^{h_1}_{g_1}]Z^2_{2,2}[^{h_2}_{g_2}]Z^3_{2,2}[^{h_1+h_2}_{g_1+g_2}]$$
$$\times\frac{1}{2}\sum_{\bar\alpha,\bar\beta=0}^{1}\frac{\bar\vartheta[^{\bar\alpha}_{\bar\beta}]^5}{\bar\eta^5}\frac{\bar\vartheta[^{\bar\alpha+h_1}_{\bar\beta+g_1}]}{\bar\eta}\frac{\bar\vartheta[^{\bar\alpha+h_2}_{\bar\beta+g_2}]}{\bar\eta}\frac{\bar\vartheta[^{\bar\alpha-h_1-h_2}_{\bar\beta-g_1-g_2}]}{\bar\eta}. \tag{9.6.2}$$

We will classify the massless spectrum in multiplets of $\mathcal{N}=1_4$ supersymmetry. We obtain de facto the $\mathcal{N}=1_4$ supergravity multiplet. Next we consider the gauge group of this vacuum. It originates in the untwisted sector. The orbifold group here contains the Z_2 of the orbifold as a subgroup. We can therefore obtain the gauge group by imposing the extra $\mathbb{Z}_2$ projection on the gauge group of the $\mathcal{N}=2_4$ vacuum of section 9.4. The graviphoton, the vector partner of the dilaton, and the two U(1)'s coming from the T^2 are now projected out. The second E_8 survives. The extra $\mathbb{Z}_2$ projection on $E_7\times SU(2)$ gives $E_6\times U(1)\times U(1)'$. The adjoint of E_6 can be written as the adjoint of O(10) plus the O(10) spinor plus a U(1) (singlet).

Therefore, the gauge group of this vacuum is $E_8\times E_6\times U(1)\times U(1)'$ and we have the associated vector multiplets. There is also the linear multiplet containing the antisymmetric tensor and the dilaton.

We now consider the rest of the states that form $\mathcal{N}=1_4$ chiral multiplets. Notice first that there are no massless multiplets charged under the E_8.

The charges of chiral multiplets under $E_6\times U(1)\times U(1)'$ and their multiplicities are given in the tables 9.1 and 9.2 below. You are invited to verify them in exercise 9.10 on page 288.

Table 9.1 Nonchiral Massless States of the $\mathbb{Z}_2\times\mathbb{Z}_2$ Orbifold.

E_6	U(1)	U(1)′	Sector	Multiplicity
27	1/2	1/2	Untwisted	1
27	−1/2	1/2	Untwisted	1
27	0	−1	Untwisted	1
1	−1/2	3/2	Untwisted	1
1	1/2	3/2	Untwisted	1
1	1	0	Untwisted	1
1	1/2	0	Twisted	32
1	1/4	3/4	Twisted	32
1	1/4	−3/4	Twisted	32

Table 9.2 Chiral Massless States of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold.

E_6	U(1)	U(1)′	Sector	Multiplicity
27	0	1/2	Twisted	16
27	1/4	−1/4	Twisted	16
27	−1/4	−1/4	Twisted	16
1	0	3/2	Twisted	16
1	3/4	−3/4	Twisted	16
1	−3/4	−3/4	Twisted	16

As we can see, the spectrum of the theory is chiral. For example, the number of **27**'s minus the number of $\overline{\mathbf{27}}$'s is 3×16. The theory is free of gauge anomalies as it can be checked using the formulas of section 7.9 on page 176.

More complicated orbifolds give rise to different gauge groups and spectra. A guide of such constructions is provided at the end of this chapter.

9.7 Calabi-Yau Manifolds

We provide in this section information about a special class of complex manifolds, Calabi-Yau manifolds. As will become evident shortly, such manifolds are an indispensable tool in string compactifications.

We will start by introducing briefly the idea of cohomology for a generic real manifold. The exterior derivative (defined in appendix B) is nilpotent, $d^2 = 0$. We may therefore introduce a cohomology similar to the definition of physical states using the nilpotent BRST operator in section 3.7 on page 40.

A p-form A_p is called *closed*, if it is annihilated by the exterior derivative: $dA_p = 0$. It is called *exact*, if it can be written as the exterior derivative of a $(p-1)$-form: $A_p = dA_{p-1}$. Any closed form is locally exact but not globally. The pth de Rham cohomology group $H^p(K)$ of a D-dimensional manifold K is the space of closed p-forms modulo the space of exact p-forms. This is a group that depends only on the topology of K. Its dimension is known as the pth Betti number b_p. The Euler number of the manifold is given by the alternating sum

$$\chi(K) = \sum_{p=0}^{D} (-1)^p b_p. \tag{9.7.1}$$

The Laplacian on p-forms can be written in terms of the exterior derivative and the Hodge star operator as

$$\Box \equiv \star d \star d + d \star d \star = (d + \star\, d\star)^2. \tag{9.7.2}$$

A harmonic p-form satisfies $\Box A_p = 0$. It can be shown that the harmonic p-forms are in one-to-one correspondence with the generators of $H^p(K)$. The $\star$ operator maps every harmonic p-form to a harmonic $(D-p)$-form, so that $b_P = b_{D-p}$.

An almost complex manifold has a (1,1) tensor $J^i{}_j$, known as the almost complex structure, that squares to minus one:

$$J^i{}_j J^j{}_k = -\delta^i{}_k. \tag{9.7.3}$$

It can be used to define complex coordinates at any given point, since it plays the role of the imaginary number i locally. An interesting question is whether the definition of complex coordinates at a point, extends to a local neighborhood. This happens when the Nijenhuis tensor

$$N^k{}_{ij} = J^l{}_i(\partial_l J^k{}_j - \partial_j J^k{}_l) - J^l{}_j(\partial_l J^k{}_i - \partial_i J^k{}_l) \tag{9.7.4}$$

vanishes. In that case the manifold is called a *complex manifold*. Such a manifold can be covered by patches of complex coordinates (defined via the complex structure) with holomorphic transition functions. In any given patch we can choose

$$J^a{}_b = i\delta^a{}_b, \quad J^{\bar{a}}{}_{\bar{b}} = -i\delta^{\bar{a}}{}_{\bar{b}}. \tag{9.7.5}$$

On complex manifolds the notion of holomorphic functions is independent of the coordinates.

A *Hermitian metric* on a complex manifold is one for which

$$g_{ab} = g_{\bar{a}\bar{b}} = 0. \tag{9.7.6}$$

We may have a finer definition of forms: a (p, q)-form is a $(p + q)$-form with p antisymmetrized holomorphic indices and q antiholomorphic antisymmetrized indices. The exterior derivative can also be separated as

$$d = \partial + \bar{\partial}, \quad \partial = dz^a \partial_a, \quad \bar{\partial} = d\bar{z}^{\bar{a}} \bar{\partial}_{\bar{a}}. \tag{9.7.7}$$

∂ takes a $(p, q) \to (p + 1, q)$ while $\bar{\partial}$ takes a $(p, q) \to (p, q + 1)$. Moreover

$$\partial^2 = \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{9.7.8}$$

∂ and $\bar{\partial}$ can be used to define a refined cohomology on a complex manifold, the Dolbeault cohomology groups $H^{p,q}_{\bar{\partial}}(K)$ of dimension $h^{p,q}$, containing the (p, q) forms that are $\bar{\partial}$-closed but not $\bar{\partial}$-exact. Using the natural inner product for (p, q) forms we define adjoints $\partial^\dagger, \bar{\partial}^\dagger$ for $\partial, \bar{\partial}$ and construct the two Laplacians

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial, \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}. \tag{9.7.9}$$

The $\Delta_{\bar{\partial}}$-harmonic (p, q)-forms are in one-to-one correspondence with the generators of $H^{p,q}_{\bar{\partial}}(K)$.

On a complex manifold we may impose a stronger condition, namely, that the complex structure $J^i{}_j$ is covariantly constant. In this case we obtain a *Kähler manifold*. From J we can also construct the Kähler two-form $k_{ij} = g_{ik}J^k{}_j$. It is a closed form, $dk = 0$. In holomorphic coordinates we have that

$$k_{a\bar{b}} = -ig_{a\bar{b}} = -k_{\bar{b}a}, \quad k_{ab} = k_{\bar{a}\bar{b}} = 0, \tag{9.7.10}$$

where g is the Hermitian metric. The Kähler form is a closed (1,1) form. This means that

$$\partial k = \bar{\partial} k = 0, \tag{9.7.11}$$

from which it follows that locally

$$k = -i\partial\bar{\partial}K. \tag{9.7.12}$$

K is a zero-form (function) known as the Kähler potential. It is not uniquely determined since any transformation $K \to K + F + \bar{F}$ where F is holomorphic does not change the Kähler form. From relation (9.7.10) we obtain a local expression for the metric

$$g_{a\bar{b}} = g_{\bar{b}a} = \frac{\partial^2 K}{\partial z^a \partial z^{\bar{b}}}, \tag{9.7.13}$$

from which the Christoffel connections can be calculated. The only nonzero ones are

$$\Gamma^a_{b\,c} = g^{a\bar{d}}\partial_b g_{c\bar{d}}, \quad \Gamma^{\bar{a}}_{\bar{b}\,\bar{c}} = g^{\bar{a}d}\partial_{\bar{b}} g_{\bar{c}d}. \tag{9.7.14}$$

The only nonzero components of the Riemann tensor are $R_{a\bar{b}c\bar{d}}$ and the cyclic identity gives

$$R_{a\bar{b}c\bar{d}} = R_{c\bar{b}a\bar{d}} = R_{a\bar{d}c\bar{b}}. \tag{9.7.15}$$

The Ricci tensor can be calculated to be

$$R_{a\bar{b}} = -\partial_a \partial_{\bar{b}} \log\det g. \tag{9.7.16}$$

Ricci flatness leads to the Monge-Ampère equation.

9.7.1 Holonomy

The notion of holonomy is central in geometry. A Riemannian manifold of dimension D has a spin connection ω that is generically an SO(D) gauge field. This implies, in analogy with standard gauge fields, that a field ϕ transported around a path γ, transforms to $W\phi$ where

$$W = P\, e^{\int_\gamma \omega \cdot dx}. \tag{9.7.17}$$

ω above is taken in the same SO(D) representation as ϕ and P stands for path ordering. The SO(D) matrices W form a group $\mathrm{H} \subset \mathrm{SO}(D)$. It is called the holonomy group of the manifold. Generically $\mathrm{H} = \mathrm{SO}(D)$, but there are special cases where this is not so.

Consider the possibility that the manifold admits a covariantly constant spinor ζ: $\nabla_i \zeta = 0$. As in gauge theories, a covariantly constant field has trivial holonomy: by definition it does not change along a path. This means that for any group element $W \in \mathrm{H}$, $\mathrm{W}\,\zeta = \zeta$. We are interested in finding what subgroup H can have this property.

In the $D = 6$ case, SO(6) is locally equivalent to SU(4).The spinor and conjugate spinor representations of SO(6) are the fundamental **4** and antifundamental $\bar{\mathbf{4}}$ representations of SU(4). Without loss of generality we can assume that ζ transforms as the **4**.

This special spinor, can always be brought to the form $(0, 0, 0, \zeta_0)$ by a SU(4) rotation. In this frame, it is obvious that the subgroup of SU(4) that preserves the spinor is SU(3) and acts on the first three components. Because $\mathbf{4} \to \mathbf{3} + \mathbf{1}$, a manifold that has an SU(3) holonomy, has necessarily just one covariantly constant spinor. If there are more, the holonomy group must be smaller. For example, for two distinct covariantly constant spinors, the holonomy group must be SU(2). This is the case for the manifold $K3 \times T^2$ that we will meet later.

It will be shown in the next section, that the existence of a single covariantly-constant spinor will be associated with the presence of $\mathcal{N} = 1_4$ supersymmetry in the heterotic string.

9.7.2 Consequences of SU(3) holonomy

Once we have a covariantly constant spinor, we can construct several closed forms as bilinears in this spinor. One is the two-form, $k_{ij} = \bar{\zeta}\Gamma_{ij}\zeta$ where Γ_{ij} are the antisymmetrized product of the SO(6) Γ-matrices, and the associated complex structure $J^i{}_j = g^{ik}k_{kj}$. The second is a three-form $\Omega_{ijk} = \zeta^T\Gamma_{ijk}\zeta$. The one-form $\Omega_i = \zeta^T\Gamma_i\zeta$ vanishes because of the six-dimensional Fierz identities.

We first focus on $J^i{}_j$. It is an SO(6) matrix acting on vectors. By construction it is real, traceless and SU(3) invariant. There is a unique such matrix in SO(6) up to normalization and group conjugation

$$J = \begin{pmatrix} 0 & +1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{9.7.18}$$

Therefore, J satisfies $J^2 = -1$ and is an almost complex structure. It can be used to define complex coordinates over K. From its construction in terms of ζ, J is covariantly constant, and the Nijenhuis tensor thus vanishes: K is a complex manifold.

Since the two-form k_{ij} constructed in terms of the spinor, is closed, the manifold K is also a Kähler manifold according to our previous definition. A Kähler manifold does not admit a unique metric, but two different Kähler metrics g and g' are related as

$$g'_{a\bar{b}} = g_{a\bar{b}} + \partial_a\partial_{\bar{b}}\phi, \tag{9.7.19}$$

where ϕ is an arbitrary function on the manifold.

A generic six-dimensional Kähler manifold has U(3) holonomy. To obtain the restricted SU(3) holonomy, we must impose extra conditions on the manifold K. The spin connection of a Kähler manifold is a U(3) gauge field. The U(1) part is a gauge field that we will call A. $F = dA$ is a closed two-form, an element of H^2(K). Its class is known as the first Chern class of the manifold c_1(K). For SU(3) holonomy such a gauge field must have a vanishing field strength, F. This can happen if this U(1) bundle (the canonical bundle) is topologically trivial. In this case, it has a global section. Moreover, $c_1(\mathrm{K}) = 0$.

We will now show that a vanishing first Chern class implies Ricci flatness of K. Remember that the $U(1) \subset SO(6)$ in question is (9.7.18), namely, the complex structure J. Given an antisymmetric matrix (in the Lie algebra of SO(6)), its U(1) part is given by $\mathrm{tr}(JM) = J^i{}_j M^j{}_i$. Consider now the Riemann form generated by the Riemann tensor. Its U(1) part is

$$F_{ij} = \mathrm{tr}(JR_{ij}) = R_{ij;kl}J^{kl}. \tag{9.7.20}$$

From the previous equation, the nonzero components of F are

$$F_{a\bar{b}} = -F_{\bar{b}a} = R_{a\bar{b}}{}^{k}{}_{l} J^{l}{}_{k} = iR_{a\bar{b}}{}^{c}{}_{c} - iR_{a\bar{b}}{}^{\bar{c}}{}_{\bar{c}}. \tag{9.7.21}$$

Since

$$R_{a\bar{b}}{}^{c}{}_{c} = R_{a\bar{b}\bar{d}c} g^{\bar{d}c} = -R_{a\bar{b}c\bar{d}} g^{c\bar{d}} = -R_{a\bar{b}}{}^{\bar{d}}{}_{\bar{d}}, \tag{9.7.22}$$

we obtain

$$F_{a\bar{b}} = -F_{\bar{b}a} = 2iR_{a\bar{b}}{}^{c}{}_{c} = -2iR_{a\bar{b}}. \tag{9.7.23}$$

What we have shown is, that the U(1) component of the holonomy, in U(1)×SU(3)⊂SO(6), is generated by the Ricci form. Thus, Ricci flatness implies SU(3) holonomy. The converse is also true by the Yau theorem. Such spaces, are known as Calabi-Yau (CY) manifolds.

We have also constructed a closed (covariantly constant) three-form Ω_{ijk} on K. This is expected for the following reason: the vector of SO(6) decomposes as $\mathbf{6} \to \mathbf{3} + \bar{\mathbf{3}}$ under SU(3). We construct an SU(3) singlet out of the antisymmetrized product of three **3**s, which on the other hand would transform under U(3). The existence of SU(3) holonomy is equivalent to the existence of such a covariantly constant (3,0) form Ω_{ijk}. In fact, a manifold of SU(3) holonomy has a unique non-vanishing (3,0) form that is covariantly constant. It is a section of the (topologically trivial) canonical bundle on K.

We now proceed to discuss the Dolbeault cohomology of compact CY manifolds.

Reality implies that $h^{p,q} = h^{q,p}$ and Poincaré duality $h^{p,q} = h^{3-p,3-q}$. $h^{0,0} = 1$ corresponding to the constant solution of the Laplacian on any connected compact manifold. Since there are no harmonic one-forms, $h^{1,0} = h^{0,1} = 0$. The relation $h^{p,0} = h^{3-p,0}$ valid for CY manifolds[8] then implies that $h^{2,0} = h^{0,2} = 0$. Finally, the uniqueness of the (3,0) form implies $h^{3,0} = h^{0,3} = 1$. We arrive at the following Hodge diamond, characteristic of CY manifolds:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array} . \tag{9.7.24}$$

The Euler number is

$$\chi = 2(h^{1,1} - h^{2,1}). \tag{9.7.25}$$

9.7.3 The CY moduli space

Once we have a CY manifold, there may be continuous deformations that preserve this property. This will give rise to a moduli space. Its structure is important in order to understand the effective field theory of CY compactifications.

We will start by describing the *deformations of the complex structure J* defined as

$$J^{i}{}_{k} J^{k}{}_{j} = -\delta^{i}{}_{j}, \quad N^{i}{}_{jk} = 0, \tag{9.7.26}$$

[8] This isomorphism is obtained by contracting with the (3,0)-form Ω.

where N was defined in (9.7.4). An infinitesimal deformation of the complex structure

$$\tilde{J}^i_j = J^i{}_j + \tau^i{}_j \tag{9.7.27}$$

must satisfy to leading order (9.7.26). The first of the equation sets (in complex coordinates) $\tau^a{}_b = \tau^{\bar{a}}{}_{\bar{b}} = 0$. Moreover, the only nonzero components of the Nijenhuis tensor are

$$N^a{}_{\bar{b}\bar{c}} = \bar{\partial}_{\bar{b}}\tau^a{}_{\bar{c}} - \bar{\partial}_{\bar{c}}\tau^a{}_{\bar{b}}, \tag{9.7.28}$$

and its complex conjugate. We may view $\tau^a{}_{\bar{b}}$ as a (0,1)-form with values in the holomorphic tangent bundle T. Then, the vanishing of N can be written as

$$\bar{\partial}\tau^a = 0. \tag{9.7.29}$$

This says that τ is an element of $H^1(T)$, the closed one-forms with values in the tangent bundle. We may now consider the (2,1) form

$$\eta_{ab\bar{c}} = \Omega_{abd}\tau^d{}_{\bar{c}}. \tag{9.7.30}$$

Since Ω is covariantly constant, η is a harmonic form if and only if τ is. Thus, $H^1(T) \sim H^{2,1}(K)$, and the non-trivial complex structure deformations are in one-to-one correspondence with the (2,1)-harmonic forms. They form a moduli space, called a *complex structure moduli space* $\mathcal{M}_C$ of complex dimension $h^{2,1}$. It can be shown that they are related to deformations of the metric δg_{ab} and $\delta g_{\bar{a}\bar{b}}$ which preserve the CY condition.

There is another perturbation of the metric, namely, $\delta g_{a\bar{b}}$. The condition for this to preserve Ricci flatness is that $\delta g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}$ is harmonic. Thus the number of independent such deformations is $h^{1,1}$. These are known as deformations of the Kähler structure and their moduli space, *Kähler moduli space* $\mathcal{M}_K$.

Thus the total moduli space of CY metrics is a direct product $\mathcal{M}_K \times \mathcal{M}_C$ of real dimension $h^{1,1} + 2h^{2,1}$.

In string theory (compactified on a CY manifold) the metric comes always together with the two-index antisymmetric tensor and the dilaton. The two-index antisymmetric tensor, B, being a two-form, will give another $h^{1,1}$ real moduli (scalars), as well as a four-dimensional two-tensor, that in four dimensions is equivalent to a pseudoscalar. The $h^{1,1}$ moduli of B combine with the Kähler moduli and complexify the Kähler moduli space. From now on by $\mathcal{M}_K$ we will denote the complexified Kähler moduli space of real dimension $2h^{1,1}$.

Both $\mathcal{M}_K$ and $\mathcal{M}_C$ are themselves Kähler manifolds. The Kähler potential for $\mathcal{M}_K$ is given by

$$\mathcal{K} = -\log \int_K J \wedge J \wedge J, \tag{9.7.31}$$

while for $\mathcal{M}_C$

$$\mathcal{K}_C = -\log \left(i \int_K \Omega \wedge \bar{\Omega} \right). \tag{9.7.32}$$

In fact, these manifolds are special Kähler manifolds, whose geometry is determined from a holomorphic function $\mathcal{F}$, the prepotential. As we will see later, this is related to

$\mathcal{N}=2_4$ supersymmetry (see appendix I on page 533). In special geometry, the Kähler potential can be obtained from the holomorphic prepotential $\mathcal{F}$ as

$$K=-\log\left[i(\bar{z}^i\partial_{z_i}\mathcal{F}-z^i\partial_{\bar{z}_i}\bar{\mathcal{F}})\right]. \tag{9.7.33}$$

9.8 $\mathcal{N}=1_4$ Heterotic Compactifications

We have seen how orbifolds provide solvable compactifications of string theory. The disadvantage of the orbifold approach is that it describes explicitly a small subspace of the relevant moduli space. To obtain a potentially wider view of the space of $\mathcal{N}=1_4$ compactifications, we must work perturbatively in α' (σ-model approach).

In the effective field theory approach (to leading order in α'), we assume that some bosonic fields acquire expectation values that satisfy the equations of motion, while the expectation values of the fermions are zero (to preserve $D=4$ Lorentz invariance). Generically, such a background breaks all the supersymmetries of flat ten-dimensional space. Some supersymmetry will be preserved, if the associated variation of the fermion fields vanish. This gives a set of first order equations. If they are satisfied for at least one supersymmetry, then the full equations of motion will also be satisfied to leading order in α'. Another way to state this is by saying that every compact manifold that preserves at least one SUSY, is a solution of the equations of motion.

We will consider here the case of the heterotic string on a space that is locally $M_4\times K$ with M_4 the four-dimensional Minkowski space and K some six-dimensional compact manifold. We split indices into Greek indices for M_4 and Latin indices for K.

The ten-dimensional Γ-matrices can be constructed from the $D=4$ matrices γ^μ, and the internal matrices γ^m, $m=4,5,\ldots,9$ as

$$\Gamma^\mu=\gamma^\mu\otimes\mathbf{1}_6,\quad \Gamma^m=\gamma^5\otimes\gamma^m, \tag{9.8.1}$$

$$\gamma^5=\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma},\quad \gamma=\frac{i}{6!}\sqrt{\det g}\,\epsilon_{mnrpqs}\gamma^{mnrpqs}. \tag{9.8.2}$$

γ is the analog of γ^5 for the internal space.

