Preface

This book grew from an article I wrote in 2008 for the centenary of Felix Klein's *Elementary Mathematics from an Advanced Standpoint*. The article reflected on Klein's view of elementary mathematics, which I found to be surprisingly modern, and made some comments on how his view might change in the light of today's mathematics. With further reflection I realized that a discussion of elementary mathematics today should include not only some topics that are elementary from the twenty-first-century viewpoint, but also a more precise explanation of the term "elementary" than was possible in Klein's day.

So, the first goal of the book is to give a bird's eye view of elementary mathematics and its treasures. This view will sometimes be "from an advanced standpoint," but nevertheless as elementary as possible. Readers with a good high school training in mathematics should be able to understand most of the book, though no doubt everyone will experience some difficulties, due to the wide range of topics. Bear in mind what G. H. Hardy (1942) said in a review of the excellent book *What is Mathematics?* by Courant and Robbins (1941): "a book on mathematics without difficulties would be worthless."

The second goal of the book is to explain what "elementary" means, or at least to explain why certain pieces of mathematics seem to be "more elementary" than others. It might be thought that the concept of "elementary" changes continually as mathematics advances. Indeed, some topics now considered part of elementary mathematics are there because some great advance *made* them elementary. One such advance was the use of algebra in geometry, due to Fermat and Descartes. On the other hand, some concepts have remained persistently difficult. One is the concept of real number, which has caused headaches since the time of Euclid. Advances in logic in the twentieth century help to explain why the real numbers remain an "advanced" concept, and this idea will be gradually elaborated in the second half of the book. We will see how elementary mathematics collides with the real number concept from various directions, and how logic identifies the advanced nature of the real numbers—and, more generally, the nature of *infinity*—in various ways.

Those are the goals of the book. Here is how they are implemented. Chapter 1 briefly introduces eight topics that are important at the elementary level—arithmetic, computation, algebra, geometry, calculus, combinatorics, probability, and logic—with some illustrative examples. The next eight chapters develop these topics in more detail, laying down their basic principles, solving some interesting problems, and making connections between them. Algebra is used in geometry, geometry in arithmetic, combinatorics in probability, logic in computation, and so on. Ideas are densely interlocked, even at the elementary level! The mathematical details are supplemented by historical and philosophical remarks at the end of each chapter, intended to give an overview of where the ideas came from and how they shape the concept of elementary mathematics.

Since we are exploring the scope and limits of elementary mathematics we cannot help crossing the boundary into advanced mathematics on occasion. We warn the reader of these incursions with a star (*) in the titles of sections and subsections that touch upon advanced concepts. In chapter 10 we finally cross the line in earnest, with examples of *non*-elementary mathematics in each of the eight topics above. The purpose of these examples is to answer some questions that arose in the elementary chapters, showing that, with just small steps into the infinite, it is possible to solve interesting problems beyond the reach of elementary methods.

What is new in this book—apart from a hopefully fresh look at elementary mathematics—is a serious study of what it means for one theorem to be "more advanced" or "deeper" than others. In the last 40 years the subject of *reverse mathematics* has sought to classify theorems by the strength of axioms needed to prove them, measuring "strength" by how much the axioms assume about infinity. With this methodology, reverse mathematics has classified many theorems in basic analysis, such as the completeness of the real numbers, Bolzano-Weierstrass theorem, and Brouwer fixed point theorem. We can now say that these theorems are definitely "more advanced" than, say, elementary number theory, because they depend on stronger axioms.

So, if we wish to see what lies just beyond elementary mathematics, the first place to look is analysis. Analysis clarifies not only the scope of elementary calculus, but also of other fields where infinite processes occur. These include algebra (in its fundamental theorem) and combinatorics (in the Kőnig infinity lemma, which is also important in topology and logic). Infinity may not be the only characteristic that defines advanced mathematics, but it is probably the most important, and the one we understand the best.

Lest it seem that logic and infinity are formidable topics for a book about elementary mathematics, I hasten to add that we approach them very gently and gradually. Deeper ideas will appear only when they are needed, and the logical foundations of mathematics will be presented only in chapter 9—at which stage I hope that the reader will understand their value. In this respect (and many others) I agree with Klein, who said:

> In fact, mathematics has grown like a tree, which does not start at its tiniest rootlets and grow merely upward, but rather sends its roots deeper and deeper and at the same time and rate that its branches and leaves are spreading upward.
>
> Klein (1932), p.15

In chapter 9 we pursue the roots of mathematics deep enough to see, I hope, those that nourish elementary mathematics, and some that nourish the higher branches.

I expect that this book will be of interest to prospective mathematics students, their teachers, and to professional mathematicians interested in the foundations of our discipline. To students about to enter university, this book gives an overview of things that are useful to know before proceeding further, together with a glimpse of what lies ahead. To those mathematicians who teach at university level, the book can be a refresher course in the topics we want our students to know, but about which we may be (ahem) a little vague ourselves.

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