Preface

The field of partial differential equations (PDE for short) has a long history going back several hundred years, beginning with the development of calculus. In this regard, the field is a traditional area of mathematics, although more recent than such classical fields as number theory, algebra, and geometry. As in many areas of mathematics, the theory of PDE has undergone a radical transformation in the past hundred years, fueled by the development of powerful analytical tools, notably, the theory of functional analysis and more specifically of function spaces. The discipline has also been driven by rapid developments in science and engineering, which present new challenges of modeling and simulation and promote broader investigations of properties of PDE models and their solutions.

As the theory and application of PDE have developed, profound unanswered questions and unresolved problems have been identified. Arguably the most visible is one of the Clay Mathematics Institute Millennium Prize problems[1] concerning the Euler and Navier-Stokes systems of PDE that model fluid flow. The Millennium problem has generated a vast amount of activity around the world in an attempt to establish well-posedness, regularity and global existence results, not only for the Navier-Stokes and Euler systems but also for related systems of PDE modeling complex fluids (such as fluids with memory, polymeric fluids, and plasmas). This activity generates a substantial literature, much of it highly specialized and technical. Meanwhile, mathematicians use analysis to probe new applications and to develop numerical simulation algorithms that are provably accurate and efficient. Such capability is of considerable importance, given the explosion of experimental and observational data and the spectacular acceleration of computing power.

Our text provides a gateway to the field of PDE. We introduce the reader to a variety of PDE and related techniques to give a sense of the breadth and depth of the field. We assume that students have been exposed to elementary ideas from ordinary differential equations (ODE) and analysis; thus, the book is appropriate for advanced undergraduate or beginning graduate mathematics students. For the student preparing for research, we provide a gentle introduction to some current theoretical approaches to PDE. For the applied mathematics student more interested in specific applications and models, we present tools

1. www.claymath.org/millennium/.

of applied mathematics in the setting of PDE. Science and engineering students will find a range of topics in the mathematics of PDE, with examples that provide physical intuition.

Our aim is to familiarize the reader with modern techniques of PDE, introducing abstract ideas straightforwardly in special cases. For example, struggling with the details and significance of Sobolev embedding theorems and estimates is more easily appreciated after a first introduction to the utility of specific spaces. Many students who will encounter PDE only in applications to science and engineering or who want to study PDE for just a year will appreciate this focused, direct treatment of the subject. Finally, many students who are interested in PDE have limited experience with analysis and ODE. For these students, this text provides a means to delve into the analysis of PDE before or while taking first courses in functional analysis, measure theory, or advanced ODE. Basic background on functions and ODE is provided in Appendices A–C.

To keep the text focused on the analysis of PDE, we have not attempted to include an account of numerical methods. The formulation and analysis of numerical algorithms is now a separate and mature field that includes major developments in treating nonlinear PDE. However, the theoretical understanding gained from this text will provide a solid basis for confronting the issues and challenges in numerical simulation of PDE.

A student who has completed a course organized around this text will be prepared to study such advanced topics as the theory of elliptic PDE, including regularity, spectral properties, the rigorous treatment of boundary conditions; the theory of parabolic PDE, building on the setting of elliptic theory and motivating the abstract ideas in linear and nonlinear semigroup theory; existence theory for hyperbolic equations and systems; and the analysis of fully nonlinear PDE.

We hope that you, the reader, find that our text opens up this fascinating, important, and challenging area of mathematics. It will inform you to a level where you can appreciate general lectures on PDE research, and it will be a foundation for further study of PDE in whatever direction you wish.

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