The supersymmetry variations of fermions in the heterotic string were given (in the Einstein frame) in appendix H.5 on page 530. Using the decomposition above, they can be written as

$$\delta\psi_\mu\sim\nabla_\mu\epsilon+\frac{e^{-\Phi/2}}{96}\left(\gamma_\mu\gamma_5\otimes H\right)\epsilon, \tag{9.8.3}$$

$$\delta\psi_m\sim\nabla_m\epsilon+\frac{e^{-\Phi/2}}{96}\left(\gamma_m H-12H_m\right)\epsilon, \tag{9.8.4}$$

$$\delta\lambda\sim-(\gamma^m\partial_m\Phi)\epsilon+\frac{1}{12}e^{-\Phi/2}H\epsilon, \tag{9.8.5}$$

$$\delta\chi^a\sim-\frac{1}{4}e^{-\Phi/4}F^a_{mn}\gamma^{mn}\epsilon, \tag{9.8.6}$$

where ψ is the gravitino, λ is the dilatino, and χ^a are the gaugini; ϵ is a spinor (the parameter of the supersymmetry transformation). Furthermore, we used

$$H=H_{mnr}\,\gamma^{mnr},\quad H_m=H_{mnr}\,\gamma^{nr}. \tag{9.8.7}$$

If, for some value of the background fields, the equations $\delta(\text{fermions}) = 0$ admit a solution, namely, a nontrivial, globally defined spinor ϵ, then the background is $\mathcal{N} = 1_4$ supersymmetric. If more than one solution exists, then we will have extended supersymmetry. For simplicity, we will make the assumption here that $H_{mnr} = 0$.

We assume a factorized spinor *Ansatz* $\epsilon = \chi \otimes \xi$. Vanishing of (9.8.3) when $H_{mnr} = 0$ implies that the four-dimensional spinor χ is constant. The vanishing of (9.8.4) implies that the internal manifold K must admit a Killing spinor ξ,

$$\nabla_m \xi = 0. \tag{9.8.8}$$

The vanishing of the dilatino variation (9.8.5) implies that the dilaton must be constant.

Applying one more covariant derivative to (9.8.8) and antisymmetrizing we obtain

$$[\nabla_m, \nabla_n]\xi = \frac{1}{4} R_{rs;mn}\gamma^{rs}\,\xi = 0. \tag{9.8.9}$$

Since $R_{rs;mn}\gamma^{rs}$ is the generator of the holonomy of the manifold, (9.8.9) implies that the holonomy group is smaller than the generic one, O(6). By multiplying (9.8.9) by γ^n and using the properties of the Riemann tensor we also obtain Ricci flatness ($R_{mn} = 0$). The holonomy is thus reduced to SU(3)$\subset$ SU(4)$\sim$O(6) so that the spinor decomposes as $4 \to 3 + 1$. Moreover the manifold has to be a Kähler manifold. Finally the background (internal) gauge fields must satisfy

$$F^a_{mn}\gamma^{mn}\,\xi = 0, \tag{9.8.10}$$

which again implies that $F^a_{mn}\gamma^{mn}$ acts as an SU(3) matrix.

Equation (7.9.33) on page 182 becomes

$$R^{rs}{}_{[mn}R_{pq]rs} = \frac{1}{30} F^a_{[mn}F^a_{pq]}. \tag{9.8.11}$$

We now take into account the discussion of the previous section to conclude that a compactification of the heterotic string on a CY manifold (SU(3) holonomy) with a gauge bundle satisfying (9.8.10) and (9.8.11) gives $\mathcal{N} = 1_4$ supersymmetric vacua.

9.8.1 The low-energy $\mathcal{N} = 1_4$ heterotic spectrum

We may now proceed with analyzing the effective theory of the heterotic string compactified on a CY manifold.

We must choose a gauge bundle on the CY manifold. A simple way to solve (9.8.10) and (9.8.11) is to embed the spin connection $\omega \in$ SU(3) into the gauge connection $A \in$ O(32) or $E_8 \times E_8$. The only embedding of SU(3) in O(32) that satisfies (9.8.11) is the one in which O(32) $\ni 32 \to 3 + \bar{3} +$ singlets $\in$ SU(3). In this case O(32) is broken down to U(1) $\times$ O(26) (this is the subgroup that commutes with SU(3)).

The U(1) is "anomalous," namely, the sum of the U(1) charges $\rho = \sum_i q^i$ of the massless states is not zero. This anomaly is only apparent, since the underlying string theory is not anomalous. What happens is that the Green-Schwarz mechanism implies that there is a one-loop coupling of the form $B \wedge F$. This gives a mass to the U(1) gauge field. The associated gauge symmetry is therefore broken at low-energy. This is discussed further in section 10.4. The leftover gauge group O(26) has only nonchiral representations.

More interesting is the case of $E_8 \times E_8$. E_8 has a maximal $SU(3) \times E_6$ subgroup, under which the adjoint of E_8 decomposes as $E_8 \ni \mathbf{248} \to (\mathbf{8}, \mathbf{1}) \otimes (\mathbf{3}, \mathbf{27}) \otimes (\bar{\mathbf{3}}, \overline{\mathbf{27}}) \otimes (\mathbf{1}, \mathbf{78}) \in SU(3) \times E_6$. Embedding the spin connection in one of the E_8 in this fashion solves (9.8.11). The unbroken gauge group in this case is $E_6 \times E_8$. Let N_L be the number of massless left-handed Weyl fermions in four dimensions transforming in the **27** of E_6 and N_R the same number for the $\overline{\mathbf{27}}$. The number of net chirality (number of "generations") is $|N_L\text{-}N_R|$; it can be obtained by applying the Atiyah-Singer index theorem on the CY manifold. The **27**'s transform as the **3** of SU(3) and the $\overline{\mathbf{27}}$ transform in the $\bar{\mathbf{3}}$ of SU(3). Thus, the number of generations is the index of the Dirac operator on K for the fermion field $\psi_{\alpha A}$, where α is a spinor index and A is a **3** index. It can be shown that the index of the Dirac operator, and thus the number of generations, is equal to $|\chi(K)/2|$, where $\chi(K)$ is the Euler number of the manifold K.

The compactification of the $E_8 \times E_8$ theory provides a low-energy theory involving the E_6 gauge group that is known to be phenomenologically attractive. Moreover, below the string scale, there are no particles charged under both E_8's. Therefore, the other E_8 forms the "hidden sector": it contains particles that interact to the observable ones only via gravity and other universal interactions. This sector seems very weakly coupled to normal particles to have observable consequences. However, it can trigger supersymmetry breaking. Its strong self-interactions may force gaugini to condense, breaking supersymmetry. The breaking of supersymmetry can then be transmitted to the observable sector by the gravitational interaction.

The considerations in this section are correct to leading order in α'. At higher orders we expect (generically) corrections. It turns out that most of the statements above survive these corrections.

9.9 K3 Compactification of the Type-II String

As another example, we will consider the compactification of type II theory on the K3 manifold down to six dimensions. K3 denotes the class of four-dimensional compact, Ricci-flat, Kähler manifolds without isometries. Such manifolds have $SU(2) \subset O(4)$ holonomy and are also hyper-Kähler. The hyper-Kähler condition is equivalent to the existence of three integrable complex structures that satisfy the SU(2) algebra.[9]

It can be shown that a left-right symmetric $\mathcal{N} = (1,1)_2$ supersymmetric σ-model on such manifolds is exactly conformally invariant and has extended $\mathcal{N} = (4,4)_2$ superconformal symmetry (see section 4.13.3 on page 81). Moreover, K3 has two covariantly constant spinors, so that the type-II theory compactified on it, has $\mathcal{N} = 2_6$ supersymmetry in six dimensions (and $\mathcal{N} = 4_4$ if further compactified on a two-torus).

It is useful for later purposes to briefly describe the cohomology of K3. There is a harmonic zero-form that is constant (since the manifold is compact and connected). There

[9] See also the discussion in appendix I.2 on page 535.

are no harmonic one-forms or three-forms. There is one (2,0) and one (0,2) harmonic forms as well as 20 (1,1) forms. The (2,0), (0,2), and one of the (1,1) Kähler forms are self-dual, the other 19 (1,1) forms are anti-self-dual. There is a unique four-form (the volume form).

We will consider first the type-IIA theory and derive the massless bosonic spectrum in six dimensions. To find the massless states originating from the ten-dimensional metric G, we make the following decomposition

$$G_{MN} \sim h_{\mu\nu}(x) \otimes \phi(y) + A_\mu(x) \otimes f_m(y) + \Phi(x) \otimes h_{mn}(y), \tag{9.9.1}$$

where x denotes the six-dimensional noncompact flat coordinates and y are the internal (K3) coordinates. Also $\mu = 0, 1, \ldots, 5$ and $m = 1, 2, 3, 4$ is a K3 index. Applying the ten-dimensional equations of motion to the metric G, we obtain that $h_{\mu\nu}$ (the six-dimensional graviton) is massless if

$$\Box_y\, \phi(y) = 0. \tag{9.9.2}$$

The solutions to this equation are the harmonic zero-forms on K3, and there is only one of them. Thus, there is one massless graviton in six dimensions. $A_\mu(x)$ is massless if $f_m(y)$ is covariantly constant on K3. Thus, it must be a harmonic one-form and there are none on K3. Consequently, there are no massless vectors coming from the metric. $\Phi(x)$ is a massless scalar if $h_{mn}(y)$ satisfies the Lichnerowicz equation

$$-\Box h_{mn} + 2R_{mnrs}h^{rs} = 0, \quad \nabla^m h_{mn} = g^{mn}h_{mn} = 0. \tag{9.9.3}$$

The solutions of this equation can be constructed out of the three self-dual harmonic two-forms S_{mn} and the 19 anti-self-dual two-forms A_{mn}. Being harmonic, they satisfy the following equations (R_{mnrs} is anti-self-dual)

$$\Box f_{mn} - R_{mnrs}f^{rs} = \Box f_{mn} + 2R_{mrsn}f^{rs} = 0, \tag{9.9.4}$$

$$\nabla_m f_{np} + \nabla_p f_{mn} + \nabla_n f_{pm} = 0, \quad \nabla^m f_{mn} = 0. \tag{9.9.5}$$

Using these equations and the self-duality properties, it can be verified that solutions to the Lichnerowicz equation are given by

$$h_{mn} = A_m^p S_{pn} + A_n^p S_{pm}. \tag{9.9.6}$$

Thus, there are $3 \cdot 19 = 57$ massless scalars. There is an additional massless scalar (the volume of K3) corresponding to constant rescalings of the K3 metric, that obviously preserves the Ricci-flatness condition. We obtain in total 58 scalars. The ten-dimensional dilaton also gives an extra massless scalar in six dimensions.

There is a similar expansion for the two-index antisymmetric tensor:

$$B_{MN} \sim B_{\mu\nu}(x) \otimes \phi(y) + B_\mu(x) \otimes f_m(y) + \Phi(x) \otimes B_{mn}(y). \tag{9.9.7}$$

The masslessness condition implies that the zero-, one-, and two-forms (ϕ, f_m, B_{mn}, respectively) must be harmonic. We therefore obtain one massless two-index antisymmetric tensor and 22 scalars in six dimensions.

From the R-R sector we have a one-form that gives a massless vector in six dimensions. We also have a three-form that gives a massless three-form, and 22 vectors in six dimensions. A massless three-form in six dimensions is equivalent to a massless vector via a Poincaré duality transformation.

In total we have a graviton, an antisymmetric tensor, 24 vectors, and 81 scalars. The two gravitini in ten dimensions give rise to two Weyl gravitini in six dimensions. Their internal wave-functions are proportional to the two covariantly constant spinors that exist on K3. The gravitini preserve their original chirality. They have therefore opposite chirality. The relevant representations of $\mathcal{N} = (1,1)_6$ supersymmetry in six dimensions are

- The vector multiplet. It contains a vector, two Weyl spinors of opposite chirality and four scalars.
- The supergravity multiplet. It contains the graviton, two Weyl gravitini of opposite chirality, four vectors, an antisymmetric tensor, a scalar, and four Weyl fermions of opposite chirality.

We conclude that the six-dimensional massless content of type-IIA theory on K3 consists of the supergravity multiplet and 20 U(1) vector multiplets. $\mathcal{N} = (1,1)_6$ supersymmetry is sufficient to fix the two-derivative low-energy couplings of the massless fields. The bosonic part is (in the string frame)

$$S_{\text{K3}}^{IIA} = \int d^6x\sqrt{-\det G_6}\, e^{-2\Phi}\left[R + \nabla^\mu\Phi\nabla_\mu\Phi - \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} + \frac{1}{8}\text{Tr}(\partial_\mu\hat{M}\partial^\mu\hat{M}^{-1})\right]$$
$$- \frac{1}{4}\int d^6x\sqrt{-\det G}(\hat{M}^{-1})_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{1}{16}\int d^6x\epsilon^{\mu\nu\rho\sigma\tau\upsilon}B_{\mu\nu}F^I_{\rho\sigma}\hat{L}_{IJ}F^J_{\tau\upsilon}, \tag{9.9.8}$$

where $I = 1, 2, \ldots, 24$. Φ is the six-dimensional dilaton.

Supersymmetry and the fact that there are 20 vector multiplets restricts the $4 \cdot 20$ scalars to live on the coset space $O(4,20)/O(4) \times O(20)$. The scalars are therefore parameterized by the matrix $\hat{M}$ as in (D.4) on page 514 with $p = 4$, where $\hat{L}$ is the invariant O(4,20) metric. The action (9.9.8) is invariant under the continuous O(4,20) global symmetry. Here $H_{\mu\nu\rho}$ does not contain any Chern-Simons term. Note also the absence of the dilaton-gauge field coupling. This is due to the fact that the gauge fields come from the R-R sector.

Observe that type-IIA theory on K3 gives exactly the same massless spectrum as the heterotic string theory compactified on T^4. The low-energy actions (9.1.8) and (9.9.8) are different, though. As we will see in chapter 11, there is a nontrivial and interesting relation between the two.

Now consider the type-IIB theory compactified on K3 down to six dimensions. The NS-NS sector bosonic fields (G, B, Φ) are the same as in the type-IIA theory and we obtain again a graviton, an antisymmetric tensor, and 81 scalars.

From the R-R sector we have another scalar, the axion, which gives a massless scalar in $D = 6$. There is another two-index antisymmetric tensor, which gives, in six dimensions, a two-index antisymmetric tensor and 22 scalars. Finally there is the self-dual four-index antisymmetric tensor, which gives three self-dual two-index antisymmetric tensors and 19 anti-self-dual two-index antisymmetric tensors and a scalar. Since we can split a two-index antisymmetric tensor into a self-dual and an anti-self-dual part we can summarize the bosonic spectrum in the following way: a graviton, five self-dual and 21 anti-self-dual antisymmetric tensors, and 105 scalars.

Here, unlike the type-IIA case we obtain two massless Weyl gravitini of the same chirality. They generate a chiral $\mathcal{N} = (2,0)_6$ supersymmetry. The relevant massless representations are

- The SUGRA multiplet. It contains the graviton, five self-dual antisymmetric tensors, and two left-handed Weyl gravitini.
- The tensor multiplet. It contains an anti-self-dual antisymmetric tensor, five scalars, and two Weyl fermions of chirality opposite to that of the gravitini.

The total massless spectrum forms the supergravity multiplet and 21 tensor multiplets. The theory is chiral but anomaly-free. The scalars live on the coset space $O(5,21)/O(5) \times O(21)$ and there is a global O(5,21) symmetry. Since the theory involves self-dual tensors, there is no covariant action principle, but we can write covariant equations of motion.

9.10 $\mathcal{N} = 2_6$ Orbifolds of the Type-II String

In section 9.9 we considered the compactification of the ten-dimensional type II string on the four-dimensional manifold K3. This provided a six-dimensional theory with $\mathcal{N} = 2_6$ supersymmetry. Upon toroidal compactification on an extra T^2 we obtain a four-dimensional theory with $\mathcal{N} = 4_4$ supersymmetry.

We will now consider a $\mathbb{Z}_2$ orbifold compactification to six dimensions with $\mathcal{N} = 2_6$ supersymmetry. We will also argue that it provides an alternative description of the geometric compactification on K3, considered earlier.

The $\mathbb{Z}_2$ orbifold transformation will act on the T^4 by reversing the sign of all four coordinates (and similarly for the world-sheet fermions on both the left and the right). This projects out half of the original gravitini. The partition function is

$$Z_{6-d}^{II-\lambda} = \frac{1}{2}\sum_{h,g=0}^{1} \frac{Z_{(4,4)}\left[{h \atop g}\right]}{\tau_2^2\eta^4\bar{\eta}^4} \times \frac{1}{2}\sum_{a,b=0}^{1} (-1)^{a+b+ab}\, \frac{\vartheta^2\left[{a \atop b}\right]\vartheta\left[{a+h \atop b+g}\right]\vartheta\left[{a-h \atop b-g}\right]}{\eta^4}$$
$$\times \frac{1}{2}\sum_{\bar{a},\bar{b}=0}^{1} (-1)^{\bar{a}+\bar{b}+\lambda\bar{a}\bar{b}}\, \frac{\bar{\vartheta}^2\left[{\bar{a} \atop \bar{b}}\right]\bar{\vartheta}\left[{\bar{a}+h \atop \bar{b}+g}\right]\bar{\vartheta}\left[{\bar{a}-h \atop \bar{b}-g}\right]}{\bar{\eta}^4}. \tag{9.10.1}$$

$Z_{4,4}\left[{h \atop g}\right]$ are the $T^4/\mathbb{Z}_2$ orbifold blocks in (9.4.6) and $\lambda = 0,1$ corresponds to type-IIB and type-2A, respectively.

We now focus on the massless bosonic spectrum. In the untwisted NS-NS sector we obtain the graviton, antisymmetric tensor, the dilaton (in six dimensions) and 16 scalars (the moduli of the $T^4/\mathbb{Z}_2$). In the NS-NS twisted sector we obtain $4{\cdot}16$ scalars. The total number of scalars (apart from the dilaton) is $4{\cdot}20$. Thus, the massless spectrum of the NS-NS sector is the same as that of the K3 compactification in section 9.9.

In the R-R sector we will have to distinguish IIA from IIB. In the type-IIA theory, we obtain seven vectors and a three-form from the R-R untwisted sector and another 16 vectors from the R-R twisted sector. In type IIB we obtain four two-index antisymmetric tensors and eight scalars from the R-R untwisted sector and 16 anti-self-dual two-index antisymmetric tensors and 16 scalars from the R-R twisted sector. Again this agrees with the K3 compactification.

To further motivate the fact that we are describing a CFT realization of the string moving on the K3 manifold, let us look more closely at the cohomology of the orbifold. We will use the two complex coordinates that describe the T^4, $z^{1,2}$. The T^4 has one zero-form, the constant, two (1,0) one-forms (dz^1, dz^2), two (0,1) one-forms $(d\bar{z}^1, d\bar{z}^2)$, one (2,0) form $(dz^1 \wedge dz^2)$ one (0,2) form $(d\bar{z}^1 \wedge d\bar{z}^2)$, and four (1,1) forms $(dz^i \wedge d\bar{z}^j)$. Finally there are four three-forms and one four-form.

Under the orbifolding $\mathbb{Z}_2$, the one- and three-forms are projected out and we are left with a zero-form, a four-form, a (0,2), (2,0), and 4 (1,1) forms. However the $\mathbb{Z}_2$ action has 16 fixed points on T^4, which become singular in the orbifold. To make a regular manifold we excise a small neighborhood around each singular point. The boundary is $S^3/\mathbb{Z}_2$ and we can paste a Ricci-flat manifold with the same boundary. The relevant manifold with this property is the Eguchi-Hanson gravitational instanton. This is the simplest of a class of four-dimensional noncompact hyper-Kähler manifolds known as asymptotically locally Euclidean (ALE) manifolds. These manifolds asymptote at infinity to a cone over S^3/Γ, with Γ one of the simple finite subgroups of $SU(2)$. The $SU(2)$ action on S^3 is the usual group action (remember that S^3 is the group manifold of $SU(2)$). This action induces an action of the finite subgroup Γ. The finite simple $SU(2)$ subgroups have an *A*-*D*-*E* classification. The *A* series corresponds to the $\mathbb{Z}_N$ subgroups. The Eguchi-Hanson space corresponds to $N = 2$. The *D* series corresponds to the dihedral D_N subgroups of $SU(2)$, which are $\mathbb{Z}_N$ groups augmented by an extra $\mathbb{Z}_2$ element. Finally, the three exceptional cases correspond to the tetrahedral, octahedral, and icosahedral groups.

The Eguchi-Hanson space carries an anti-self-dual $(1, 1)$ form. Thus, in total, we will obtain 16 of them. We have eventually obtained the cohomology of the K3 manifold, at a submanifold of the moduli space where the metric has conical singularities. We can also compute the Euler number. Suppose we have a manifold M that we divide by the action of an abelian group G of order g; we excise a set of fixed points F and we paste some regular manifold N back. Then the Euler number is given by

$$\chi = \frac{1}{g}[\chi(M) - \chi(F)] + \chi(N). \tag{9.10.2}$$

Here $\chi(T^4) = 0$, F is the set of 16 fixed points with $\chi = 1$ each, while $\chi = 2$ for each of the 16 Eguchi-Hanson instantons, so that in total $\chi(T^4/\mathbb{Z}_2) = 24$. This is indeed the Euler number of K3.

The orbifold can be desingularized by moving away from zero S^2 volumes. This procedure is called a "blow-up" of the orbifold singularities. In the orbifold CFT description, it corresponds to marginal perturbations by the orbifold twist operators. In string theory language this corresponds to changing the expectation values of the scalars that are generated by the 16 orbifold twist fields. Note that at the orbifold limit, although the K3 geometry is singular, the associated string theory is not. The reason is that the shrinking spheres that become singularities have an NS two-form flux trapped in. The string couples to the flux and this prevents the development of divergences. There are points in the K3 moduli space though, where string theory does become singular. We will return in sections 11.9.1 and 11.10 to the interpretation of such singularities.

Before we move on, we will briefly describe other T^4 orbifolds associated to K3. They are of the $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ type. They are equivalent to associated orbifold limits of K3.

We use complex coordinates for the internal T^4 as

$$z^1 = x^6 + ix^7, \quad z^2 = x^8 + ix^9, \tag{9.10.3}$$

with the torus identifications $z^i \sim z^i + 1 \sim z^i + i$ for $\mathrm{N} = 2, 4$ and $z^i \sim z^i + 1 \sim z^i + e^{i\frac{\pi}{3}}$ for $\mathrm{N} = 3, 6$. Such identifications specify submoduli spaces of the T^4 moduli space for which $\mathbb{Z}_\mathrm{N}$ is a symmetry, that you may explore in exercise 9.21 on page 289. The $\mathbb{Z}_\mathrm{N}$ then acts as

$$(z^1, z^2) \to (e^{2\pi i/\mathrm{N}}\, z^1, e^{-2\pi i/\mathrm{N}}\, z^2). \tag{9.10.4}$$

In exercise 9.22 you are invited to construct the one-loop partition functions of these orbifolds, and read off the associated massless spectra.

9.11 CY Compactifications of Type-II Strings

We will study in this section some simple aspects of the compactification of type-II string theory in four dimensions on a CY manifold. We have already seen in section 9.8 on page 245 that in the compactification of the heterotic string on a CY manifold, the $\mathcal{N} = 1_{10}$ supersymmetry was reduced to an $\mathcal{N} = 1_4$ supersymmetry. The ten-dimensional gravitino gave a single massless gravitino in four dimensions. The type-II string has two gravitini in ten dimensions. Consequently, upon compactification on a CY manifold we obtain two massless gravitini and $\mathcal{N} = 2_4$ supersymmetry. In such a compactification, one of the supersymmetries is originating in the left-moving sector and the other in the right-moving sector.

We will now derive the massless spectrum of such compactifications. An important ingredient is the number of various harmonic forms of a CY threefold as discussed in section 9.7 on page 239. There is a single zero-form and no one-forms. There are $h^{1,1}$ (1,1)-forms and no (2,0)- or (0,2)-forms. A characteristic of CY manifolds is that there are unique (3,0)- and (0,3)-forms Ω and $\bar{\Omega}$. Ω is used to define the period integrals of the manifold. There are also $h^{2,1}$ (2,1)- and (1,2)-forms. The rest of the forms are given by Poincaré duality.

Let us first describe the massless spectrum of type-IIA theory compactified on a CY manifold. In the NS-NS sector, the ten-dimensional metric gives rise to a four-dimensional metric and $(h^{1,1} + 2h^{1,2})$ scalars (see section 9.7.3 on page 243). The $h^{1,1} + 2h^{1,2}$ scalars are the moduli of the CY manifold.

The NS antisymmetric tensor gives rise to a four-dimensional antisymmetric tensor (equivalent to an axion) as well as $h^{1,1}$ scalars, while the dilaton gives an extra scalar. So far in the NS-NS sector we have a metric as well as $2h^{1,1} + 2h^{1,2} + 2$ scalars.

In the R-R sector, the three-form gives $h^{1,1}$ vectors and $(2h^{1,2} + 2)$ scalars (descending from the three-forms), while the vector gives a vector in four dimensions. In total, apart from the supergravity multiplet, we have $\mathrm{N}_V = h^{1,1}$ vector multiplets and $\mathrm{N}_H = h_{12} + 1$

hypermultiplets. An important observation here is that, in contrast to the heterotic string, the dilaton belongs to a hypermultiplet.

Since the scalars of the vector multiplets are associated with the (1,1)-forms, the classical vector moduli space is the same as the moduli space of complexified Kähler structures, $k + iB$. Moreover, $\mathcal{N} = 2_4$ supersymmetry forbids neutral couplings between vector multiplets and hypermultiplets. Since the dilaton (string coupling) is in a hypermultiplet, this means that the tree-level geometry of the vector-multiplet moduli space M_V is exact! Notice that all vectors come from the R-R sector and thus have no perturbative charged states. On the other hand, the hypermultiplets are $h_{21} + 1$ in number. One contains the dilaton, while the others come from the metric and antisymmetric tensor. Therefore, the classical hypermultiplet moduli space is a product of the moduli space of complex structures and the SU(2,1)/U(2) coset parametrizing the geometry of the dilaton hypermultiplet. This space is affected by quantum corrections both perturbative and non-perturbative.

Let us now focus at the type-IIB theory compactified on a CY manifold. The NS-NS sector obviously remains similar. However, the content of the R-R sector is different. The ten-dimensional axion gives a lower-dimensional axion while the two-index antisymmetric tensor gives $h^{1,1} + 1$ scalars, the last one coming from dualizing the four-dimensional antisymmetric tensor. The self-dual four-form gives $h^{1,1}$ scalars and $h^{2,1} + 1$ vectors. The last one comes from the unique (3,0)-form of a CY. In total we have $h^{1,2}$ vector multiplets and $h^{1,1} + 1$ hypermultiplets. Thus, in type-IIB compactifications the vector moduli space $\mathcal{M}_V$ parametrizes the space of complex structures of the CY manifold. The hypermultiplet moduli space parameterizes the complexified Kähler structures. As in the type-IIA case, the dilaton is part of a hypermultiplet.

9.12 Mirror Symmetry

We have seen in the previous section that type-IIA and type-IIB theory compactified on a CY manifold are related by exchanging the complex structure and Kähler moduli spaces. This is reminiscent of the action of toroidal T-duality described in the end of section 7.2 on page 157. We will see that this resemblance is more than a coincidence.

Before we delve into three-complex-dimensional CY manifolds we will warm up by looking at one-complex-dimensional CY manifolds. Here the holonomy should be by definition SU(1) and since this is trivial the manifold is flat. Thus, a compact CY_1 is a T^2. The CFT on the torus has four moduli: the metric and the antisymmetric tensor

$$G = \frac{T_2}{U_2}\begin{pmatrix} 1 & U_1 \\ U_1 & |U|^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & T_1 \\ -T_1 & 0 \end{pmatrix}. \tag{9.12.1}$$

Since U defines the complex coordinates on the torus as $z = \sigma_1 + U\sigma_2$ it is the complex structure modulus. $T = T_1 + iT_2$ is the complexified Kähler modulus.[10] For a rectangular torus with radii R_1, R_2 and no B field,

$$T = iR_1R_2, \quad U = i\frac{R_1}{R_2}. \tag{9.12.2}$$

[10] The Kähler form is the volume form on the two-torus and T_2 is the volume of T^2.

From this we can see that a single T-duality in the second direction implements $T \leftrightarrow U$ interchange. Moreover, if this torus forms part of the type-II string compactification manifold, then as we have argued in section 7.2 this T-duality interchanges IIA $\leftrightarrow$ IIB. We therefore observe a similar phenomenon, a IIA/IIB interchange accompanied by an interchange of Kähler and complex structure moduli.

We now return to CY_3. We start by describing more closely the world-sheet superconformal field theory that is relevant in type-II CY compactifications. Since the string theory background is $M_4 \times K$, the $\mathcal{N} = (2,2)_2$ world-sheet theory on the CY has $(c, \bar{c}) = (9, 9)$. This should be generated by the supersymmetric σ-model on the CY.

In section 4.13.2 on page 79 we have described in some detail the general structure of $\mathcal{N} = (2,2)_4$ superconformal theories. An important class of states are the chiral primary states with $\Delta = \frac{q}{2}$ and the antichiral primary states with $\Delta = -\frac{q}{2}$. We have shown that they both form a ring under OPE, the chiral (c) and the antichiral (a) rings.

An important ingredient is the spectral flow (4.13.24) on page 80, that maps NS states to R states and vice versa. In particular, the $(\Delta, q) = (0, 0)$ NS vacuum is mapped to the states $\left(\frac{3}{2}, \pm 3\right)$, carrying the maximal possible U(1) charge. An important constraint that is imposed by space-time supersymmetry is that the spectrum of the U(1) charge must be integral in the NS sector. This is required in order to guarantee locality with the space-time supercharges. Thus, the charge can take integer values in the range $-3, \ldots, 3$.

Therefore, in the NS sector the chiral primaries have

$$(\Delta, q) = (0, 0), \left(\frac{1}{2}, \pm 1\right), (1, \pm 2), \left(\frac{3}{2}, \pm 3\right). \tag{9.12.3}$$

Only $\left(\frac{1}{2}, \pm 1\right)$ will give massless states in the type-II compactification.

We now take into account also the right-moving part of the theory. Then we have four chiral rings: (c,c), (c,a), (a,c), (a,a). The two last ones are related by charge conjugation to the two first. The question we would like to answer is this: What is the relationship between the two independent chiral rings (c,c) and (a,c) and the geometry of the CY manifold?

The (c,c) ring contains (massless) states with charges $(q, \bar{q}) = (1, 1)$, while the (c,a) ring contains $(q, \bar{q}) = (1, -1)$ massless states. All of them have conformal weights $\left(\frac{1}{2}, \frac{1}{2}\right)$ and generate massless states. We will now compare them with the cohomology of the related CY manifold.

The (c,c) ring contains the unique state $(q, \bar{q}) = (3, 0)$ with the maximal U(1) charge which should correspond to the (3,0) Ω form, as well as its conjugate $(0, 3)$ that should correspond to $\bar{\Omega}$. It also contains the (3,3) states that should correspond to $\Omega \wedge \bar{\Omega}$.

The (1,1) states of the (c,c) ring should correspond to the complex structure moduli. This can be seen as follows. Let $\psi^i, \bar{\psi}^i$, $i = 1, 2, 3$ be the left-moving world-sheet fermions, while $\lambda^i, \bar{\lambda}^i$, $i = 1, 2, 3$ are the right-moving world-sheet fermions. The left and right U(1) currents are $J_L = \psi^i \bar{\psi}^i$, $J_R = \lambda^i \bar{\lambda}^i$. The lowest dimension field corresponding to the (1,1) state is $g_{ij} \psi^i \lambda^j$. We obtain the top state in the superfield by acting with $G^+_{-1/2} \bar{G}^-_{-1/2}$ to obtain $g_{ij} \partial X^i \bar{\partial} X^j$. This is the complex structure deformation operator in the σ-model.

On the other hand the (−1,1) states of the (a,c) ring by spectral flow can be mapped to (1,1) moduli. They can be written as $g_{i\bar{j}} \psi^i \bar{\lambda}^{\bar{j}}$ whose top component is $g_{i\bar{j}} \partial X^i \bar{\partial} \bar{X}^{\bar{j}}$ and corresponds to Kähler deformations.

This correspondence between the chiral rings, massless states and the cohomology of the CY manifold can be made more precise by identifying $G_0^+ \sim \partial$, $\bar{G}_0^+ \sim \bar{\partial}$ in the R sector and $G^+_{-1/2} \sim \partial$, $\bar{G}^+_{-1/2} \sim \bar{\partial}$ in the NS sector. There is a Hilbert space decomposition which parallels the Hodge decomposition for forms.

We conclude that the (c,c) chiral ring is associated with the complex structure moduli and the (a,c) ring with the Kähler moduli.

The simple observation is that the relative sign of the right U(1) current is a matter of convention and can be changed at will. This is an obvious symmetry of the CFT. However, the implications for the geometry are far reaching. The change of sign, interchanges the roles of the Complex structure and Kähler moduli spaces. This is known as *mirror symmetry*.

Define a mirror CY manifold K* as a CY space with cohomology

$$h^{p,q}_{K^*} = h^{3-p,q}_K. \tag{9.12.4}$$

K and K^* are said to form a mirror pair.

Mirror symmetry in CFT is the statement that the supersymmetric σ-models on K and K^* give rise to the same CFT.

Once this $\mathcal{N} = (2, 2)_2$ CFT is embedded in type-II string theory, the mirror symmetry transformation interchanges type IIA and type-IIB because it is similar to T-duality. This is in agreement with our observations in the previous section.

9.13 Absence of Continuous Global Symmetries

An important result in string theory is the absence of continuous global symmetries. Physicists for a long time had a prejudice against continuous global symmetries. The rough argument is that one needs to rotate fields all over space-time at once. This is at odds with the "spirit" of relativity. Moreover, it is plausible that gravity in the quantum regime involves baby-universe processes. This leads to the conclusion that such global symmetries will be spoiled by quantum gravity, since global charge will leak out to baby universes and will never be retrieved.

We will give here an argument which indicates that all internal symmetries must be local symmetries in string theory.

We start from bosonic strings and consider a continuous symmetry with a conserved charge which acts on the physical spectrum of the theory. This guarantees the existence of a local current,

$$Q = \frac{1}{2\pi i} \oint (dz J_z - d\bar{z} j_{\bar{z}}). \tag{9.13.1}$$

If such a symmetry is continuous and appears in the compact sector of the CFT then it is conformal. That is the current J_z is a (1,0) operator while $\bar{J}_{\bar{z}}$ is a (0,1) operator. Then the following states are massless gauge bosons in space-time

$$A_\mu \sim J_z \bar{\partial} X^\mu : e^{ip\cdot x} :, \quad B_\mu \sim \bar{J}_{\bar{z}} \partial X^\mu : e^{ip\cdot x} : . \tag{9.13.2}$$

Thus, the symmetry is also local. It is not necessary in general that there will be two gauge bosons. Sometimes the symmetry is purely left moving and it will be associated with a single gauge boson. This is also what happens in the open string case.

When the world-sheet is supersymmetric, one can write (9.13.1) in superspace

$$Q = \frac{1}{2\pi i} \oint (dz d\theta J - d\bar{z} d\bar{\theta} \bar{J}), \tag{9.13.3}$$

where by superconformal invariance J is a (1/2,0) superfield and $\bar{J}$ is a (0,1/2) superfield. We can again construct gauge bosons

$$a_\mu \sim J_z \bar{\psi}^\mu : e^{ip\cdot x} : , \quad b_\mu \sim \bar{J}_{\bar{z}} \psi^\mu : e^{ip\cdot x} : . \tag{9.13.4}$$

This also generalizes to the heterotic case.

The two assumptions made so far are important. The loophole consists in the existence of a conserved current

$$\partial_z \bar{J} + \partial_{\bar{z}} J = 0, \quad Q = \int_{t=\text{constant}} dx J^0, \tag{9.13.5}$$

whose charge is conserved and commutes with L_0, but $J, \bar{J}$ are not conformal operators. This can happen in noncompact CFTs and the prototype example is provided by the Lorentz symmetry of the string. The currents are

$$J_\tau^{\mu\nu} = X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu, \quad J_\sigma^{\mu\nu} = X^\mu X'^\nu - X^\nu X'^\mu. \tag{9.13.6}$$

These currents generate a symmetry, the associated charge is conserved, and it commutes with the Virasoro algebra. However, the local currents $J_z^{\mu\nu}$, $\bar{J}_{\bar{z}}^{\mu\nu}$ are not good conformal operators due to IR divergences. No gauge bosons are associated with these currents. Although no general proof exists, no such occurrence seems to exist in a compact CFT.

The other possibility is a "compact" CFT with a σ-model description and a continuous symmetry whose current is not conformal in the sense described above. This is the case of a large class of parafermionic CFTs. What happens in this case is that nonperturbative world-sheet effects break the continuous symmetry to a discrete one. You are invited to work out the simplest case in exercise (9.66) on page 293.

We will comment on another case that is worth mentioning: that of *approximate global symmetries*. It is typical in orientifold vacua of the type-II string (generalizations of the type-I string) for the gauge group to contain several anomalous U(1) factors. The anomaly is canceled via a lower-dimensional version of the Green-Schwarz mechanism involving a pseudoscalar (axion). This breaks the gauge symmetry and gives a mass to the gauge field. However, in some regions of the moduli space the global part of the gauge symmetry remains intact in perturbation theory. It is broken by instanton effects to a discrete symmetry but this breaking can be made arbitrarily small at sufficiently weak coupling.

In exercise 13.53 on page 469 you are invited to use holography in order to uncover another reason for the absence of continuous global symmetries in a large class of string theory vacua.

9.14 Orientifolds

In sections 7.3 and 7.6 on page 170 the construction of the unoriented (type-I) string theory was described. It was performed through quotiening the IIB theory by the orientation reversal transformation Ω. This is the simplest example of an *orientifold*. It is a generalization of an orbifold, where the symmetry group involves also orientation reversal, generically combined with other symmetry transformations. In this language, the type-I orientifold group is $G = \{1, \Omega\}$.

In this section we will construct more general orientifolds by hybridization of the orbifold concept and orientation reversal. They are important vacua of string theory containing both open and closed strings. D-branes and orientifold planes also enter in an essential manner.

We will consider orientifolds that break half of the supersymmetry of the original ten-dimensional theory. They may be viewed as compactifications of the type-I theory on orbifold limits of the K3 manifold. Although not phenomenologically relevant as such, they are simple enough to illustrate the issues involved.

9.14.1 K3 orientifolds

In section 9.10 we presented in detail the $\mathbb{Z}_2$ orbifold of type-II string theory. This described the string compactification to six dimensions on a $\mathbb{Z}_2$ orbifold limit of K3. We also briefly described the $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ orbifolds that are equivalent to other limits of K3.

Our present aim is to analyze unoriented strings moving in an orbifold K3 compactification. This will be implemented by adding the orientation reversal to the orbifold group. The construction of the closed part of the theory was described in section 9.10 on page 250. We will describe in detail here the construction of the open string sectors since this involves novel features.

The orbifold action on the T^4 coordinates was specified in (9.10.4). There are two distinct orientifold groups possible:

$$Y_{\rm N} = \{1, \Omega, g_k, \Omega_k\}, \quad k = 1, 2, \ldots, {\rm N}, \quad g_k \equiv e^{2\pi i k/{\rm N}}, \quad \Omega_k \equiv e^{2\pi i k/{\rm N}}\Omega, \tag{9.14.1}$$

and

$$W_{\rm N} = \{1, g_{2k-2}, \Omega_{2k-1}\}, \quad k = 1, 2, \ldots, \frac{\rm N}{2}, \quad {\rm N}\ \text{even}. \tag{9.14.2}$$

Both $Y_{\rm N}$ and $W_{\rm N}$ form groups since Ω commutes with the orbifold elements and $\Omega^2 = 1$.

Another point to stress is that for Ω to be a symmetry of the T^4 lattice sum, we must put restrictions on the moduli. We will take here the internal components $B_{ij} = 0$.[11]

We will now elaborate the action of the orientifold groups on the states in the open string sector, on D-branes. A generic state can be written as $\lambda_{ij}|X, ij\rangle$ where i, j label the end points of the open strings, λ is a CP matrix, and X collectively labels the world-sheet oscillators that are involved in that state.

The orientifold elements have two possible actions on a generic D-brane state. In addition to the obvious action on the oscillator states, they also act on the CP indices with a matrix representation of the orientifold group. It is generated via matrices γ_g

$$g_k : \ |X, ij\rangle \to \epsilon_k\, (\gamma_k)_{ii'}\, |g_k \cdot X, i'j'\rangle\, (\gamma_k^{-1})_{j'j}, \tag{9.14.3}$$

$$\Omega_k : \ |X, ij\rangle \to \epsilon_{\Omega_k}\, (\gamma_{\Omega_k})_{ii'}\, |\Omega_k \cdot X, j'i'\rangle\, (\gamma_{\Omega_k}^{-1})_{j'j}, \tag{9.14.4}$$

[11] In exercise 9.28 you are requested to find all values of the T^4 moduli so that Ω is a symmetry.

where $\epsilon_k, \epsilon_{\Omega_k}$ are signs. Note that the Ω_k elements interchange also the string end points. The group property $g_k = (g_1)^k$ and $g_N = 1$ implies

$$\gamma_k = \pm(\gamma_1)^k, \quad (\gamma_k)^N = \pm 1. \tag{9.14.5}$$

Furthermore, the condition that Ω^2

$$\Omega^2 : \ |X, ij\rangle \to \epsilon_\Omega^2 \, (\gamma_\Omega (\gamma_\Omega^T)^{-1})_{ii'} \, |X, i'j'\rangle \, (\gamma_\Omega^T \gamma_\Omega^{-1})_{j'j}, \tag{9.14.6}$$

is equal to the identity requires that

$$\gamma_\Omega = \zeta \gamma_\Omega^T, \quad \zeta^2 = 1. \tag{9.14.7}$$

Note that the adjoint action on the CP indices implies that the representation of the orientifold group on the CP sector is defined up to a sign.

These transformations do not completely fix the orientifold group transformations. There can be several CP matrices γ up to basis change that satisfy the group algebra. As we have seen however in section 5.3 on page 133, at the one-loop level, extra constraints emerge from tadpole cancellation.

We will now consider the implementation of the orientifold action at one-loop order. We focus on the $\mathbb{Z}_2$ case for simplicity.

9.14.2 The Klein bottle amplitude

The Klein bottle amplitude arises from the orientation projection in the closed string sector. In the operator formulation, according to our discussion in sections 5.3 on page 133 and 7.6 on page 170, the amplitude can be written as

$$Z_K = \mathrm{Tr}^{U+T}_{\text{NS-NS+R-R}} \left[\frac{\Omega}{2} \cdot \frac{1+g}{2} \cdot \frac{1+(-1)^{F_L}}{2} \, e^{-2\pi t (L_0 + \bar{L}_0 - c/12)} \right]. \tag{9.14.8}$$

The trace is taken both in the $\mathbb{Z}_2$ untwisted and twisted sector. As usual, because of the Ω insertion, only the left-right symmetric sectors (NS-NS and R-R) contribute to the trace. Only the left GSO projection was inserted for the same reason. g is the $\mathbb{Z}_2$ orbifold element. To evaluate these traces, we require the action of the orientation reversal on the bosonic oscillators, given in (3.4.2) on page 33 as well as on the fermionic ones,

$$\Omega \, \psi_r \, \Omega^{-1} = \bar{\psi}_r, \quad \Omega \, \bar{\psi}_r \, \Omega^{-1} = -\psi_r. \tag{9.14.9}$$

The extra minus sign is inserted in order for the product $\psi_r \bar{\psi}_r$ to be orientation invariant. This is mostly for convenience: this choice does not affect the GSO-invariant states.

We now compute the traces. We start from the T^4 lattice states. Since the orientation reversal acts on momenta and windings as

$$\Omega \, |m_i, n_i\rangle = |m_i, -n_i\rangle, \tag{9.14.10}$$

only momenta survive the Klein bottle trace when no $\mathbb{Z}_2$ element g is inserted

$$\langle -m_i, -n_i| \, \Omega \, |m_i, n_i\rangle = \prod_{i=1}^{4} \delta_{n_i,0}. \tag{9.14.11}$$

On the other hand, since $g\,|m_i,n_i\rangle = |-m_i,-n_i\rangle$ we obtain

$$\langle -m_i,-n_i|\,\Omega\,g\,|m_i,n_i\rangle = \prod_{i=1}^{4}\delta_{m_i,0}. \tag{9.14.12}$$

Concerning the action of Ω on the bosonic and fermionic oscillators, we obtain a nonzero contribution in the trace only if the state has the same left and right oscillators. This effectively sets $L_0+\bar{L}_0 \to 2L_0$ for such symmetric states.

It is useful at this point to introduce the $\mathbb{Z}_2$-twisted GSO-projected partition functions

$$T[^h_g](\tau) \equiv \sum_{a,b=0}^{1} (-1)^{a+b}\,\vartheta^2[^a_b](\tau)\,\vartheta[^{a+h}_{b+g}](\tau)\,\vartheta[^{a-h}_{b-g}](\tau). \tag{9.14.13}$$

Putting everything together, we find in the untwisted sector

$$\begin{aligned}\Lambda_{K_1} &= \frac{1}{4}\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}^U\left[\Omega\cdot\frac{1+(-1)^{F_L}}{2}\,e^{-2\pi t\left(L_0+\bar{L}_0-c/12\right)}\right]\\ &= i\frac{V_6\sqrt{G}}{2(2\pi\ell_s)^6}\int_0^\infty \frac{dt}{8t}\,\frac{T[^0_0](2it)}{t^5\,\eta^{12}(2it)}\sum_{m^i\in\mathbb{Z}}\exp\left[-\frac{\pi}{t}G_{ij}m^i m^j\right],\end{aligned} \tag{9.14.14}$$

where $i=1,2,3,4$.

It is important in this computation, to start with the lattice sum in the Hamiltonian form. It is this form that is proper in the operator formalism and the windings and momenta in equations (9.14.11) and (9.14.12) are those of the Hamiltonian form. Once the projection to windings or momenta only is made, we may then Poisson-resum at will. In (9.14.14) we have in fact Poisson-resumed the lattice sum over all momenta.

Taking the decompactification limit for the T^4, $\sqrt{G}\to\infty$, we obtain the associated ten-dimensional type-I Klein bottle amplitude in (7.6.5) on page 171 up to a factor of two, originating from the $\mathbb{Z}_2$ projection.

To obtain the same trace with the $\mathbb{Z}_2$ element g inserted, we may note that for states that are left-right symmetric and therefore survive the Ω projection, the g action is trivial. Therefore, the only nontrivial consequence of the insertion of g is to keep the T^4 windings instead of the momenta as documented in (9.14.12)

$$\begin{aligned}\Lambda_{K_2} &= \frac{1}{4}\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}^U\left[\Omega\cdot g\cdot\frac{1+(-1)^{F_L}}{2}\,e^{-2\pi t\left(L_0+\bar{L}_0-c/12\right)}\right]\\ &= i\frac{V_6}{2(2\pi\ell_s)^6}\int_0^\infty \frac{dt}{8t}\,\frac{T[^0_0](2it)}{t^3\,\eta^{12}(2it)}\sum_{n^i\in\mathbb{Z}}\exp\left[-\pi t G_{ij}n^i n^j\right].\end{aligned} \tag{9.14.15}$$

We now turn to the twisted sector. Here the partition functions, before Ω projection, can be found in section 9.10 on page 250. Note that there is no lattice sum here because twisted states are localized, and therefore carry no windings or momenta. Only symmetric states survive the Ω projection, so that $L_0+\bar{L}_0 \to 2L_0$. The g insertion in the trace is trivial, since left and right pieces transform similarly under the g projection after the Ω projection.[12]

[12] For a general orbifold, the insertion of the Ωg element, implies that this sector is equivalent to the sector with $L_0+\bar{L}_0 \to 2L_0$ and the insertion of g^2 in the trace.

This implies that we may take $g \to 1$ in the trace. We therefore find

$$\Lambda_{K_3} = \frac{1}{4}\int_0^\infty \frac{dt}{t}\,\mathrm{Tr}^T\left[\Omega\cdot(1+g)\cdot\frac{1+(-1)^{F_L}}{2}\,e^{-2\pi t(L_0+\bar{L}_0-c/12)}\right]$$
$$= i\frac{2^4 V_6}{(2\pi\ell_s)^6}\int_0^\infty \frac{dt}{8t}\,\frac{T[^1_0](2it)}{t^3\,\eta^6(2it)\vartheta_4^2(2it)}. \tag{9.14.16}$$

We may now collect the full Klein bottle amplitude, and transform it to the transverse (closed string) channel along the lines of section 5.3.3 on page 138 in order to expose the tadpoles. We use

$$\ell = \frac{\pi}{2t},\quad \vartheta_2(2it) = \frac{1}{\sqrt{2t}}\vartheta_4\left(i\frac{\ell}{\pi}\right),\quad \vartheta_3(2it) = \frac{1}{\sqrt{2t}}\vartheta_3\left(i\frac{\ell}{\pi}\right),\quad \eta(2it) = \frac{1}{\sqrt{2t}}\eta\left(i\frac{\ell}{\pi}\right), \tag{9.14.17}$$

to find

$$\Lambda_K = i\frac{2V_6}{\pi(2\pi\ell_s)^6}\int_0^\infty d\ell\,\frac{T[^0_0]\left(i\frac{\ell}{\pi}\right)}{\eta^{12}\left(i\frac{\ell}{\pi}\right)}\left[V_4\sum_{m^i\in\mathbb{Z}}\exp[-2\ell\,G_{ij}m^im^j] + \frac{1}{V_4}\sum_{n_i\in\mathbb{Z}}\exp[-2\ell\,G^{ij}n_in_j]\right]$$
$$+\,i\frac{2^4 V_6}{\pi(2\pi\ell_s)^6}\int_0^\infty d\ell\,\frac{T[^0_1]\left(i\frac{\ell}{\pi}\right)}{\eta^6\left(i\frac{\ell}{\pi}\right)\vartheta_2^2\left(i\frac{\ell}{\pi}\right)}, \tag{9.14.18}$$

where we Poisson-resummed the winding contribution and set $V_4 = \sqrt{G}$ (dimensionless).

We may now extract the diverged part of the Klein bottle, i.e., the tadpole

$$\mathcal{T}_K = i\frac{2^{10} V_6}{32\pi(2\pi\ell_s)^6}\int_0^\infty d\ell\left[V_4 + \frac{1}{V_4}\right]. \tag{9.14.19}$$

We note that the twisted sector contribution does not give rise to a tadpole. This occurs only in $\mathbb{Z}_2$ sectors of orientifolds.

Due to supersymmetry, the R tadpoles are opposite in sign to the NS ones. We have kept all contributions even though their sum formally vanishes, since it will not vanish in more complicated amplitudes.

The tadpole contribution linear in V_4 is the one that survives the decompactification to ten dimensions. It does indeed agree with the ten-dimensional result in (7.6.6) on page 172 once a factor of 2 coming from the $\mathbb{Z}_2$ projection is accounted for.

The tadpole will be canceled by the insertion of D9-branes filling all ten dimensions. The term inversely proportional to V_4 is related to the previous one by inverting the volume of T^4. As this operation turns D9-branes to D5-branes, the tadpole must be canceled by the addition of D5-branes.

We therefore conclude that the tadpoles are due to O9- and O5-planes.

9.14.3 D-branes on $T^4/\mathbb{Z}_2$

We now turn to the open sector. According to the previous section, we must include D9- and D5-branes. Although there are no options on D9-branes, since they fill all ten dimensions, there are options for D5-branes. They will be stretching in the six noncompact dimensions.

They are also pointlike on T^4. The orbifold now acts on the transverse positions of the branes. Therefore, there are two main options to consider.

We may consider a group of branes sitting at a fixed point of the orbifold action. In such a case there is no further restriction on the transverse position. We may also consider a group of branes at a generic position x^i on T^4. Orbifold invariance imposes that we also include a mirror brane group at the position $-x^i$.

Branes placed at an orbifold fixed point, are sometimes fixed to it. Such branes are also known as "fractional" branes. One reason for this is that to move off the fixed point they must split in mirror pairs and sometimes this is impossible. An equivalent reason is that the scalar fields, corresponding to the transverse brane coordinates are all projected out by the orbifold projection. Another reason is that their world-volume fields are charged under vectors localized on the orbifold planes.

Not every set of branes localized at an orbifold fixed point represents fractional branes. In the orientifold we are considering, the D5-branes will have vanishing twisted tadpoles and therefore will not be fractional.

In order to accommodate the orbifold action on the CP factors of D9- and D5-branes we must introduce matrices $\gamma_{g,9}$ and $\gamma_{g,5}$. They satisfy the constraints, (9.14.5)–(9.14.7) coming from the orbifold group property.

It is important to determine the signs entering in the orientifold projections. According to the detailed discussion in section 7.3 on page 162, in the NS sector there is an ϵ phase for each of the 9-9, 5-5, and 9-5 strings as follows

$$\Omega\,|9-9,p\,;ij\rangle_{\rm NS} = \epsilon_{99}\,(\gamma_{\Omega,9})_{ii'}\,|9-9,p\,;,j'i'\rangle_{\rm NS}\,(\gamma_{\Omega,9})^{-1}_{j'j}, \tag{9.14.20}$$

$$\Omega\,|5-5,p\,;ij\rangle_{\rm NS} = \epsilon_{55}\,(\gamma_{\Omega,5})_{ii'}|5-5,p\,;,j'i'\rangle_{\rm NS}\,(\gamma_{\Omega,5})^{-1}_{j'j}. \tag{9.14.21}$$

Similar arguments as in section 7.3 fix

$$\epsilon_{99}^2 = \epsilon_{55}^2 = -1, \quad \gamma_{\Omega,5,9} = \zeta_{5,9}\gamma^T_{\Omega,5,9}, \quad \zeta_5^2 = \zeta_9^2 = 1. \tag{9.14.22}$$

In the 5-9, 9-5 sectors, however, we may write

$$\begin{aligned}\Omega\,|5-9,p\,;ij\rangle_{\rm NS} &= \epsilon_{59}(\gamma_{\Omega,5})_{ii'}\,|9-5,p\,;,j'i'\rangle_{\rm NS}\,(\gamma_{\Omega,9})^{-1}_{j'j},\\ \Omega\,|9-5,p\,;ij\rangle_{\rm NS} &= \epsilon_{59}(\gamma_{\Omega,9})_{ii'}\,|5-9,p\,;,j'i'\rangle_{\rm NS}\,(\gamma_{\Omega,5})^{-1}_{j'j}.\end{aligned} \tag{9.14.23}$$

Imposing $\Omega^2 = 1$ we obtain

$$\epsilon_{59}^2\zeta_5\zeta_9 = 1. \tag{9.14.24}$$

The phase ϵ_{59} captures the transformation properties under Ω of the SO(4) twisted spinor as well of the NS open string vacuum. If two 9-5 states interact, they may produce a 5-5 or a 9-9 state. Therefore, a nontrivial coupling of two 9-5 states to the massless 9-9 or 5-5 states should be allowed. This implies that $\epsilon_{59}^2 = -1$. Therefore, from (9.14.24), the CP projection is opposite for five-branes compared to that of nine-branes,

$$\zeta_5\zeta_9 = -1. \tag{9.14.25}$$

In particular, the type-I D_5-branes have symplectic gauge group, a fact supported by other considerations in section 11.7.2. Similar considerations apply in the R sector. You are asked in exercise 9.29 on page 289 to carefully work them out.

We will now describe the light open string spectrum. For the 9-9 strings we have the following bosonic states: The vectors

$$\psi^{\mu}_{-1/2}|p;ij\rangle\lambda_{ij}, \quad \lambda = \gamma_{g,9}\,\lambda\,\gamma_{g,9}^{-1}, \quad \lambda = -\gamma_{\Omega,9}\,\lambda^T\,\gamma_{\Omega,9}^{-1} \tag{9.14.26}$$

are singlets under the SO(4) R-symmetry that rotates the four transverse dimensions. The scalars

$$\psi^{i}_{-1/2}|p;ij\rangle\lambda_{ij}, \quad \lambda = -\gamma_{g,9}\,\lambda\,\gamma_{g,9}^{-1}, \quad \lambda = -\gamma_{\Omega,9}\,\lambda^T\,\gamma_{\Omega,9}^{-1} \tag{9.14.27}$$

transform in the vector of SO(4). The fermionic states originating in the R sector can be obtained from the fact that the theory has $\mathcal{N} = 1_6$ supersymmetry and will not be consider further in this section.

D_5 branes can be localized at a fixed point a, with associated CP matrices $\gamma_{g,5a}$ and $\gamma_{\Omega,5a}$ or at a generic point x^i, together with a copy at the image point $-x^i$ with a CP matrix $\gamma_{\Omega,5x}$. For the low-lying spectrum of the 5a-5b strings we obtain

$$\psi^{\mu}_{-1/2}|p;ij\rangle\lambda_{ij}, \quad \lambda = \gamma_{g,5a}\,\lambda\,\gamma_{g,5b}^{-1}, \quad \lambda = -\gamma_{\Omega,5a}\,\lambda^T\,\gamma_{\Omega,5b}^{-1}, \tag{9.14.28}$$

$$\psi^{i}_{-1/2}|p;ij\rangle\lambda_{ij}, \quad \lambda = -\gamma_{g,5a}\,\lambda\gamma_{g,5b}^{-1}, \quad \lambda = \gamma_{\Omega,5a}\,\lambda^T\gamma_{\Omega,5b}^{-1}. \tag{9.14.29}$$

A point to stress here is that the Ω action on the DD directions is the opposite from NN, as explained in section 7.3. If $a = b$, these states are massless. For $a \neq b$, they have a mass proportional to the distance between the fixed points. Consider now the massless spectrum of the $5x$-$5x$ strings

$$\psi^{\mu}_{-1/2}|p;ij\rangle\lambda_{ij}, \quad \lambda = -\gamma_{\Omega,5x}\,\lambda^T\,\gamma_{\Omega,5x}^{-1}, \tag{9.14.30}$$

$$\psi^{i}_{-1/2}|p;ij\rangle\lambda_{ij}, \quad \lambda = \gamma_{\Omega,5x}\,\lambda^T\gamma_{\Omega,5x}^{-1}. \tag{9.14.31}$$

Note that the $\mathbb{Z}_2$ transformation g, relates them to the $5(-x)$-$5(-x)$ strings and poses no other constraint. All such strings so far give pairs of a vector and a hypermultiplet of $\mathcal{N} = 1_6$ supersymmetry.

Consider now the 9-$5a$ strings. These have DN boundary conditions along the T^4 directions. Therefore the massless (bosonic) state is a space-time scalar but an internal SO(4) spinor

$$|s, s'; ij\rangle\lambda_{ij}, \quad \lambda = \gamma_{g,9}\,\lambda\,\gamma_{g,5a}^{-1}. \tag{9.14.32}$$

There are two such scalars. The Ω projection relates these states to the states of the $5a$-9 strings and therefore provides no further constraints. We obtain hypermultiplets in this sector.

Similarly, for 9-$5x$ strings we have

$$|s, s'; ij\rangle\lambda_{ij}. \tag{9.14.33}$$

The $\mathbb{Z}_2$ projection relates them to the 9-$5(-x)$ strings, and the Ω projection to the $5x$-9 strings.

9.14.4 The cylinder amplitude

We may now proceed to evaluate the cylinder amplitude. We should remember the following general properties: NN directions have only momenta, DD only windings, and DN none of the above.

In operator form, the amplitude is

$$\Lambda_C = \int_0^\infty \frac{dt}{2t}\,\mathrm{Tr}^{99+55+95+59}_{\mathrm{NS,R}}\left[\frac{1}{2}\cdot\frac{1+g}{2}\cdot\frac{1+(-1)^F}{2}\,e^{-2\pi t(L_0-c/24)}\right]. \tag{9.14.34}$$

We start with the untwisted contributions. The 9-9 strings contribute

$$\begin{aligned}\Lambda_{C^U_{99}} &= i\frac{V_6 V_4}{2^6(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{1,9})^2\int_0^\infty \frac{dt}{8t}\,\frac{T[^0_0](it)}{t^5\,\eta^{12}(it)}\sum_{m^i\in Z} e^{-(\pi/t)G_{ij}m^im^j}\\ &= i\frac{V_6 V_4}{2^9\pi(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{1,9})^2\int_0^\infty d\ell\,\frac{T[^0_0]\left(i\frac{\ell}{\pi}\right)}{\eta^{12}\left(i\frac{\ell}{\pi}\right)}\sum_{m^i\in Z} e^{-\ell\, G_{ij}m^im^j},\end{aligned} \tag{9.14.35}$$

where, as usual for the cylinder, $\ell=\pi/t$ and we included the 1/2 from the Ω projection and the 1/2 from the $\mathbb{Z}_2$ projection. $\gamma_{1,9}$ is the unit matrix in the nine-brane sector and therefore $\mathrm{Tr}(\gamma_{1,9})=N_9$, the number of D9-branes. We also set $V_4=\sqrt{G}$. This amplitude decompactifies properly to recover (7.6.7) on page 172 as expected, up to an extra factor of 1/2 coming from the $\mathbb{Z}_2$ orbifold projection.

We now consider the (untwisted) contribution of the 5-5 strings. We will label the D5-branes by the index a. A subset will be localized at the (16) orbifold fixed points. We will label the fixed points with the letter I. The lattice sum is here a winding sum. We must sum over all paths connecting the D5-branes. Let the brane coordinates on T^4 be X^i_a with $i=1,2,3,4$ and a labeling the particular set of D5-branes. The compact coordinates are normalized so as to have integer periodicity. Then, on T^4 the distance between the two sets is $G_{ij}(X^i_a-X^i_b+n^i)(X^j_a-X^j_b+n^j)$ where n^i are arbitrary integers (windings). Using (2.3.37) on page 24 we may write the cylinder contribution of this configuration as

$$\begin{aligned}\Lambda_{C^U_{5_a5_b}} &= i\frac{V_6}{2^6(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{1,5_a})\mathrm{Tr}(\gamma_{1,5_b})\int_0^\infty \frac{dt}{8t}\,\frac{T[^0_0](it)}{t^3\,\eta^{12}(it)}\sum_{n^i\in Z} e^{-\pi t G_{ij}(X^i_a-X^i_b+n^i)(X^j_a-X^j_b+n^j)}\\ &= i\frac{V_6}{2^9\pi(2\pi\ell_s)^6V_4}\mathrm{Tr}(\gamma_{1,5_a})\mathrm{Tr}(\gamma_{1,5_b})\int_0^\infty d\ell\,\frac{T[^0_0]\left(i\frac{\ell}{\pi}\right)}{\eta^{12}\left(i\frac{\ell}{\pi}\right)}\sum_{n_i\in Z} e^{-\ell\, G^{ij}n_in_j-2\pi i n_i(X^i_a-X^i_b)},\end{aligned} \tag{9.14.36}$$

where as before $\mathrm{Tr}(\gamma_{1,5_a})=N^a_5$ is the number of D5-branes located at X^i_a.

Lastly, we have the (untwisted) contributions of the 9-5_a strings. Here the torus coordinates have DN boundary conditions and are therefore $\mathbb{Z}_2$ twisted. Therefore, the oscillator trace here can be obtained from the chiral $h=1, g=0$ part of the closed string orbifold in section 9.10 on page 250. The amplitude then is

$$\begin{aligned}\Lambda_{C^U_{9-5_a}} &= i\frac{V_6}{2^5(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{1,9})\mathrm{Tr}(\gamma_{1,5_a})\int_0^\infty \frac{dt}{8t}\,\frac{T[^1_0](it)}{t^3\,\eta^6(it)\vartheta_4^2(it)}\\ &= i\frac{V_6}{2^8\pi(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{1,9})\mathrm{Tr}(\gamma_{1,5_a})\int_0^\infty d\ell\,\frac{T[^0_1]\left(i\frac{\ell}{\pi}\right)}{\eta^6\left(i\frac{\ell}{\pi}\right)\vartheta_2^2\left(i\frac{\ell}{\pi}\right)}.\end{aligned} \tag{9.14.37}$$

We have included a factor of 2, due to the two orientations of the 9-5 strings. The transverse channel 9-5 contribution is zero in both the NS and the R sectors, because this is so for $T[^0_1]$. This is accidental for $\mathbb{Z}_2$ orientifold sectors.

We now move to the twisted contributions, which arise by inserting the $\mathbb{Z}_2$ element g in the cylinder trace. For the 9-9 strings we obtain

$$\Lambda_{C^T_{99}} = i\frac{V_6}{2^6(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{g,9})^2\int_0^\infty \frac{dt}{8t}\frac{T[^0_1](it)}{t^3\,\eta^8(it)}\left(\frac{2\eta(it)}{\vartheta_2(it)}\right)^2$$
$$= i\frac{V_6}{2^7\pi(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{g,9})^2\int_0^\infty d\ell\,\frac{T[^1_0]\left(i\frac{\ell}{\pi}\right)}{\eta^6\left(i\frac{\ell}{\pi}\right)\vartheta_4^2\left(i\frac{\ell}{\pi}\right)}, \tag{9.14.38}$$

where the last contribution comes from the T^4 bosons. This effectively follows from the $h=0, g=1$ chiral part of the closed $T^4/\mathbb{Z}_2$ partition function in section 9.10.

Consider now 5_a-5_b strings. In order for the trace to be nonzero, $a=b$ and the associated D5-branes should be located at the orbifold fixed points. The presence of the four DD directions does not otherwise affect the trace:

$$\Lambda_{C^T_{5_I-5_I}} = i\frac{V_6}{2^4(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{g,5_I})^2\int_0^\infty \frac{dt}{8t}\frac{T[^0_1](it)}{t^3\,\eta^6(it)\vartheta_2^2(it)}$$
$$= i\frac{V_6}{2^7\pi(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{g,5_I})^2\int_0^\infty d\ell\,\frac{T[^1_0]\left(i\frac{\ell}{\pi}\right)}{\eta^6\left(i\frac{\ell}{\pi}\right)\vartheta_4^2\left(i\frac{\ell}{\pi}\right)}. \tag{9.14.39}$$

Finally we consider the 9-5 strings. The presence of four DN directions effectively twists the four bosonic and fermionic coordinates. Therefore the oscillator trace here can be obtained from the chiral $h=1, g=1$ part of the closed string orbifold in 9.10. The presence of the $\mathbb{Z}_2$ element in the trace implies that only D5-branes localized at the orbifold fixed points can contribute:

$$\Lambda_{C^T_{9-5_I}} = i\frac{V_6}{2^5(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{g,9})\mathrm{Tr}(\gamma_{g,5_I})\int_0^\infty \frac{dt}{8t}\frac{T[^1_1](it)}{t^3\,\eta^6(it)\vartheta_3^2(it)}$$
$$= i\frac{V_6}{2^8\pi(2\pi\ell_s)^6}\mathrm{Tr}(\gamma_{g,9})\mathrm{Tr}(\gamma_{g,5_I})\int_0^\infty d\ell\,\frac{T[^1_1]\left(i\frac{\ell}{\pi}\right)}{\eta^6\left(i\frac{\ell}{\pi}\right)\vartheta_3^2\left(i\frac{\ell}{\pi}\right)}. \tag{9.14.40}$$

We have again multiplied by a factor of 2, to account for the two possible orientations.

The cylinder tadpoles extracted from (9.14.40) are

$$\mathcal{T}_C = i\frac{V_6}{2^5\pi(2\pi\ell_s)^6}\int_0^\infty d\ell\left[V_4\,(\mathrm{Tr}[\gamma_{1,9}])^2 + \frac{\left(\sum_a \mathrm{Tr}[\gamma_{1,5_a}]\right)^2}{V_4}\right.$$
$$\left. + \frac{1}{16}\sum_{I=1}^{16}\left(\mathrm{Tr}[\gamma_{g,9}] - 4\mathrm{Tr}[\gamma_{g,5_I}]\right)^2\right]. \tag{9.14.41}$$

The minus sign in the 9-5 twisted contribution is due to the $\mathbb{Z}_2$ element g in the associated trace.

9.14.5 The Mbius strip amplitude

We now turn to the Möbius strip, which implements the Ω projection in the open sector. We must calculate the same traces as on the cylinder but with an extra insertion of Ω,

$$\Lambda_M = \int_0^\infty \frac{dt}{2t} \mathrm{Tr}^{99+55}_{\mathrm{NS,R}} \left[\frac{\Omega}{2} \cdot \frac{1+g}{2} \cdot \frac{1+(-1)^F}{2} e^{2\pi t(L_0 - c/24)} \right]. \tag{9.14.42}$$

Since Ω changes the orientation of the string, 9-5 strings do not contribute to the trace. For the same reason, only strings starting and ending on the same D_5-brane contribute. For the CP factors, using (9.14.4) we may evaluate the trace as in (5.3.24) on page 139.

We start from the untwisted sector. The contribution of the 9-9 strings is

$$\begin{aligned}\Lambda_{M^U_{99}} &= -i \frac{V_6 V_4}{2^6 (2\pi \ell_s)^6} \mathrm{Tr}(\gamma^T_{\Omega,9} \gamma^{-1}_{\Omega,9}) \int_0^\infty \frac{dt}{8t} \frac{\hat{T}[^0_0](it)}{t^5\, \hat{\eta}^{12}(it)} \sum_{m^i \in Z} e^{-(\pi/t) G_{ij} m^i m^j} \\ &= -i \frac{V_6 V_4}{2^3 \pi (2\pi \ell_s)^6} \mathrm{Tr}(\gamma^T_{\Omega,9} \gamma^{-1}_{\Omega,9}) \int_0^\infty d\ell \frac{\hat{T}[^0_0]\left(i\frac{\ell}{\pi}\right)}{\hat{\eta}^{12}\left(i\frac{\ell}{\pi}\right)} \sum_{m^i \in Z} e^{-4\ell\, G_{ij} m^i m^j},\end{aligned} \tag{9.14.43}$$

where for the Möbius strip, $\ell = \pi/(4t)$ and the overall sign is a convention. The various characters have been replaced with careted characters as is standard for the Möbius strip. They are defined in (7.6.10) on page 172 and some of their properties presented in (C.28) and (C.29) on page 510. We also used in the second line, the transformation properties of the fermionic characters from appendix C on page 507.

For the 5_a-5_a strings, according to (7.3.5) and (7.3.6) on page 163, the T^4 directions have an extra minus sign because they now carry DD boundary conditions. This is equivalent to an insertion of the $\mathbb{Z}_2$ element g in the trace. We obtain

$$\begin{aligned}\Lambda_{M^U_{5_a 5_a}} &= i \frac{V_6}{2^6 (2\pi \ell_s)^6} \mathrm{Tr}(\gamma^T_{\Omega,5_a} \gamma^{-1}_{\Omega,5_a}) \int_0^\infty \frac{dt}{8t} \frac{\hat{T}[^0_1](it)}{t^3\, \hat{\eta}^8(it)} \left(\frac{2\hat{\eta}(it)}{\hat{\vartheta}_2(it)} \right)^2 \\ &= i \frac{V_6}{2^4 \pi (2\pi \ell_s)^6} \mathrm{Tr}(\gamma^T_{\Omega,5_a} \gamma^{-1}_{\Omega,5_a}) \int_0^\infty d\ell \frac{\hat{T}[^0_1]\left(i\frac{\ell}{\pi}\right)}{\eta^6\left(i\frac{\ell}{\pi}\right) \vartheta_2\left(2i\frac{\ell}{\pi}\right) \vartheta_4\left(2i\frac{\ell}{\pi}\right)},\end{aligned} \tag{9.14.44}$$

where we have used $\hat{\vartheta}^2_2(it) = 2\vartheta_2(2it)\vartheta_4(2it)$ and $\hat{T}[^0_1](it) = -\hat{T}[^0_1]\left(i\frac{\ell}{\pi}\right)$.

We now proceed to calculate the traces in the twisted sector,

$$\begin{aligned}\Lambda_{M^T_{99}} &= i \frac{V_6}{2^6 (2\pi \ell_s)^6} \mathrm{Tr}(\gamma^T_{g\Omega,9} \gamma^{-1}_{g\Omega,9}) \int_0^\infty \frac{dt}{8t} \frac{\hat{T}[^0_1](it)}{t^3\, \hat{\eta}^8(it)} \left(\frac{2\hat{\eta}(it)}{\hat{\vartheta}_2(it)} \right)^2 \\ &= i \frac{V_6}{2^4 \pi (2\pi \ell_s)^6} \mathrm{Tr}(\gamma^T_{g\Omega,9} \gamma^{-1}_{g\Omega,9}) \int_0^\infty d\ell \frac{\hat{T}[^0_1]\left(i\frac{\ell}{\pi}\right)}{\eta^6\left(i\frac{\ell}{\pi}\right) \vartheta_2\left(2i\frac{\ell}{\pi}\right) \vartheta_4\left(2i\frac{\ell}{\pi}\right)}.\end{aligned} \tag{9.14.45}$$

Before computing the twisted trace for the 5-5 strings we first observe that not only the 5_I-5_I strings contribute, as on the cylinder but also the 5_x-5_{-x} strings for any $x \in T^4$. To see this, consider a 5-5 string stretched between points x and y on T^4. Since in these directions the boundary conditions are DD, the expansion (2.3.28) on page 23 is relevant with center-of-mass coordinate x and winding $w \sim y - x$. We have the following actions on the string ground state:

$$\Omega|x, w\rangle = |y, -w\rangle, \quad g|x, w\rangle = |-x, -w\rangle, \tag{9.14.46}$$

where, as usual, Ω interchanges the end points of the string. Therefore,

$$\langle x', w'|g \cdot \Omega|x, w\rangle = \langle x', w'|-y, w\rangle = \delta(y + x')\delta(w - w') = \delta(y + x')\delta(x + y'). \tag{9.14.47}$$

The trace vanishes unless $y = -x$, that is the string stretches from an arbitrary D5-brane to its image under the $\mathbb{Z}_2$ transformation g. Note that this property ceases to be true for other $\mathbb{Z}_N$ orbifold actions.

We may now evaluate the trace as

$$\begin{aligned}
\Lambda_{M^T_{5_a5_a}} &= i\frac{V_6}{2^6(2\pi\ell_s)^6}\mathrm{Tr}\left(\gamma^T_{g\Omega,5_a}\gamma^{-1}_{g\Omega,5_a}\right)\int_0^\infty \frac{dt}{8t}\frac{\hat{T}[^0_0](it)}{t^3\,\hat{\eta}^{12}(it)}\sum_{n^i\in Z} e^{-\pi t G_{ij}(2X^i_a+n^i)(2X^j_a+n^j)} \\
&= -i\frac{V_6}{2^3\pi(2\pi\ell_s)^6\,V_4}\mathrm{Tr}\left(\gamma^T_{g\Omega,5_a}\gamma^{-1}_{g\Omega,5_a}\right)\int_0^\infty d\ell\,\frac{\hat{T}[^0_0]\left(i\frac{\ell}{\pi}\right)}{\hat{\eta}^{12}\left(i\frac{\ell}{\pi}\right)}\sum_{n_i\in Z} e^{-4\ell\, G^{ij}n_in_j - 4\pi n_iX^i_a},
\end{aligned} \tag{9.14.48}$$

where the projection is reversed in the DD directions.

We collect the tadpoles as

$$\mathcal{T}_M = -i\frac{2V_6}{\pi(2\pi\ell_s)^6}\int_0^\infty d\ell\left[V_4\,\mathrm{Tr}(\gamma^T_{\Omega,9}\gamma^{-1}_{\Omega,9}) + \frac{\sum_a \mathrm{Tr}\left(\gamma^T_{g\Omega,5_a}\gamma^{-1}_{g\Omega,5_a}\right)}{V_4}\right], \tag{9.14.49}$$

where the contributions proportional to $T[^0_1]\left(i\frac{\ell}{\pi}\right)$ vanish identically for the $\mathbb{Z}_2$ orbifold.

9.14.6 Tadpole cancellation

We are now ready to discuss the cancellation of tadpoles. Due to the unbroken supersymmetry, the NS and R tadpoles are equal and opposite. Collecting the various contributions from (9.14.19), (9.14.41), and (9.14.49) we obtain

$$\begin{aligned}
\mathcal{T} = \frac{iV_6}{32\pi(2\pi\ell_s)^6}\int_0^\infty d\ell\Bigg[&\left(2^{10} + (\mathrm{Tr}[\gamma_{1,9}])^2 - 2^6\mathrm{Tr}[\gamma^T_{\Omega,9}\gamma^{-1}_{\Omega,9}]\right)V_4 \\
&+\frac{\left(2^{10} + (\sum_a \mathrm{Tr}[\gamma_{1,5_a}])^2 - 2^6\sum_a \mathrm{Tr}[\gamma^T_{g\Omega,5_a}\gamma^{-1}_{\Omega,5_a}]\right)}{V_4} + \frac{1}{16}\sum_{I=1}^{16}\left(\mathrm{Tr}[\gamma_{g,9}] - 4\mathrm{Tr}[\gamma_{g,5_I}]\right)^2\Bigg].
\end{aligned} \tag{9.14.50}$$

Tadpole cancellation conditions thus require the cancellation of the ten-form R-R charge

$$2^{10} + (\mathrm{Tr}[\gamma_{1,9}])^2 - 2^6\mathrm{Tr}[\gamma^T_{\Omega,9}\gamma^{-1}_{\Omega,9}] = 0, \tag{9.14.51}$$

six-form R-R charge

$$2^{10} + \left(\sum_a \mathrm{Tr}[\gamma_{1,5_a}]\right)^2 - 2^6\sum_a \mathrm{Tr}[\gamma^T_{g\Omega,5_I}\gamma^{-1}_{g\Omega,5_I}] = 0, \tag{9.14.52}$$

and the twisted-form R-R charges,

$$\mathrm{Tr}[\gamma_{g,9}] - 4\mathrm{Tr}[\gamma_{g,5_I}] = 0, \quad \forall\ I = 1, 2, \ldots, 16. \tag{9.14.53}$$

We will now try to find a simple solution to these conditions. We assume that all D_5-branes are located at a single fixed point, that we will take to be the origin. As shown in (9.14.7) we must have

$$\gamma_{\Omega,9} = \zeta_9\, \gamma^T_{\Omega,9}, \quad \gamma_{g\Omega,9} = \tilde{\zeta}_9\, \gamma^T_{g\Omega,9}, \quad \gamma_{\Omega,5} = \zeta_5\, \gamma^T_{\Omega,5}, \quad \gamma_{g\Omega,5} = \tilde{\zeta}_5\, \gamma^T_{g\Omega,5}. \tag{9.14.54}$$

Then (9.14.51), (9.14.52) become

$$(N_9 - 32\zeta_9)^2 = 0, \quad (N_5 - 32\tilde{\zeta}_5)^2 = 0, \tag{9.14.55}$$

with obvious solution

$$N_9 = N_5 = 32, \quad \zeta_9 = 1, \quad \tilde{\zeta}_5 = 1. \tag{9.14.56}$$

Moreover, from (9.14.25), $\zeta_5 = -\zeta_9 = -1$.

We may therefore take

$$\gamma_{\Omega,9} = \gamma_{g\Omega,5} = \mathbf{1}_{32}, \quad \gamma_{g,9} = \gamma_{g\Omega,9} = \gamma_{g,5} = \gamma_{\Omega,5} = \begin{pmatrix} 0 & i\mathbf{1}_{16} \\ -i\mathbf{1}_{16} & 0 \end{pmatrix}, \tag{9.14.57}$$

where the subscripts stand for the dimension of the matrix blocks. It can be directly verified that these matrices satisfy the group relations and also satisfy the remaining twisted tadpole conditions (9.14.53). In exercise 9.33 on page 289 you are asked to investigate other solutions to the tadpole conditions.

Note also, that for this solution to the tadpole conditions, the twisted tadpoles vanish. This implies that the D_5-branes are not fractional branes. They are expected to be allowed to move off the orbifold fixed points.

9.14.7 The open string spectrum

We have determined the consistent projection in the open spectrum, by asking for the absence of tadpoles. We may now solve the projection conditions of section 9.14.3 to obtain the open string (massless) spectrum. We will split the CP matrices λ into 16×16 blocks, to accommodate the structure of the projection matrices in (9.14.57).

In the 9-9 sector, solving (9.14.26) we find that the vectors have

$$\lambda_V = \begin{pmatrix} A & S \\ -S & A \end{pmatrix}, \tag{9.14.58}$$

where A stands for a Hermitian antisymmetric matrix and S for a Hermitian symmetric matrix. Such matrices form the Lie algebra of the U(16) group. Therefore, taking into account the fermions, we have a U(16) vector multiplet of $\mathcal{N} = 1_6$ supersymmetry.

For the 9-9 scalars, solving (9.14.27) we obtain

$$\lambda_S = \begin{pmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{pmatrix}, \tag{9.14.59}$$

where again $A_{1,2}$ are Hermitian antisymmetric matrices. We therefore obtain two antisymmetric representations of U(16): $\mathbf{120} + \overline{\mathbf{120}}$. The scalars come in multiples of four (transforming as the **4** of the R-symmetry SO(4)) We therefore obtain two hypermultiplets transforming in the **120** of U(16).[13]

For the spectrum of 5-5 strings emerging from the 32 D_5-branes all in one of the fixed points we must solve (9.14.28) and (9.14.29). The solution is the same as in the 9-9 sector and we obtain another U(16) vector multiplet as well as two hypermultiplets in the **120**.

In the 5-9 sector we must solve (9.14.32). The solution is

$$\lambda_{95} = \begin{pmatrix} H_1 & H_2 \\ -H_2 & H_1 \end{pmatrix}, \tag{9.14.60}$$

where $H_{1,2}$ are Hermitian matrices. We therefore obtain the $(\mathbf{16},\overline{\mathbf{16}})$ and $(\overline{\mathbf{16}},\mathbf{16})$ representations of U(16)×U(16). Taking into account the multiplicity of scalars, this is a single hypermultiplet transforming as a $(\mathbf{16},\overline{\mathbf{16}})$.

We have assumed a very special D_5 brane configuration where all of them are on a single fixed point. We expect to be able to move them away, to other fixed points or in pairs in the bulk of T^4. Consider $2n_a$ D_5-branes at the ath fixed point. This number must be even so that (9.14.57) makes sense. Consider also n_x branes at point x and the same number at its image $-x$. The solution to the tadpole conditions gives a gauge group

$$\mathrm{U}(16) \times \prod_{a=1}^{16} \mathrm{U}(n_a) \prod_{x} \mathrm{Sp}\,(2n_x), \quad \sum_{a=1}^{16} n_a + \sum_{x} n_x = 16, \tag{9.14.61}$$

where the U(16) factor originates from the 9-9 strings. There are two 9-9 hypermultiplets transforming in the **120** of U(16). There are also two hypermultiplets transforming in the antisymmetric representation for each $\mathrm{U}(n_a)$ group. There is one hypermultiplet in the $(\mathbf{16}, \bar{\mathbf{n}}_a)$ for each $\mathrm{U}(n_a)$ factor. There is one hypermultiplet in the antisymmetric representation plus a singlet for each symplectic factor. Finally, there is one hypermultiplet in the $(\mathbf{16}, m_x)$ for each symplectic factor.

You are invited to derive this spectrum in exercise 9.34 by solving the tadpole conditions and implementing the projections. In exercise 9.35 on page 290 you are asked to give a field theory derivation of the same massless spectrum by Higgsing the U(16)×U(16) gauge symmetry.

9.15 D-branes at Orbifold Singularities

An important ingredient of the Standard Model of the fundamental interactions is the chirality of the particle spectrum. As already discussed in the case of the heterotic string in section 9.6 on page 237, to obtain a four-dimensional chiral spectrum the supersymmetry of the string vacuum should be at most $\mathcal{N} = 1_4$.

In orientifolds, as we will argue in section 9.17, matter is expected to arise from the open string sector, localized on D-branes. An attractive way to produce a chiral spectrum, is to place D-branes at an orbifold singularity as we will now show.

[13] The hypermultiplet being nonchiral, we do not need to distinguish a representation from its conjugate. In fact if a complex scalar transforms in the representation R, the second complex scalar transforms in the representation $\bar{R}$. The same applies to the two Weyl fermions of the hypermultiplet.

When a D-brane is placed transverse to an orbifold singularity, the orbifold projection acts directly on its world-volume spectrum. By an appropriate choice of projection, the spectrum will be chiral. This is to be contrasted with D-branes placed in a generic bulk point. Such D-branes, in order to be invariant under the orbifold projections, must have mirror copies placed in related points. The orbifold projection in this case gives a spectrum that is identical to one of the original D-brane copies. Therefore the spectrum is not chiral in this case, due to the effective extended supersymmetry that remains.

We will therefore analyze branes transverse to orbifold singularities. The orbifold action being local, we may ignore global issues when we discuss the invariant spectrum. Global issues will become important when we wish to implement tadpole cancellation.

We will examine orbifold fixed points whose local structure is $\mathbb{R}^6/\mathbb{Z}_N$ for some integer N. We will therefore consider D-branes transverse to a $\mathbb{R}^6/\mathbb{Z}_N$ singularity.

As we have seen in the previous section, the branes we simply obtain during orientifold compactifications of the type-I string are D_9- and D_5-branes. These can be dualized to D_3- and D_7-branes and it is in this incarnation that we will describe our brane configuration.

We will first consider n D_3-branes transverse to the $\mathbb{R}^6/\mathbb{Z}_N$ singularity. We split the ten-dimensional indices into the four-dimensional Minkowski ones denoted by $\mu, \nu, \dots$ and the six internal ones that we package in three complex pairs and label as $k, l, \dots$. The $\mathbb{Z}_N$ rotation acts on the internal $\mathbb{R}^6$. It equivalently acts on the SO(6) R-symmetry quantum numbers of the massless D-brane fields. The vectors A_μ transform in the singlet, the fermions in the spinor and the scalars in the vector.

Complexifying the scalars[14] in pairs, the $\mathbb{Z}_N$ rotation acts on them as

$$R_\theta = \text{diag}\left(e^{2\pi i b_1/N}, e^{-2\pi i b_1/N}, e^{2\pi i b_2/N}, e^{-2\pi i b_2/N}, e^{2\pi i b_3/N}, e^{-2\pi i b_3/N} \right), \tag{9.15.1}$$

with $b_i \in \mathbb{Z}_N$. In exercise 9.40 on page 290 you are asked to show that on the four-dimensional spinor representation of SO(6), the rotation acts as

$$S_\theta = \text{diag}\left(e^{2\pi i a_1/N}, e^{2\pi i a_2/N}, e^{2\pi i a_3/N}, e^{2\pi i a_4/N} \right), \tag{9.15.2}$$

with

$$a_1 = \frac{b_2 + b_3 - b_1}{2}, \quad a_2 = \frac{b_1 - b_2 + b_3}{2}, \quad a_3 = \frac{b_1 + b_2 - b_3}{2}, \quad a_4 = -\frac{b_1 + b_2 + b_3}{2}. \tag{9.15.3}$$

We can parametrize the action of the rotation on the CP indices without loss of generality using the matrices

$$\gamma_{3,\theta} = \text{diag}\left(\mathbf{1}_{n_0}, \theta\, \mathbf{1}_{n_1}, \dots, \theta^{N-1}\, \mathbf{1}_{n_{N-1}}\right), \tag{9.15.4}$$

where $\theta = e^{2\pi i/N}$ is the generating $\mathbb{Z}_N$ rotation, $n = \sum_{i=0}^{N-1} n_i$ and $\mathbf{1}_n$ is the unit $n \times n$ matrix.

The orbifold action on the gauge boson state is

$$A_\mu \sim \psi^\mu_{-1/2}|0;\ \lambda\,\rangle \to \psi^\mu_{-1/2}|0;\ \gamma_{3,\theta}\,\lambda\,\gamma_{3,\theta}^{-1}\,\rangle, \tag{9.15.5}$$

where the matrix λ keeps track of the CP indices: $|0;\ \lambda\,\rangle \equiv \lambda^{ij}|0; ij\rangle$.

[14] These are in one-to-one correspondence with the six transverse coordinates of the D_3-branes.

Therefore, the gauge bosons must satisfy $\lambda = \gamma_{3,\theta}\,\lambda\,\gamma_{3,\theta}^{-1}$. The solutions to this equation are $n_i \times n_i$ block diagonal matrices: the invariant gauge bosons are in the adjoint of $\prod_{i=0}^{N-1} U(n_i)$.

The three complex scalars Φ_k obtained from the complexification of the six real scalars transform as

$$\Phi_k \sim \psi^k_{-1/2}|0;\ \lambda\,\rangle \to e^{-2\pi i b_k/N}\ \psi^k_{-1/2}|0;\ \gamma_{3,\theta}\,\lambda\,\gamma_{3,\theta}^{-1}\,\rangle. \tag{9.15.6}$$

The invariant scalars must therefore satisfy $\lambda = e^{2\pi i b_k/N}\gamma_{3,\theta}\,\lambda\,\gamma_{3,\theta}^{-1}$. In exercise 9.41 you are asked to solve this condition explicitly and show that the invariant scalars transform in the following representation of the gauge group

$$\text{scalars} \to \oplus_{k=1}^{3} \oplus_{i=0}^{N-1} \left(n_i, \bar{n}_{i-b_k}\right). \tag{9.15.7}$$

Finally the fermions are labeled as

$$\psi_a \sim |\lambda\,; s_1, s_2, s_3, s_4\rangle, \tag{9.15.8}$$

where $s_i = \pm\frac{1}{2}$ are spinorial indices, with $\sum_{i=1}^{4} s_i = \text{odd}$ (due to the GSO projection). The states with $s_4 = -\frac{1}{2}$ correspond to left-handed, four-dimensional Weyl fermions while $s_4 = \frac{1}{2}$ corresponds to right-handed, four-dimensional Weyl fermions. The $s_{1,2,3}$ spinor quantum numbers are R-symmetry spinor quantum numbers. We can thus label the 8 on-shell fermion states as $|\lambda;\alpha, s_4\rangle$ where $\alpha = 1, 2, 3, 4$ is the R-spinor quantum number. The fermions then transform as

$$|\lambda;\alpha, s_4\rangle \to e^{2\pi i a_\alpha/N}\ |\ \gamma_{3,\theta}\,\lambda\,\gamma_{3,\theta}^{-1}\,;\alpha, s_4\,\rangle, \tag{9.15.9}$$

and the invariant fermions must satisfy $\lambda = e^{2\pi i a_\alpha/N}\gamma_{3,\theta}\,\lambda\,\gamma_{3,\theta}^{-1}$. The solution to this equation gives left-handed Weyl fermions in the following representation of the gauge group:

$$\text{left-handed fermions} \to \oplus_{\alpha=1}^{4} \oplus_{i=0}^{N-1} \left(n_i, \bar{n}_{i+a_\alpha}\right), \tag{9.15.10}$$

a representation that is generically chiral.

When $\sum_{i=1}^{3} b_i = 0$ so that $a_4 = 0$ we have an $\mathcal{N} = 1_4$ supersymmetric configuration (the rotation $\in SU(3) \subset SO(6)$). The associated fixed point is known as an $\mathcal{N} = 1_4$ orbifold singularity. The a_4 fermions become the gaugini, while the $a_{1,2,3}$ fermions are the $\mathcal{N} = 1_4$ supersymmetric partners of the scalars.

We now add D7-branes. They are in general needed to cancel the twisted tadpoles. There are three generically distinct ways of adding the D7-branes. They may be transverse to the third plane (and therefore wrap the 1 and 2 complex internal dimensions). They could also be transverse to the first or second plane. We will discuss only the first case explicitly, leaving the other two cases to the reader as an exercise (9.42). We therefore place m $D7_3$-branes that we take to be transverse to the last complex coordinate (the third plane).

For the 7_3-7_3 strings, the story is similar, with a new CP matrix parametrized as

$$\gamma_{7_3,\theta} = \begin{cases} \text{diag}\left(\mathbf{1}_{m_0}, \theta\,\mathbf{1}_{m_1}, \ldots, \theta^{N-1}\,\mathbf{1}_{m_{N-1}}\right), & b_3\ \text{even}, \\ \text{diag}\left(\theta\,\mathbf{1}_{m_0}, \theta^3\,\mathbf{1}_{m_1}, \ldots, \theta^{2N-1}\,\mathbf{1}_{m_{N-1}}\right), & b_3\ \text{odd}. \end{cases} \tag{9.15.11}$$

The extra fields that are localized on the D_3-brane world-volume come from the 3-7_3 and 7_3-3 strings. For such strings, there are four DN directions which provide four zero modes in the NS sector (directions 4,5,6,7), while from the NN and DD directions we have zero modes in the R sector (directions 2,3,8,9).

The invariant (complex) scalars (NS sector) must satisfy

$$\lambda_{3-7_3} = e^{-i\pi(b_1+b_2)/N}\ \gamma_{3,\theta}\ \lambda_{3-7_3}\ \gamma_{7_3,\theta}^{-1}, \tag{9.15.12}$$

$$\lambda_{7_3-3} = e^{-i\pi(b_1+b_2)/N}\ \gamma_{7_3,\theta}\ \lambda_{7_3-3}\ \gamma_{3,\theta}^{-1}, \tag{9.15.13}$$

with spectrum

$$\begin{aligned} &\oplus_{i=0}^{N-1}\left[\left(n_i,\bar{m}_{i-(b_1+b_2)/2}\right)+\left(m_i,\bar{n}_{i-(b_1+b_2)/2}\right)\right], \quad b_3 \ \text{even}, \\ &\oplus_{i=0}^{N-1}\left[\left(n_i,\bar{m}_{i-(b_1+b_2+1)/2}\right)+\left(m_i,\bar{n}_{i-(b_1+b_2-1)/2}\right)\right], \quad b_3 \ \text{odd}. \end{aligned} \tag{9.15.14}$$

The invariant fermions coming from the R sector must satisfy

$$\lambda_{3-7_3} = e^{i\pi b_3/N}\ \gamma_{3,\theta}\ \lambda_{3-7_3}\ \gamma_{7_3,\theta}^{-1}, \tag{9.15.15}$$

$$\lambda_{7_3-3} = e^{i\pi b_3/N}\ \gamma_{7_3,\theta}\ \lambda_{7_3-3}\ \gamma_{3,\theta}^{-1}, \tag{9.15.16}$$

with spectrum

$$\begin{aligned} &\oplus_{i=0}^{N-1}\left[\left(n_i,\bar{m}_{i+b_3/2}\right)+\left(m_i,\bar{n}_{i+b_3/2}\right)\right], \qquad b_3 \ \text{even}, \\ &\oplus_{i=0}^{N-1}\left[\left(n_i,\bar{m}_{i+(b_3-1)/2}\right)+\left(m_i,\bar{n}_{i+(b_3+1)/2}\right)\right], \quad b_3 \ \text{odd}. \end{aligned} \tag{9.15.17}$$

We observe that such brane configurations provide a generically chiral spectrum of four-dimensional fermions. Model building involves putting together such sets of branes on a compact orbifold so that the tadpoles are canceled. It turns out that several of the U(1) factors of the gauge group have triangle anomalies.These are canceled by a variation of the Green-Schwarz mechanism, which at the same time renders the U(1)'s massive.

9.16 Magnetized Compactifications and Intersecting Branes

So far we have seen how compactification on tori, combined with orbifold projections reduce the space-time supersymmetry, in our quest for realistic vacua of string theory.

In this section, we will describe another method of breaking supersymmetry during compactification. It involves turning on constant internal magnetic fields. Considering the internal manifold to be a torus this provides with vacua with reduced supersymmetry, where calculations can be performed.

In the case of closed string theory, turning on a constant internal magnetic field must be accompanied by a nontrivial deformation of the metric, in order to satisfy the classical equations of motion. Although such exact solutions exist, model building is complicated.

If the internal magnetic field originates in the open sector, the gravitational back-reaction appears in the next order of perturbations theory (at one loop). It is therefore easier to tune the appropriate brane configurations.

It turns out that T-duality changes magnetized branes into intersecting branes and vice versa. This gives an alternative (geometric) view of some important effects, like chirality generation in such compactifications.

In the sequel, we will analyze magnetized and intersecting branes in simple contexts in order to illustrate the important effects.

9.16.1 Open strings in an internal magnetic field

We will consider open strings compactified on T^6. We take for simplicity values for the T^6 moduli in order for the torus to have the factorized form $T^2 \times T^2 \times T^2$. We consider a D_p brane wrapping one of the tori, in the x^4-x^5 plane. We turn on a constant magnetic field H, in the Cartan of the D-brane gauge group U(n):

$$A_4 = 0, \quad A_5 = Hx^4. \tag{9.16.1}$$

This is the magnetic monopole solution on T^2. The flux quantization condition implies that

$$n\,(2\pi \ell_s R_4)(2\pi \ell_s R_5)\, qH = 2\pi m \to (2\pi \ell_s^2) qH = \frac{1}{R_4 R_5}\frac{m}{n}, \quad m, n \in \mathbb{Z}, \tag{9.16.2}$$

where q is the minimum charge and m, n are relatively prime. We have assumed that the brane wraps the T^2 n times. We have also assumed that the T^2 is orthogonal with the two radii being $R_{4,5}$.

It is obvious from (9.16.2), that an internal magnetic field is not a continuous modulus of the compactification. It is inversely proportional to the volume and is characterized by a rational number m/n.

We now consider an open string with one (or both) end points on the D_p-brane under consideration. One or both end points will in general carry electric charges $q_{L,R}$ under the magnetic field. The charge that couples to the magnetic field is $q = q_L + q_R$.

Before quantizing this open string exactly, we would like to look at the modifications to the massless spectrum due to the magnetic field. The first obvious modification affects the momenta on T^2. They no longer commute, rather their commutator is proportional to the gauge field as in the Landau problem,

$$[p_4, p_5] = iqH. \tag{9.16.3}$$

For various fields on T^2, the modification to the mass formula has the form

$$\delta M^2 = \left(N + \frac{1}{2}\right)|2qH| + 2qH\Sigma_{45}, \quad N = 0, 1, 2, \ldots, \tag{9.16.4}$$

where N labels the Landau levels and Σ_{45} is the projection of the angular operator on the 45 plane. For fermions,[15] $\Sigma_{45} = \frac{i}{4}[\Gamma^4, \Gamma^5]$. The lowest level is degenerate as we will show below.

Consider a spin-1/2 state. The $\Sigma_{45} = 1/2$ component, has, according to (9.16.4) a lowest mass of $\delta M^2 = 2|qH|$ $(N = 0)$ while the $\Sigma_{45} = -1/2$ component is massless

[15] For example, for spin 1/2, $\delta M^2 = (\Gamma^4 p_4 + \Gamma^5 p_5)^2$.

at the lowest Landau level. Therefore, we have massless chiral fermions ($\Sigma_{45} = -1/2$, $N = 0$) and at the first massive level an equal number of massive Dirac fermions ($\Sigma_{45} = -1/2,\ N = 1 \oplus \Sigma_{45} = 1/2,\ N = 0$). The generation of chirality can be understood from the index theorem, since the Dirac index is proportional to the integral of the magnetic field on the two-torus,

$$\text{Index}(\partial) = \frac{q}{2\pi} \int dx^4 dx^5 F_{45}. \tag{9.16.5}$$

Consider now an internal massless vector. The state with helicity on T^2, $\Sigma_{45} = 1$ is massive, while the one with helicity $\Sigma_{45} = -1$ has a mass at the lowest Landau level, $\Delta M^2 = -|qH|$ and is tachyonic. This is the well-known Nielsen-Olesen instability of field theory due to constant chromomagnetic fields.

The presence of the magnetic field breaks supersymmetry. This is obvious from the fact that the masses depend nontrivially on the spin component. This breaking is spontaneous since $\text{Str}[\delta M^2] = 0$.

Consider now independent magnetic fields $H_I,\ I = 1, 2, 3$, on each of the three T^2's. Then, scalars are all massive with lowest masses

$$\delta M_0^2 = \sum_{I=1}^{3} |q_I H_I|. \tag{9.16.6}$$

Fermions have a single massless chiral mode, with $\Sigma_{45} = \Sigma_{67} = \Sigma_{89} = -1/2$. All others are massive with minimum masses

$$\delta M_{1/2}^2 = 2|q_I H_I|, \quad 2(|q_I H_I| + |q_J H_J|), \quad 2\sum_{I=1}^{3} |q_I H_I|. \tag{9.16.7}$$

Note that chirality in four dimensions requires that all $q_I H_I$ are nonzero.

Finally, the vectors have minimal masses

$$\delta M_1^2 = |q_1 H_1| + |q_2 H_2| - |q_3 H_3|, \quad |q_1 H_1| - |q_2 H_2| + |q_3 H_3|, \quad -|q_1 H_1| + |q_2 H_2| + |q_3 H_3|. \tag{9.16.8}$$

Depending on the values of the magnetic fields, the masses in (9.16.8) maybe positive or tachyonic. In the second case, that may be used to trigger spontaneous symmetry breaking in the open sector. If one of the masses in (9.16.8) vanishes, some supersymmetry remains unbroken.

After this field-theoretic description of the effect of the internal magnetic fields on the massless sector, we now turn to a stringy description. We will describe the string quantization of the coordinates of the 4-5 plane, the others being similar. We start with the action for the X^4 and X^5 and the partner fermions

$$\begin{aligned} S = {} & \frac{1}{4\pi \ell_s^2} \int d\tau \int_0^\pi d\sigma \left[\partial_\alpha X^I \partial^\alpha X^I - \frac{i}{2}\psi^I(\partial_\tau + \partial_\sigma)\psi^I - \frac{i}{2}\bar{\psi}^I(\partial_\tau - \partial_\sigma)\bar{\psi}^I \right] \\ & + q_L H_L \int d\tau \left[X^4 \partial_\tau X^5 - \frac{i}{4}(\psi^4\psi^5 + \bar{\psi}^4\bar{\psi}^5) \right]\Big|_{\sigma=0} \\ & + q_R H_R \int d\tau \left[X^4 \partial_\tau X^5 - \frac{i}{4}(\psi^4\psi^5 + \bar{\psi}^4\bar{\psi}^5) \right]\Big|_{\sigma=\pi}. \end{aligned} \tag{9.16.9}$$

The boundary terms incorporate the presence of the magnetic field. We allowed for different magnetic fields at the two endpoints, since strings can start and end at different magnetized branes. We vary the action, being careful to keep the boundary terms in the σ direction. After integrations by parts, we obtain the usual bulk equations

$$\Box X^I = 0, \quad (\partial_\tau + \partial_\sigma)\psi^I = (\partial_\tau - \partial_\sigma)\bar{\psi}^I = 0, \tag{9.16.10}$$

together with the boundary conditions

$$\partial_\sigma X^4 - \beta_L\, \partial_\tau X^5\big|_{\sigma=0} = 0, \quad \partial_\sigma X^5 + \beta_L\, \partial_\tau X^4\big|_{\sigma=0} = 0, \tag{9.16.11}$$

$$\psi^4 - \bar{\psi}^4 + \beta_L(\psi^5 + \bar{\psi}^5)\big|_{\sigma=0} = 0, \quad \psi^5 - \bar{\psi}^5 - \beta_L(\psi^4 + \bar{\psi}^4)\big|_{\sigma=0} = 0, \tag{9.16.12}$$

$$\partial_\sigma X^4 + \beta_R\, \partial_\tau X^5\big|_{\sigma=\pi} = 0, \quad \partial_\sigma X^5 - \beta_R\, \partial_\tau X^4\big|_{\sigma=\pi} = 0, \tag{9.16.13}$$

$$\psi^4 - \beta_R\psi^5 + (-1)^a(\bar{\psi}^4 + \beta_R\bar{\psi}^5)\big|_{\sigma=\pi} = 0, \quad \psi^5 - \beta_R\psi^4 + (-1)^a(\bar{\psi}^5 - \beta_R\bar{\psi}^4)\big|_{\sigma=\pi} = 0, \tag{9.16.14}$$

where $a = 0$ for the NS sector and $a = 1$ for the R sector, in accordance with section 4.16.2 on page 88. We also defined

$$\beta_{L,R} \equiv 2\pi q_{L,R}\, H_{L,R}\, \ell_s^2. \tag{9.16.15}$$

Note that magnetic fields interpolate between Neumann and Dirichlet boundary conditions. For example, in the limit $\beta_L \to 0$, the $\sigma = 0$ end point has Neumann boundary conditions. In the opposite limit $\beta_L \to \infty$, the boundary conditions can be satisfied only when $\partial_\tau X^{4,5} = 0$, i.e., for Dirichlet boundary conditions.

Defining the complex coordinates

$$X_\pm = (X^4 \pm iX^5)/\sqrt{2}, \quad \psi_\pm = (\psi^4 \pm i\psi^5)/\sqrt{2}, \quad \bar{\psi}_\pm = (\bar{\psi}^4 \pm i\bar{\psi}^5)/\sqrt{2}, \tag{9.16.16}$$

we may rewrite the boundary conditions as

$$\partial_\sigma X_\pm \pm i\beta_L\, \partial_\tau X_\pm\big|_{\sigma=0} = 0, \quad \partial_\sigma X_\pm \mp i\beta_R\, \partial_\tau X_\pm\big|_{\sigma=\pi} = 0, \tag{9.16.17}$$

$$\left(\psi - \frac{1 + i\beta_L}{1 - i\beta_L}\bar{\psi}\right)\Bigg|_{\sigma=0} = 0, \quad \left(\psi + (-1)^a\frac{1 - i\beta_R}{1 - i\beta_R}\bar{\psi}\right)\Bigg|_{\sigma=\pi} = 0. \tag{9.16.18}$$

The boundary conditions are linear and are easily solved,

$$X_\pm = x^\pm + i\sqrt{2}\ell_s \sum_{n\in Z} \frac{a^\pm_{n\mp\epsilon}}{n \mp \epsilon} e^{-i(n\mp\epsilon)\tau}\, \cos\left[(n \mp \epsilon)\sigma \pm \theta_L\right], \tag{9.16.19}$$

$$\psi_\pm = \sum_{\mathbb{Z}+\frac{1-a}{2}} b^\pm_{n\mp\epsilon}\, e^{i(n\mp\epsilon)(\tau-\sigma)\pm i\theta_L}, \quad \bar{\psi}_\pm = \sum_{\mathbb{Z}+\frac{1-a}{2}} b^\pm_{n\mp\epsilon}\, e^{i(n\mp\epsilon)(\tau+\sigma)\pm i\theta_L}. \tag{9.16.20}$$

We have set

$$\theta_{L,R} = \arctan(\beta_{L,R}), \quad \epsilon = \frac{1}{\pi}\left[\theta_L + \theta_R\right]. \tag{9.16.21}$$

The Hermiticity relations are

$$(a^-_{n+\epsilon})^\dagger = a^+_{-n-\epsilon}, \quad (b^-_{n+\epsilon})^\dagger = b^+_{-n-\epsilon}. \tag{9.16.22}$$

Note that the oscillator expansions are identical to those of the twisted sector of an orbifold, with twist angle $2\pi\epsilon$.

As in the orbifold case, $X_\pm$ carry no momentum and the oscillator frequencies are shifted from integer ones. Unlike the orbifold case, the phase here is continuous, and there is no summation over orbifold sectors.

The oscillator expansions must be supplemented by canonical commutations relations that as usual read

$$[a^+_{n-\epsilon}, a^-_{m+\epsilon}] = (n-\epsilon)\delta_{m+n}, \quad \{b^+_{n-\epsilon}, b^-_{m+\epsilon}\} = \delta_{m+n}. \tag{9.16.23}$$

The commutator of the zero modes, however, is a bit unusual. We will evaluate the equal-time commutator of the coordinates using the commutation relations in (9.16.23),

$$[X_+(\tau,\sigma), X_-(\tau,\sigma')] = [x^+, x^-] + 2\ell_s^2\, J(\sigma,\sigma'), \tag{9.16.24}$$

$$J(\sigma,\sigma') = \sum_{n\in\mathbb{Z}} \frac{\cos[(n-\epsilon)\sigma + \theta_L]\cos[(n-\epsilon)\sigma' + \theta_L]}{n-\epsilon}. \tag{9.16.25}$$

This function has the property of being piecewise constant. This can be ascertained by evaluating $\partial_\sigma J$ and showing that apart from jumps at $\sigma = \sigma' = 0, \pi$, it is constant. Using

$$\sum_{n\in\mathbb{Z}} \frac{1}{n-\epsilon} = -\pi\cot(\pi\epsilon), \quad \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{n-\epsilon} = -\frac{\pi}{\sin(\pi\epsilon)}, \tag{9.16.26}$$

we may evaluate

$$J(0,0) = \frac{\pi(\beta_L\beta_R - 1)}{(1+\beta_L^2)(\beta_L+\beta_R)}, \quad J(\pi,\pi) = \frac{\pi(\beta_L\beta_R - 1)}{(1+\beta_R^2)(\beta_L+\beta_R)},$$
$$J(0,\pi) = -\frac{\pi}{(\beta_L+\beta_R)}. \tag{9.16.27}$$

We must now impose that the commutator (9.16.24) vanishes, except at the end points. This fixes uniquely the zero-mode commutator to

$$[x^+, x^-] = \frac{2\pi\ell_s^2}{\beta_L+\beta_R}. \tag{9.16.28}$$

Moreover, at the end points the commutator (9.16.24) does not vanish. Rather,

$$[X_+(\tau,0), X_-(\tau,0)] = \frac{2\pi\ell_s^2\,\beta_L}{1+\beta_L^2}, \quad [X_+(\tau,\pi), X_-(\tau,\pi)] = \frac{2\pi\ell_s^2\,\beta_R}{1+\beta_R^2}. \tag{9.16.29}$$

We therefore observe that the end points of the string in the magnetic field, do not commute anymore. The associated effective theory can in fact be rewritten in terms of a noncommutative field theory, but we will not explore this further.

We may now discuss the spectrum. The vacuum is defined in analogy with orbifolds. We take without loss of generality $0 < \epsilon < \frac{1}{2}$. In the NS sector

$$a^+_{n-\epsilon}|0\rangle = 0, \quad n > 0, \quad a^-_{n+\epsilon}|0\rangle = 0, \quad n \geq 0, \tag{9.16.30}$$

$$b^+_{r-\epsilon}|0\rangle = 0, \quad r > 0, \quad b^-_{r+\epsilon}|0\rangle = 0, \quad r > 0. \tag{9.16.31}$$

In the R sector we have instead

$$b^+_{n-\epsilon}|0\rangle = 0, \quad n > 0, \quad b^-_{n+\epsilon}|0\rangle = 0, \quad n \geq 0. \tag{9.16.32}$$

The L_0 eigenvalue on the vacuum in the NS sector (associated with the two dimensions in question) is $\epsilon(1-\epsilon)/2$ from the bosonic part and $\epsilon^2/2$ from the fermionic part. Therefore,

$$L_0|0\rangle_{\mathrm{NS},\epsilon} = \frac{\epsilon}{2}\,|0\rangle_{\mathrm{NS},\epsilon}. \tag{9.16.33}$$

On the other hand, the magnetic contributions cancel in the R sector ground state.

We observe that $a^+_{-n-\epsilon}$ and $b^+_{-r-\epsilon}$ raise the helicity on the plane by 1, and shift the L_0 eigenvalue by ϵ, while $a^-_{-n+\epsilon}, b^-_{-r-\epsilon}$ lower the helicity eigenvalue by 1, and shift the L_0 eigenvalue by $-\epsilon$. The operator $a^+_{-\epsilon}$ in particular creates the Landau states upon multiple action on the ground state. We may therefore write the generic state as

$$|\psi\rangle = \prod_{i=0}(a^+_{-i-\epsilon})^{N_i}\prod_{j=1}(a^-_{-j+\epsilon})^{\bar{N}_i}\prod_{i=0}(b^+_{-i-1/2-\epsilon})^{n_i}\prod_{j=0}(b^-_{-j-1/2+\epsilon})^{\bar{n}_i}|0\rangle_\epsilon, \tag{9.16.34}$$

with L_0 eigenvalue

$$L_0|\psi\rangle = L_0|\psi\rangle_{\epsilon=0} + \left(N_0 + \frac{1}{2}\right)\epsilon + \Sigma_{45}\epsilon, \tag{9.16.35}$$

and

$$\Sigma_{45} = \sum_{i=1}^{\infty}(N_i - \bar{N}_i + n_i - \bar{n}_i), \tag{9.16.36}$$

as expected.

An important question is the multiplicity of the ground state. This is determined by the commutation relations of the coordinate zero modes in (9.16.28). We define normalized angular coordinates as $X^i = (2\pi\ell_s R_i)\theta^i$, so that when going once around the circles we have $\theta^i \to \theta^i + 1$. We now translate (9.16.28) to

$$[\theta^4, \theta^5] = -\frac{i}{2\pi}\,\frac{n_L n_R}{m_L n_R + m_R n_L}, \tag{9.16.37}$$

using (9.16.2). Due to the periodicity on T^2, the appropriate operators are $e^{2\pi i\theta^{4,5}}$ instead of $\theta^{4,5}$. We will have to treat one of them as momentum and the other as a coordinate. Then we can generate states, by acting with one set of operators on the vacuum. Since both $\theta^{4,5}$ commute with all other oscillators, they will provide with an overall degeneracy of the vacuum. Using (9.16.37) we may compute

$$e^{2\pi I_{LR}i\theta^4}\,e^{2\pi i\theta^5}\,e^{-2\pi iI_{LR}\theta^4} = e^{2\pi i\theta^5}, \quad e^{2\pi I_{LR}i\theta^5}\,e^{2\pi i\theta^4}\,e^{-2\pi iI_{LR}\theta^5} = e^{2\pi i\theta^4}, \tag{9.16.38}$$

with

$$I_{\mathrm{LR}} = m_L n_R + m_R n_L. \tag{9.16.39}$$

It is therefore obvious that the operators $e^{2\pi ik\theta^4}$, $k = 0, 1, \ldots, I_{\mathrm{LR}} - 1$, generate independent states characterized by distinct eigenvalues of the “momentum” operator $e^{2\pi i\theta^5}$. We conclude that we must have I_{LR} ground states.

Putting everything together we obtain the shift of the low-lying string energy levels

$$\delta\mathcal{M}^2_{\text{string}} \sim (2n+1)|\epsilon| + 2\epsilon\,\Sigma_{45}. \tag{9.16.40}$$

For weak magnetic fields this agrees with the field theory expectation (9.16.4).

It is useful to compute the string partition sum in the presence of a magnetic field on a plane. For the two bosonic coordinates, taking into account the frequency shifts,

$$\mathrm{Tr}[e^{-2\pi t(L_0-c/24)}] = \frac{q^{\epsilon(1-\epsilon)/2-1/12}}{(1-q^{\epsilon})\prod_{n=1}^{\infty}(1-q^{n+\epsilon})(1-q^{n-\epsilon})} = i\frac{q^{-\epsilon^2/2+1/24}\prod_{n=1}^{\infty}(1-q^n)}{\vartheta_1\left(it\epsilon|it\right)}, \tag{9.16.41}$$

with $q = e^{-2\pi t}$. The analogous trace on the fermions in the NS sector is

$$q^{\epsilon^2/2-1/24}\prod_{n=0}^{\infty}(1+q^{n+1/2+\epsilon})(1+q^{n+1/2-\epsilon}) = q^{-\epsilon^2/2+1/24}\frac{\vartheta_3\left(it\epsilon|it\right)}{\prod_{n=1}^{\infty}(1-q^n)}. \tag{9.16.42}$$

Putting together fermions and bosons we finally have

$$\mathrm{Tr}[e^{-2\pi t(L_0-c/24)}]_{\mathrm{NS}} = iI_{LR}\,\frac{\vartheta_3\left(it\epsilon|it\right)}{\vartheta_1\left(it\epsilon|it\right)}, \tag{9.16.43}$$

where we also included the degeneracy of the ground state. It is a straightforward exercise to derive the other relevant magnetized partition functions

$$\mathrm{Tr}[(-1)^F\,e^{-2\pi t(L_0-c/24)}]_{\mathrm{NS}} = iI_{\mathrm{LR}}\,\frac{\vartheta_4\left(it\epsilon|it\right)}{\vartheta_1\left(it\epsilon|it\right)}, \tag{9.16.44}$$

$$\mathrm{Tr}[e^{-2\pi t(L_0-c/24)}]_{\mathrm{R}} = iI_{\mathrm{LR}}\,\frac{\vartheta_2\left(it\epsilon|it\right)}{\vartheta_1\left(it\epsilon|it\right)}, \quad \mathrm{Tr}[(-1)^F\,e^{-2\pi t(L_0-c/24)}]_{\mathrm{R}} = I_{\mathrm{LR}}. \tag{9.16.45}$$

In particular, the last trace is nothing else than the Witten index which counts the number of ground states of the system.

9.16.2 Intersecting branes

We may now proceed to apply a T-duality transformation to one of the two T^2 coordinates of the previous section. For concreteness we will T-dualize along the X^5 coordinate. Apart from $R_5 \to 1/R_5$, the boundary conditions (9.16.11)–(9.16.14) will change, via $\partial_\sigma X^5 \leftrightarrow \partial_\tau X^5$. The new boundary conditions on the coordinates are

$$\partial_\sigma(X^4 - \beta_L\,X^5)\big|_{\sigma=0} = 0, \quad \partial_\tau(X^5 + \beta_L\,X^4)\big|_{\sigma=0} = 0, \tag{9.16.46}$$

$$\partial_\sigma(X^4 + \beta_R\,X^5)\big|_{\sigma=\pi} = 0, \quad \partial_\tau(X^5 - \beta_R\,X^4)\big|_{\sigma=\pi} = 0. \tag{9.16.47}$$

We now define rotated coordinates

$$\begin{pmatrix} X^4_{L,R} \\ X^5_{L,R} \end{pmatrix} = \begin{pmatrix} \cos\theta_{L,R} & \mp\sin(\theta_{L,R}) \\ \pm\sin(\theta_{L,R}) & \cos(\theta_{L,R}) \end{pmatrix}\begin{pmatrix} X^4 \\ X^5 \end{pmatrix}, \tag{9.16.48}$$

where the angles $\theta_{L,R}$ were defined in (9.16.21). We may now reinterpret the boundary conditions (9.16.46), (9.16.47). Let us call the branes on which the L/R end points of the open string end, the L/R-branes. Both have now only one dimension wrapping the two-torus. This is expected from the standard action of T-duality on D-branes.

The L-brane has a Neumann boundary condition along X^4_L and a Dirichlet boundary condition along X^5_L. It is therefore rotated at an angle $-\theta_L$ with respect to the X^4 axis. On the other hand the R-brane is rotated at an angle θ_R with respect to the X^4 axis.

We are therefore describing a string stretching between two intersecting branes at an angle $\theta_L + \theta_R = \pi\epsilon$. The branes intersect at a point[16] on the X^4-X^5 plane, but the intersection may also stretch in other dimensions.

The magnetic flux quantization condition (9.16.2) becomes in the T-dual version

$$(2\pi\ell_s^2)q_{L,R}H_{L,R} = \frac{R_5}{R_4}\frac{m_{L,R}}{n_{L,R}} \to \tan\theta_{L,R} = \frac{R_5}{R_4}\frac{m_{L,R}}{n_{L,R}}. \tag{9.16.49}$$

The interpretation of (9.16.49) is that the L brane is winding around the two-torus by wrapping m_L times the x^5 cycle and n_L times the x^4 cycle. The R brane is wrapping -m_R times the x^5 cycle and n_R times the x^4 cycle. $m_{L,R}$ and $n_{L,R}$ are therefore wrapping numbers of the branes on T^2. Moreover,

$$I_{LR} = m_L n_R + m_R n_L \tag{9.16.50}$$

is the (oriented) intersection number of the two branes on the T^2. It is satisfying that the number of ground states of the generalized Landau problem of the last section, namely I_{LR}, is the same as the number of geometrical brane intersections. The reason is that in the T-dual picture of intersecting branes we expect precisely this number of ground states. A string stretched between two intersecting branes will classically minimize its length (and energy) by sitting at an intersection. Upon quantization, the number of ground states of the string coordinates is equal to the number of intersections.

9.16.3 Intersecting D$_6$-branes

The simplest configuration of this setup involves an original system of D$_9$-branes on T^6. This is the type-I string on T^6. For simplicity we may take a factorizable torus $T^6 = \prod_{i=1}^3 (T^2)_i$. We may turn on different magnetic fields H^i_I on different D$_9$-branes, labeled by I on the three T^2 labeled by $i = 1, 2, 3$. If we T-dualize one coordinate from each T^2 we end up with intersecting D$_6$-branes on $(T^2)^3$. In the T-dual picture, each brane is now characterized by three angles $\theta^i_I = \arctan(H^i_I)$. They are rotated by θ^i_I in each T^2 with respect to the standard axes. The angles are related to the two winding numbers per torus (n^i_I, m^i_I), and to the complex structure U^i_2 as $\tan\theta^i_I = \frac{m^i_I}{n^i_I U^i_2}$.

In the following we shall swing back and forth between the magnetized and intersecting picture. The reason is that some features are easier to discern in one picture and others in the T-dual one.

After T-duality on each of the three tori, the Ω projection transforms to an $\Omega\mathcal{I}^3$ projection according to (8.7.4) on page 201, where the inversions act on three of the six torus coordinates. They can be thought of as an antiholomorphic involution on the three complex torus

[16] Because of the torus periodicity, there can be several intersection points.

coordinates $z_i \to \bar{z}_i$. Thus, the orientifold of the IIA string by $\Omega\mathcal{I}^3$ has an open sector that contains intersecting D_6-branes with winding numbers (n_I^i, m_I^i). In particular, for each brane a, its image under $\Omega\mathcal{I}^3$ is another brane a' with winding numbers $(n_I^i, -m_I^i)$. As mentioned earlier and advocated in exercise 9.49 on page 291, such generic configurations break supersymmetry completely.

Let us denote the number of the Ith brane by N_I. In exercise 9.58 you are asked to derive the R tadpole conditions and show that they are given by

$$\sum_I N_I n_I^1 n_I^2 n_I^3 = 16, \quad \sum_I N_I n_I^i m_I^j m_I^k = 0, \quad i \neq j \neq k \neq i. \tag{9.16.51}$$

A compact way of writing the tadpole conditions above is

$$\sum_I N_I \; \Pi_I = \Pi_{O_6}, \tag{9.16.52}$$

where Π_I is the homology cycle of the Ith brane and Π_{O_6} are the homology cycles of the Orientifold planes. They are the T-duals of the well-known ten-dimensional O_9 plane. Another way to rephrase the tadpole conditions in (9.16.51) in the magnetized picture is as follows: the first condition is the usual cancellation of the D_9 charge. The other three are the cancelations of the induced D_5 charges (see exercise 9.57) transverse to the three possible T^2s.

The orientifold projection $\Omega\mathcal{I}^3$ maps a generic brane a to its image a' which is spatially distinct. Therefore, for a generic brane, the group is expected to be $U(N_I)$. There are however two special cases. The first is a brane aligned with the axes (no original magnetic fields). This is equivalent to an unmagnetized D_9-brane and the orientifold projection is expected to give an $SO(N_I)$ group. The other extreme is a brane, where two of the original magnetic fields are infinite. As discussed in section 9.16.1, an infinite magnetic field imposes a Dirichlet boundary condition. Therefore, such branes correspond to D_5-branes. And for D_5-branes we have argued already that they must have an opposite projection compared to that of the D_9-branes (see (9.16.4)). Therefore branes equivalent to D_5-branes will have an $Sp(N_I)$ gauge group.

We will now consider a generic configuration of intersecting branes giving rise to unitary groups only, and describe the massless spectrum. We will assume for simplicity that all branes intersect pairwise non-trivially. Some general properties of the low-lying spectrum were already detailed in an earlier section:

- Strings starting and ending on the same brane do not feel the magnetic fields since they are neutral ($q_L H_L = -q_R H_R$). Here we have the full $\mathcal{N} = 4_4$ supersymmetry.
- Strings stretching between intersecting branes will have generically massive scalars and vectors and some of the fermions as explained in section 9.16.1. However, there will be a number of massless chiral four-dimensional fermions. This number is equal to the oriented intersection number of the branes.

Let us first consider strings that start at a set of branes I and their image I′.

- $\Omega\mathcal{I}^3$ maps II strings to $I'I'$ strings. Therefore we can take one set as the independent one. The massless states generate the $\mathcal{N} = 4_4$ $U(N_I)$ Yang-Mills theory.

- II' strings are mapped by $\Omega\mathcal{I}^3$ to themselves. Therefore here we obtain symmetric and antisymmetric representations. The intersection number $I_{II'} = 8\prod_{i=1}^{3} m_I^i n_I^i$ is generically nonzero so we obtain only massless chiral fermions here. You are invited in exercise 9.59 on page 292 to show that we obtain $8m_I^1 m_I^2 m_I^3$ fermions in the $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ representation of $\mathrm{U}(\mathrm{N}_I)$ as well as $4m_I^1 m_I^2 m_I^3(n_I^1 n_I^2 n_I^3 - 1)$ fermions in the $\begin{smallmatrix}\square\\ \square\end{smallmatrix}$ and $\square\square$ representations.

Consider now strings stretching between different stacks:

- The sector IJ is mapped by $\Omega\mathcal{I}^3$ to I′J′. We obtain fermions in the bifundamental $(\mathrm{N}_I,\overline{\mathrm{N}}_J)$ with multiplicity

$$I_{IJ} = \prod_{i=1}^{3}(m_I^i n_J^i - m_J^i n_I^i). \tag{9.16.53}$$

- The sector IJ′ is mapped by $\Omega\mathcal{I}^3$ to I′J. We obtain fermions in the bifundamental $(\mathrm{N}_I,\mathrm{N}_J)$ with multiplicity

$$I_{IJ} = -\prod_{i=1}^{3}(m_I^i n_J^i + m_J^i n_I^i), \tag{9.16.54}$$

where the minus sign as usual implies opposite chirality.

The spectra thus obtained can be engineered to reproduce the chiral SM spectrum. You are invited to explore this in exercise 9.60.

9.17 Where is the Standard Model?

Different classes of string vacua have distinct ways of realizing the gauge interactions that could be responsible for the SM forces. Ten-dimensional gravity is always an ingredient, coming from the closed string sector. The simplest way to convert it to four dimensional gravity is via compactification and this is what we will assume here. In section 13.13 we will describe another way of turning higher-dimensional gravity to four-dimensional, but the implementation of this idea in string theory is still in its infancy.

From (8.4.8) on page 195, upon compactification to four dimensions on a six-dimensional manifold of volume $(2\pi\ell_s)^6\, V_6$, the four-dimensional Planck scale is given at tree level by

$$M_P^2 = \frac{V_6}{2\pi g_s^2} M_s^2 = \frac{M_s^2}{g_u^2}, \tag{9.17.1}$$

where g_s is the string coupling constant and the volume V_6 is by definition dimensionless. We have implicitly defined also g_u, the effective four-dimensional string coupling constant.

9.17.1 *The heterotic string*

The ten-dimensional theory, apart from the gravitational supermultiplet, contains also a (super-)Yang-Mills (SYM) sector with gauge group $E_8 \times E_8$ or SO(32).

Here, the four-dimensional gauge fields descend directly from ten dimensions. The gauge field states are

$$|A^a_\mu\rangle = b^\mu_{-1/2}\bar{J}^a_{-1}|p\rangle, \tag{9.17.2}$$

where b^μ_r are the modes of the left-moving world-sheet fermions and their vertex operators are given in (10.1.2) on page 296. The four-dimensional action and gauge coupling constants are given by

$$S_4 = -\frac{1}{4g_I^2}\mathrm{Tr}[F_I F_I], \quad \frac{1}{g_I^2} = \frac{V_6}{4\pi g_s^2}k_I, \tag{9.17.3}$$

where the trace is in the fundamental representation and k_I is the level[17] of the associated affine algebra (see section 4.11 on page 69). You are invited to derive this relation in exercise 9.64 on page 292. Although vectors can also come from the metric, they cannot provide chirality [220]. Therefore, the essential part of the SM must come from the vectors arising from the non-supersymmetric side.

Tree-level relations like (9.17.1) or (9.17.3) are corrected in perturbation theory and the couplings run with energy. We will see this phenomenon in more detail in the next section. The tree-level couplings correspond to their values at the string (unification) scale up to some threshold corrections coming from integrating out the stringy modes. In a stable and reliable perturbative expansion, such corrections are small. There may be also corrections from KK modes. These can become important only if the KK masses are very light compared to M_s. This is typically not the case in the heterotic string.

Therefore, the order of magnitude estimates of couplings at the string scale are expected to be reliable. In order to comply with experimental data, $g_{YM} \sim \mathcal{O}(1)$ and (9.17.1), (9.17.3) imply that

$$M_P^2 = \frac{M_s^2}{g_I^2}\frac{2}{k_I}. \tag{9.17.4}$$

Typically $k_I = 1$ for almost all semirealistic heterotic vacua. Also the values of the observable coupling constants are in the 1–10^{-2} range. We deduce from (9.17.4) that the string scale and the Planck scale have the same order of magnitude. This is an interesting prediction, valid for all realistic perturbative heterotic string vacua.

The issue of supersymmetry breaking is of crucial importance in order to eventually make contact with the low-energy dynamics of the Standard Model.

There are two alternatives here, gaugino condensation (dynamical) and Scherk-Schwarz (geometrical) supersymmetry breaking described in section 9.5 on page 235.

The first possibility can be implemented in the heterotic string. However, it involves nonperturbative dynamics and consequently is not well controlled in perturbation theory. We do not know how to describe this dynamics at the string level.

If supersymmetry is broken à la Scherk-Schwarz, then the supersymmetry breaking scale is related to the size R of an internal compact dimension as

$$M_{\mathrm{SUSY}} \sim \frac{1}{R}. \tag{9.17.5}$$

[17] For abelian groups, one must first normalize the charges in order to determine the level.

A successful resolution of the hierarchy problem requires that $M_{\text{SUSY}} \sim$ a few TeV so that $M_{\text{SUSY}}/M_P \ll 1$. This implies, $R \gg M_s^{-1}$ and from (9.17.3) $g_s \gg 1$ in order to keep $g_I \sim \mathcal{O}(1)$. Thus, we are pushed in the non-perturbative regime. In chapter 11 we will find out how to handle such strong couplings regions, and therefore open new model-building possibilities.

9.17.2 Type-II string theory

The perturbative type-II string is very restrictive when it comes to nonabelian gauge groups combined with chirality.

Gauge fields may come both from the NS-NS and R-R sectors. R-R sector gauge fields generate abelian gauge groups in perturbation theory. The reason is that, as we have argued in section 7.2.1 on page 159, they cannot have minimal couplings to any perturbative state. Therefore, no perturbative string state is charged under them. To put it mildly, they are phenomenologically worthless.[18]

We will not prove in detail here why it is impossible to embed the SM spectrum in the perturbative type-II string. We will give instead the basic hints why this is so. The curious and enterprizing reader is guided to exercise 9.63 on page 292.

- To construct a compactification to four flat dimensions, the internal SCFT must have $(c_L, c_R) = (9, 9)$ and $\mathcal{N} = (1, 1)$ supersymmetry on the world-sheet.

- Gauge groups in space-time are in one-to-one correspondence with right-moving or left-moving (super)current algebras of $\text{SCFT}_{\text{internal}}$.

- If there is a nonabelian left-moving current algebra, then the R_L-NS_R sector contains only massive fermions. An immediate corollary is that all R_L-R_R bosons are also massive. Similarly, with $L \leftrightarrow R$.

- If there is an abelian left-moving current algebra, then R_L-NS_R fermions are neutral with respect to it. Worse, in such a case the R_L-NS_R fermions are nonchiral.

- The upshot of the previous statements is that all the SM gauge symmetry must come from one side of the type-II string, say the left. Moreover, the vectors will come from the NS-NS sector. All the massless fermions of the SM model will then arise from the NS_L-R_R sector.

  Together with the constraint on the central charge, this substantially limits the possible gauge groups that can appear. They are $SU(2)^6$, $SU(4)\times SU(2)$, $SO(5)\times SU(3)$, $SO(5)\times SU(2)\times SU(2)$, $SU(3)\times SU(3)$, G_2, and their subgroups. So far the gauge group of the standard model is possible.

- If we further require that massless fermionic states transform in the representations of the SM, then it turns out that it is not possible to fit them with the allowed internal central charge (9,9).

[18] This statement ceases to be true, beyond perturbation theory.

Therefore, to embed the standard model in the type-II string we must go beyond perturbation theory. This turns out to be possible [221]. However, it is difficult in this case to do detailed calculations.

9.17.3 The type-I string

In the type-I vacua, gauge symmetries can arise from D_p-branes that stretch along the four Minkowski directions and wrap their extra $p-3$ dimensions in a submanifold of the compactification manifold. Let us denote by $V_{||}$ the volume of such a submanifold in string units.

The relation of the four-dimensional Planck scale to the string scale is the same as in (9.17.1) since gravity originates in the closed string sector. However, the four-dimensional YM coupling of the D-brane gauge fields now become[19]

$$\frac{1}{g_{YM}^2} = \frac{V_{||}}{2\sqrt{2}\pi g_s}, \quad \frac{M_P^2}{M_s^2} = \frac{V_6}{\sqrt{2}g_s\ V_{||}}\frac{1}{g_{YM}^2}, \tag{9.17.6}$$

where M_P is the four-dimensional Planck scale. M_s can be much smaller than M_P while keeping the theory perturbative, $g_s < 1$, by having the volume of the space transverse to the D_p-branes $\frac{V_6}{V_{||}} \gg 1$. Therefore, in this context, the string scale M_s can be anywhere between the four-dimensional Planck scale and a few TeV without obvious experimental contradictions. The possibility of perturbative string model building with a very low string scale is intriguing and interesting for several reasons:

- If M_s is a few TeV, string effects will be visible at TeV-scale experiments in the near future. If nature turns out to work that way, the experimental signals will be forthcoming. In the other extreme case $M_s \sim M_P$, there seems to be little chance to see telltale signals of the string in TeV-scale experiments.
- Supersymmetry can be broken directly at the string scale without the need for fancy supersymmetry-breaking mechanisms (for example by direct orbifolding). Past the string scale, there is no hierarchy problem since there is no field-theoretic running of couplings.

The possibility of having the string scale and the supersymmetry-breaking scale in the TeV range renders the gauge hierarchy problem nonexistent. However, the realization of such vacua is difficult, since, as we have seen earlier, they require the presence of large internal volumes. Once supersymmetry is broken, the volume moduli will acquire potentials. The novel hierarchy problem is that such minima for the volume will be required to give $V_6 \ggg 1$. Although there are ideas in this direction, no fully successful vacuum is known yet.

[19] The origin of the factor of $\sqrt{2}$ can be found in exercise 11.23 on page 366.

9.18 Unification

The first attempt to unify the fundamental interactions beyond the SM employed the embedding of the SM group into a simple unified group. This provided tree-level relations between the different SM coupling constants of the form

$$\frac{1}{g_I^2} = \frac{k_I}{g_U^2}, \quad I = \mathrm{SU}(3), \mathrm{SU}(2), \mathrm{U}(1)_Y, \tag{9.18.1}$$

where g_U is the unique coupling constant of the unified gauge group. The k_I are group-theoretic rational numbers, that depend on the way the SM gauge group is embedded in the unified gauge group.[20] All couplings are evaluated at the scale M_{GUT} where the unified gauge group is expected to break to the SM gauge group.

In string theory, unification in its general sense is a fact: the theory has no free parameters, but expectation values that should be determined in a given ground state by minimizing the appropriate potential. This picture, often fails in string theory, when some scalars, the moduli, have no potential. However this is a characteristic of supersymmetric vacua. In nonsupersymmetric vacua, all moduli are expected to have a potential, and baring accidents, they determine, among other things, the gauge coupling constants.

Remarkably, the (measured) gauge couplings constants of the SM, when extrapolated to high energy using the (supersymmetric) renormalization group, they seem to satisfy the relation (9.18.1) at an energy $M_{\mathrm{GUT}} \sim 10^{16.1\pm0.3}$ GeV, with $1/\alpha_{\mathrm{GUT}} \equiv (4\pi)/g_U^2 \simeq 25$. This is pictured in figure 9.1. The matching is not as good if the nonsupersymmetric running is used.

Of course, some assumptions must be made, in order to make such an extrapolation. The first, is that the only particles that contribute are those of the minimal supersymmetric standard model.[21] The second is that no important thresholds are met, until $E \sim M_{\mathrm{GUT}}$.

What should we conclude from such an observation? Certainly, it is not a proof for the existence of a relation of the type (9.18.1). It is however, an intriguing piece of evidence that we cannot immediately discard.

In this section we would like to investigate what kind of gauge coupling relations we obtain in the two most promising sets of string theory vacua: heterotic and type I.

In the heterotic string theory, as we have seen, the four-dimensional gauge groups descend from the right-moving nonsupersymmetric sector. They are associated with a respective right-moving current algebra.

As we have shown in (9.17.3) a formula similar to (9.18.1) holds for the gauge couplings in the heterotic string at tree level with $g_U^2 = 4\pi g_s^2/V_6$. This type of coupling unification occurs naturally in the heterotic string.

Let us now consider the gauge couplings in type-I string theory. As already explained in section 9.17.3, we may consider the Ith gauge group factor coming from a D_{p+3}-brane.

[20] For the SU(5)-like embeddings, $k_{\mathrm{SU}(3)} = k_{\mathrm{SU}(2)} = 1$, $k_{\mathrm{U}}(1)_Y = 5/3$.

[21] Groups of particles whose presence does not affect the running of the ratio of couplings, could be allowed.

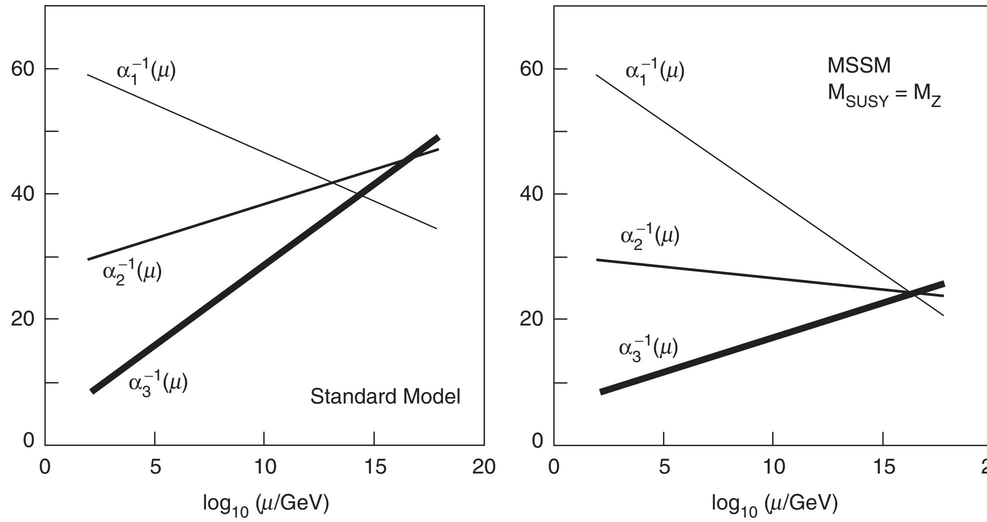

Figure 9.1 The running of gauge couplings $\alpha_i = g_i^2/(4\pi)$ in the SM and the supersymmetric standard model with supersymmetry breaking scale at M_Z. $\alpha_1 = \frac{5}{3}\alpha_Y$ where α_Y is the canonically normalized hypercharge coupling. The width of the lines is proportional to the respective experimental error.

It stretches along the four Minkowski dimensions. It also wraps $k_I \in \mathbb{Z}$ times a p-dimensional cycle of the compact six-dimensional manifold with volume $V_{||}$ in string units. The associated tree-level gauge coupling is

$$\frac{1}{g_I^2} = k_I \frac{V_{||}}{2\sqrt{2}\pi\, g_s}. \tag{9.18.2}$$

It is therefore obvious that, for gauge group factors originating from the same type of D_{p+3}-brane, we have a similar relation to (9.18.1) but the interpretation of k_I is different here. It is an (integer) wrapping number. When gauge group factors originate on different branes, then (9.18.1) ceases to hold.

The presence of other background fields, may change the relation (9.18.2) already at the tree level. For example, if twisted bulk moduli in orbifolds have nonzero expectation values, there are generic additive corrections to (9.18.2). Internal magnetic fields also alter (9.18.2). You are invited to investigate this in exercise 9.65.

We conclude that the "unification relation" (9.18.2) is not generic in type-I vacua. It will hold only if the SM gauge group originates from branes in the same stack.

An intermediate situation may arise in this case. It is known as "petite unification." It is the statement, that relation (9.18.2) is valid for a subset of the SM group factors. For example, SU(3) and $U(1)_Y$ may originate from a D_9 stack of branes while SU(2) from a D_5 stack.

We will revisit relation (9.18.2) and the associated one-loop corrections at the end of the next chapter.

Bibliography

A nice and comprehensive review of KK compactifications in supergravity and related issues can be found in [222].

Our discussion on the connection between space-time and world-sheet supersymmetry is based on [223]. In the same reference the absence of continuous global symmetries in string vacua was argued. Other general phenomenological issues in heterotic vacua are discussed in [224].

Basic references on the orbifold idea were given in the bibliography of chapter 4. Here we will provide a further guide towards the building of realistic string vacua. A large class of vacua has been constructed by utilizing free fermionic blocks in order to construct the CFT representing the internal six-dimensional compact part [225,226]. A similar construction using bosonic generalized lattices is reviewed in [73]. Using such algorithmic constructions, a partial computerized scanning gave several interesting heterotic models. The two most successful ones are [227,228]. A good review of heterotic orbifold model building in the eighties is [229]. [230] provides another review where more attention is paid to generic phenomenological properties of heterotic vacua and the structure of the relevant moduli interactions. A more extensive review for the late nineties is [231]. The PhD thesis [232] is also a detailed source of heterotic orbifold vacua.

In [233] a pedestrian description of low-energy theories relevant for string phenomenology is given. Another review that also includes higher (affine) level string model building is [234].

Coordinate-dependent compactifications were introduced in the context of field theory in [235]. They were implemented in closed string theory, to generate spontaneous supersymmetry breaking in [236,237]. The relationship between spontaneous supersymmetry breaking and freely acting orbifolds was detailed in [238]. For the open string version see [239].

Geometric compactifications of the heterotic string to four dimensions with $\mathcal{N} = 1_4$ supersymmetry are discussed in [240]. The complex geometry and CY manifold are described extensively in GSW [7] and the reviews [241,242]. A more extensive and higher level exposition is given in the book, [243]. Detailed information on the Eguchi-Hanson space and other hyper-Kähler manifolds can be found in [244]. A detailed discussion of the geometry and topology of the K3 manifold can be found in [245].

The review [68] contains a very good survey of both the geometry and the quantum geometry of CY manifolds. In particular it contains a nice description of $\mathcal{N} = (2,2)_2$ CFTs and their diverse descriptions, mirror manifolds and mirror symmetry, examples of space-time topology change and the physics of conifold transitions, that we will also describe in section 11.10. Moreover, it contains a good introduction in toric geometry.

Mirror symmetry has been interpreted as T-duality in [246]. A more complete and rigorous discussion can be found in the AMS book [247] as well as in several good reviews [248,249,68].

Orientifolds were first described in [114–119]. D-branes in orbifolds and the related quiver theories were discussed in [250]. We follow here the Hilbert space notation of [251,252] on orientifolds. A very extensive and informative review can be found in [97]. This is also a good source for references in this direction. A comprehensive description of $\mathcal{N} = 1_4$ orientifolds of standard orbifolds can be found in [253]. Our general description of D-branes at singularities follows [254].

Magnetic fields in string theory have been discussed in [167,168] where the first derivation of the DBI action was given. Magnetic fields were used to break supersymmetry and generate chirality in the context of string theory in [255]. Further discussions of magnetized compactifications/intersecting branes and SM constructions can be found in [256–260].

A review that summarizes general features of D-brane model building in terms of branes at singularities and intersecting branes is [261]. Applications of D-branes to cosmology are also discussed. Concrete model building using intersecting branes is reviewed in [262,263]. A general overview can be found in [264].

Noncommutative aspects of magnetic fields in string theory and field theory can be found in [265]. A comprehensive review with a guide to the literature can be found in [266]. A detailed discussion of the commutators of the string coordinates in a magnetic field is in [267].

A general discussion of the gauge symmetries coming from the supersymmetric side of heterotic strings as well as type-II strings and the associated constraints on the perturbative spectrum are presented in [220].

Discussions of the large extra dimensions, the decompactification problem and suggestions on how it can be avoided, can be found in [268–278]. Reviews of the string theory related developments can be found in [279,280].

Unification in field theory is reviewed in [281]. The review of [282] provides an extensive discussion of gauge coupling unification in the heterotic string as well as several other phenomenological questions. The review of [230] contains among other things, a description of nonperturbative supersymmetry breaking due to gaugino condensation.

We have not addressed here the compactifications with nontrivial fluxes and the stabilisation of moduli. A review which is a good starting point is [283] that also contains a good guide to the literature. Warped compactifications with fluxes have been discussed in [284]. De Sitter spaces in fluxed compactifications have been described in [285,286]. The generalized geometry, capable of classifying supersymmetric compactifications with fluxes can be found in [287,288].

An emerging subject, not addressed in this book, involves cosmological applications of string theory. The reviews [289–291, 261] summarize our current knowledge on the subject.

Exercises

9.1. Consider the heterotic string compactified on a circle of radius R, with all sixteen Wilson lines Y_α turned on. Use the results of appendix D on page 513 to write the modular invariant partition function. Find how the Wilson lines transform under T-duality.

9.2. Apply the results of appendix E on page 516 to derive the heterotic effective action (9.1.8) on page 221. Show the invariance (9.1.10).

9.3. Start from the ten-dimensional type-IIA effective action in (H.16) on page 527. Use toroidal dimensional reduction (you will find relevant formulas in appendix E on page 516) and derive the four-dimensional effective action. Dualize all two-forms into axions.

9.4. In the $\mathcal{N} = 4_4$ space-time supersymmetric case of section 9.2 on page 223, bosonize the remaining three currents, write the Σ, $\bar{\Sigma}$ fields as vertex operators and show that in this case the left-moving internal CFT has to be a toroidal one.

9.5. Compute the partition function of the orbifold generated by the action (9.4.1) on page 231. Show that it is not modular invariant.

9.6. Show that the partition function (9.4.8) on page 232 is modular invariant. Verify that the massless bosonic spectrum is as claimed in the text.

9.7. Show that (9.4.14) is modular invariant if $\epsilon^2/2 = 1 \bmod (4)$.

9.8. Use the definition of the second helicity supertrace B_2 in appendix J on page 537 in order to derive (9.4.17) on page 234.

9.9. Show that only one of the four gravitini survives the $\mathbb{Z}_2 \times \mathbb{Z}_2$ projection described in section 9.6 on page 237.

9.10. Derive the massless spectrum of tables 9.1 and 9.2 from the partition function (9.6.2). Show that the spectrum is anomaly-free in four dimensions.

9.11. Consider a $\mathbb{Z}_3$ orbifold of the heterotic string, with generating rotation $\theta^1 = \theta^2 = \pi/3$, $\theta^3 = -2\pi/3$ in (9.3.8) on page 229. Show that this orbifold will give a vacuum with $\mathcal{N} = 1_4$ supersymmetry. Find the appropriate action on Γ_{16} so that a modular-invariant partition function is obtained. Derive the massless spectrum of this vacuum.

9.12. Show that the Nijenhuis tensor (9.7.4) on page 240 is indeed a tensor.

9.13. Consider the complex projective space CP^{N}: A space of $\mathrm{N}+1$ complex variables, moded out by the scaling $\{Z_k\} \sim \lambda\{Z_k\}$ where λ is any nonzero complex number. Show that this space is compact, and that it is a complex manifold.

9.14. Start from equation (9.8.9) on page 246 and use the identity

$$\gamma^j \gamma^{kl} = \gamma^{jkl} + g^{jk}\gamma^l - g^{jl}\gamma^k \tag{9.1E}$$

and the properties of the Riemann tensor to show that the Ricci tensor vanishes.

9.15. Starting from the *Ansatz* (9.9.1) on page 248 for the ten-dimensional metric derive (9.9.2) and (9.9.3) on page 248.

9.16. Show that the *Ansatz* (9.9.6) on page 248 provides solutions to equations (9.9.3).

9.17. Consider compactifications of type-IIA,B theories to four dimensions. Greek indices describe the four-dimensional part, Latin ones the six-dimensional internal part. Repeat the analysis at the beginning of section 9.8 on page 245 and find the conditions for the internal fields g_{mn}, B_{mn}, Φ as well as A_m, C_{mnr} for type-IIA and χ, $B^{\text{R-R}}_{mn}$, F^+_{mnrst} for type-IIB so that the effective four-dimensional theory has $\mathcal{N} = 4_4$ supersymmetry in flat space.

9.18. Use the results of section 7.9 on page 176 to show that the O(5,21), $\mathcal{N} = (2,0)_6$ supergravity obtained by compactifying the IIB string on K3 is anomaly-free.

9.19. Derive the massless spectrum of the $T^4/\mathbb{Z}_2$ orbifold described in section 9.10 on page 250.

9.20. Compute the elliptic genus $\text{Tr}_{\text{R-R}}[(-1)^{F_L+F_R} e^{izJ_0 - i\bar{z}\bar{J}_0}]$ for the type-II K3 compactification. The trace is in the R-R sector. Hint: show that it is independent of the K3 moduli.

9.21. The $\mathbb{Z}_2$ orbifold transformation $x^i \to -x^i$ is a symmetry of the T^4 for all values of the moduli. This is not the case for $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ rotations. Find the submoduli space of $\mathbb{Z}_3$-, $\mathbb{Z}_4$-, and $\mathbb{Z}_6$-invariant T^4s.

9.22. Construct the torus partition function of the $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ orbifold compactifications with $\mathcal{N} = 1_6$ supersymmetry. Derive from this the massless spectrum and compare with the geometrical description in section 9.9 on page 247.

9.23. Describe the blowing up of the $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ orbifold points of K3.

9.24. Show that in type-II compactifications on CY manifolds, the dilaton belongs to a hypermultiplet.

9.25. Show that in type-IIA compactifications on CY manifolds, the gauge group is $\text{U}(1)^{h^{1,1}+1}$.

9.26. Consider the IIA/B theory compactified on the $T^6/\mathbb{Z}_3$ supersymmetry-preserving orbifold, with $\mathcal{N} = 2_4$ supersymmetry. Calculate the massless spectrum. Find the topological data of the CY threefold whose singular limit is the orbifold above.

9.27. Consider a collection of O_9 and O_5 planes on $T^4/\mathbb{Z}_2$. Consider their coupling to the metric and dilaton and by varying derive the tadpole conditions.

9.28. Consider the T^4 lattice sum. Find the values for the torus moduli so that this sum is invariant under the action of Ω.

9.29. Find from first principles the phases of the Ω action on the fermionic ground-states of the open strings in section 9.14.3 on page 260.

9.30. Derive from first principles (i.e., not relying on supersymmetry) the massless fermionic spectrum of open strings in the K3 orientifold of section 9.14.3 on page 260.

9.31. Derive the low-lying spectrum of the $5a - 5x$ and $5x - 5(-x)$ strings described in section 9.14.3 on page 260.

9.32. Show that the solution to the tadpole conditions (9.14.51)–(9.14.53) on page 266 and the group properties imply (9.14.57).

9.33. Consider the tadpole conditions of the $\mathbb{Z}_2$ orientifold in section 9.14.6 on page 266. Find the general solution, considering a general brane configuration.

9.34. Consider a general configuration of D_5-branes in the $T^4/\mathbb{Z}_2$ orientifold of section 9.14.3 on page 260. Solve the tadpole conditions and derive the massless spectrum that was presented in section 9.14.7 on page 267.

9.35. Consider the effective gauge theory of the $U(16)\times U(16)$ solution to the $T^4/\mathbb{Z}_2$ tadpole conditions. Giving expectation values to various scalars, show that you can obtain the more general spectrum, in (9.14.61) on page 268.

9.36. Turning on the T^4 Wilson lines, the 9-9 gauge group U(16) of the $T^4/\mathbb{Z}_2$ orientifold is Higgsed. What is the most general remaining gauge group and the associated brane configuration?

9.37. Using the results of section 7.9 on page 176 and of exercise 7.33 on page 186 show that the $U(16)\times U(16)$ $T^4/\mathbb{Z}_2$ orientifold theory is free of gravitational and nonabelian gauge anomalies.

9.38. Show that the two U(1) factors of the $U(16)\times U(16)$ gauge group of the $T^4/\mathbb{Z}_2$ orientifold have abelian as well as abelian/nonabelian mixed anomalies in six dimensions. Show how the Green-Schwarz mechanism can cancel the anomalies in this case. Verify that in the process, the two U(1)'s become massive.

9.39. Consider the open string sector of the supersymmetric $T^4/\mathbb{Z}_3$ orientifold. Derive the tadpole conditions and show that due to the absence of $\mathbb{Z}_2$ factors in the orbifold group, no D_5-branes are needed. Solve the tadpole conditions and show that the massless spectrum consists of a vector multiplet of the $U(8)\times SO(16)$ gauge group, with hypermultiplets transforming as $(\mathbf{8},\mathbf{16})$, $(\bar{\mathbf{8}},\mathbf{16})$, $(\mathbf{28},\mathbf{1})$, $(\overline{\mathbf{28}},\mathbf{1})$.

9.40. Using group theory show that when an SO(6) $\mathbb{Z}_N$ rotation acts on the vector as in (9.15.1), it acts on the spinor as in (9.14.9).

9.41. Solve the invariance condition (9.14.14) on page 259 for the scalars explicitly in order to show that they transform in representation of the gauge group given in (9.15.7) on page 270. Do the same for the fermions to derive (9.15.10).

9.42. Consider in section 9.15) on page 268 D_{7_1}-branes transverse to the first plane and D_{7_2}-branes transverse to the second plane. Derive the massless spectrum of 3-7_1, 3-7_2, and 7_i-7_j, strings with $i,j = 1,2,3$.

9.43. Consider the massless spectrum of the D_3- and D_{7_i}-branes in section 9.15 on page 268. Calculate the four-dimensional nonabelian gauge anomalies. Impose the cancellation of the nonabelian anomalies to constraint the integers n_i, m_i.

9.44. Calculate the (mixed) gauge anomalies of the U(1) factors originating from the 3-3 strings in section 9.15. Determine the axion couplings responsible for their cancellation. Which linear combinations acquire masses in the process?

9.45. Consider the configuration of branes transverse to the orbifold singularity in section 9.15. Calculate the (massless) twisted tadpoles and show that the tadpole cancellation condition is

$$\mathrm{Tr}\left(\gamma_{3,\theta}^{k}\right)\prod_{i=1}^{3}\left[2\sin\frac{\pi k b_i}{\mathrm{N}}\right]+\sum_{i=1}^{3}\mathrm{Tr}\left(\gamma_{7_i,\theta}^{k}\right)2\sin\frac{\pi k b_i}{\mathrm{N}}=0,\quad k=1,2,\ldots,\mathrm{N}-1. \tag{9.2E}$$

What is its relation to the anomaly cancellation studied in exercise 9.43?

9.46. Consider the $T^6/\mathbb{Z}_3$ supersymmetric orbifold in four dimensions. Derive and solve the tadpole conditions, to find the massless spectrum. Is the spectrum chiral?

9.47. Consider D_3- and D_7-branes at a $\mathbb{Z}_{\mathrm{N}}$ singularity. Try to construct a gauge group and a chiral spectrum of fermions as close to the Standard Model as possible.

9.48. Show that the mass formula (9.16.4) on page 272 implies that $\mathrm{Str}[\delta M^2] = 0$.

9.49. Consider a D_9-brane wrapping a magnetized $(T^2)^3$. If H_i is the magnetic field through the ith torus show that $\mathcal{N} = 1_4$ supersymmetry is preserved if $|\theta_1| + |\theta_2| - |\theta_3| = 0$ (up to cyclic permutations). This corresponds to the statement that the SO(6) rotation, generated by $(\theta_1, \theta_2, \theta_3)$, is in fact in SU(3).

9.50. Solve equations (9.16.10) on page 274 together with the boundary conditions (9.16.17), (9.16.18), and verify the mode expansions (9.16.19), (9.16.20).

9.51. Verify explicitly, relations (9.16.24)–(9.16.29) on page 275.

9.52. Use the commutation relations (9.16.23) on page 275 to calculate the equal-time commutators $[X_I(\tau,\sigma), P^J(\tau,\sigma')]$ and $[P^I(\tau,\sigma), P^J(\tau,\sigma')]$. Observe that there are boundary contributions to the momentum operators.

9.53. The spectrum of strings starting and ending on the same magnetized brane, with $q_L = -q_R$, $H_L = H_R$, is not directly affected by the magnetic field. Quantize these strings carefully to find out the subtle effect of the magnetic field on the spectrum.

9.54. Derive from first principles the magnetized partition functions (9.16.41)–(9.16.45) on page 277.

9.55. Show geometrically, that (9.16.50) on page 278 is indeed the intersection number of the two branes on T^2.

9.56. Find the quantization of the magnetic flux threading a nonorthogonal T^2 carrying a constant antisymmetric tensor background. In type-I string theory, this background is discrete. Show that in this case, this is equivalent to the fact that the integer m in (9.16.2) on page 272 can take also half-integer values.

9.57. Consider a D_9-brane wrapping a magnetized $(T^2)^2$, with magnetic fields H_1 and H_2 through the two tori. Show that the D_9-brane acquires a D_5-brane charge. Discuss its quantization. Show that when the flux is infinite, the D_9 is equivalent to a D_5 brane stretching in the transverse directions.

9.58. Consider a set of magnetized D_9-branes wrapped on $(T^2)^3$ as those considered in section 9.16.3 on page 278. Derive the tadpole conditions using the magnetized partition functions and show that they are given by (9.16.51) on page 279.

9.59. Show that the massless states of strings stretched between a magnetized brane I and its orientifold image I' are $8m_I^1 m_I^2 m_I^3$ fermions in the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ representation of $U(N_I)$ as well as $4m_I^1 m_I^2 m_I^3(n_I^1 n_I^2 n_I^3 - 1)$ fermions in the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\square\square$ representations.

9.60. Consider intersecting D_6-branes on T^6. Find a solution to the tadpole conditions, so that the gauge group is $U(3)\times U(2)\times U(1)\times U(1)$, and the chiral spectrum is that of the SM. You must identify the hypercharge as a linear combination of the U(1) generators. Show that the other U(1)'s are anomalous and therefore massive.

9.61. Consider three D-brane stacks realizing the gauge group $U(3)\times U(2)\times U(1)$. Assume that the SM fermions and scalars (including one or more right-handed neutrino singlets) as arising from strings stretched between these three branes. Find all possible ways of doing this. An important ingredient is how hypercharge is realized as a linear combination of the three U(1) symmetries present. Do the other two U(1) gauge bosons remain massless?

9.62. Consider magnetized D_9-branes on T^6. The DBI action depends nontrivially on both the magnetic fields and the geometric moduli of the T^6. Show that this provides a potential for the torus moduli at the tree level. Minimize this potential and find which of the torus moduli can be stabilized. This is a special case of the more general stabilization mechanism of moduli by turning on fluxes of (generalized) gauge fields.

9.63. Before looking up reference [220], try to prove the key points, mentioned in section 9.17.2 on page 282 using basic properties of (super)conformal field theory and current algebra.

9.64. By considering the tree-level amplitudes of heterotic gauge bosons show the relation (9.17.3) on page 281.

9.65. Investigate how (9.18.2) on page 285 changes if the relevant brane wraps a flat internal magnetized cycle.

9.66. Consider the σ-model of the $SU(2)_k$ WZW model given in exercise 6.7 on page 152. We may gauge the $U(1)_L \times U(1)_R$ affine symmetry without including a standard kinetic term for the gauge fields. This preserves conformal invariance. Gauge fix and integrate out the gauge fields to find the resulting two-dimensional sigma-model. Describe its effective background fields and symmetry. Argue using CFT arguments that the continuous U(1) remnant symmetry is broken to $\mathbb{Z}_k$.