

## Preface

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Mathematics rarely has the reputation we feel it deserves. To many, mathematics is an area that is, sadly, too difficult and too boring. It requires too much effort to learn and to understand. It's not as much fun as other subjects. In recent years there have been numerous articles written about how many American high-school students have been outperformed in mathematics and science by students from other nations. There have also been reports of a marked decrease in the number of Americans in colleges earning graduate degrees in mathematics. For whatever reason, not nearly enough talented American students have become sufficiently excited about mathematics. For many students, this is a missed opportunity. For the United States, this is a missed opportunity. There are many areas within mathematics and we happen to think that they are all exciting. Behind the many interesting theorems in each of these areas is a history of how these came about—a story of how some dedicated mathematicians discovered something of interest and importance. These theorems were often not only attractive to those who discovered them but in many cases unexpected to others. In many instances, these theorems turned out to be extraordinarily useful—both within and outside of mathematics. Our goal in this book is to introduce you to one of the many remarkable areas of mathematics. It is with pleasure that we invite you to enter

### The Fascinating World of Graph Theory.

Like every other scholarly field, mathematics is composed of a number of areas, similar in many ways, yet each having their own distinct characteristics. The areas with which you are probably most familiar include algebra, geometry, trigonometry and calculus. Learning and understanding these subjects may very well have required some effort on your part but, hopefully, it has been interesting as well. In fact, learning any subject should be fun. But where did these and all

other areas of mathematics come from? The answer to this question is that they came from people—from their curiosity, their imagination, their cleverness. Although many of these people were mathematicians, some were not. Sometimes they were students—like all of us are (or were).

It is our goal here to introduce you to a subject to which you may have had little or no exposure: the field of graph theory. While we wish to show you how interesting this area of mathematics is, we hope to convince you that mathematics itself is not only interesting but can in fact be exciting. So come with us as we take you along on what we believe will be a fascinating journey through the area of graph theory. Not only do we want to introduce you to many of the interesting topics in this area of mathematics, but it is our desire to give you an idea of how these topics may have been discovered and the kinds of problems they can be used to solve.

Among the many things we discuss here is how often a rather curious problem or question can lead not only to a mathematical solution but to an entire topic in mathematics. While it is not our intention to describe some deep or advanced mathematics here, we do want to give an idea of how we can convince ourselves that certain mathematical statements are true.

Chapter 1 begins with some curious problems, all of which can be looked at mathematically by means of the main concept of this book: graphs. Some of these problems turn out to be important historically and will be revisited when we have described enough information to solve the problems. This chapter concludes with a discussion of the fundamental concepts that occur in this area of mathematics. The last thing we do in Chapter 1 is present a theorem often called the First Theorem of Graph Theory, dealing with what happens when the degrees of all vertices of a graph are added.

Chapter 2 begins with a discussion of theorems from many areas of mathematics that have been judged among the most beautiful. We see here that not only is graph theory well represented on this list, but one mathematician in particular is especially well represented on this list. One of the theorems on this list leads us into a much-studied type of graph called regular graphs. From this, the degrees of the vertices of a graph are discussed at some length. The remainder of the chapter deals with concepts and ideas concerning the structure of graphs. And the chapter

closes with a rather mysterious problem in graph theory that no one has been able to solve.

Chapter 3 discusses the most fundamental property that a graph can possess, dealing with the idea that within the graph, travel is possible between every two locations. This brings up questions of the distance between locations in a graph and those locations that are far from or close to a given location. The chapter concludes with the rather humorous concept of Erdős numbers, based on mathematicians who have done research with mathematicians who have done research with . . . who have done research with the celebrated twentieth-century mathematician Paul Erdős.

Chapter 4 concerns the simplest structure that a connected graph can possess, leading us to the class of graphs called trees—because they often look like trees. These graphs have connections to chemistry and can assist us in solving problems where certain decisions must be made at each stage in solving the problem. This chapter ends by discussing a practical problem, one that involves the least expensive highway system that would allow us to travel between any two locations in the system.

Graph theory has a rather curious history. Most acknowledge that this area began in the eighteenth century when the brilliant mathematician Leonhard Euler was introduced to and solved a problem referred to as the Königsberg Bridge Problem and then went on to describe a considerably more complex problem. This led to a class of graphs named for Euler, which we study in Chapter 5. This chapter concludes with another well-known problem, the Chinese Postman Problem, which deals with minimizing the length of a round-trip that a letter carrier might take.

Chapter 6 discusses a class of graphs named for a famous physicist and mathematician of the nineteenth century: Sir William Rowan Hamilton. Although Hamilton had very little to do with graph theory, it was he who came up with the idea of “icosian calculus”, which led him to inventing a game that involved finding round-trips around a dodecahedron in which every vertex is encountered exactly once. Beginning midway through the twentieth century, an explosion of theoretical results involving this concept occurred. This chapter ends with a discussion of a problem of practical importance, that of finding a shortest or least costly round-trip that visits all locations of a certain type.

Problems that ask whether some collection of objects can be matched in some way to another collection of objects are plentiful—such as matching

applicants to job openings or sometimes just people to people. These kinds of problems, discussed in Chapter 7, gave rise in the late nineteenth century to the first consideration of graph theory as a theoretical area of mathematics and no doubt led to the term “graph” being used for the structure we discuss in this book. From this topic in graph theory, we can see how different types of schedulings are possible.

Chapter 8 concerns problems of whether a graph can be divided into certain other kinds of graphs, primarily cycles. Whether some specific complete graphs can be divided into triangles in some manner was the situation encountered when, in the mid-nineteenth century, the mathematician Thomas Kirkman stated and then solved a problem often referred to as Kirkman’s Schoolgirl Problem. There is a relationship between graph decomposition problems and a problem dealing with whether the vertices of a graph can be labeled with certain integers in a manner that produces a desirable labeling of its edges. This chapter ends with a tantalizing puzzle called Instant Insanity and how graphs can be utilized to solve it.

There are situations when travel involves using one-way streets and in order to model this in a graph, it is necessary to assign directions to the edges. This gives rise to the concept of an oriented graph. These structures can also be used to represent a sports tournament where assigning a direction to an edge represents the defeat of one team by another. The mathematics related to this is presented in Chapter 9. The chapter concludes with a discussion of how various voting techniques can result in often surprising outcomes.

Some interesting problems can be looked at in terms of whether certain graphs can be drawn in the plane without any of their edges crossing. This deals with the concept of planar graphs introduced in Chapter 10. There is a rich theory with these graphs, which is discussed in this chapter. One of the problems discussed here is the Brick-Factory Problem, which originated in a labor camp during World War II.

One of the most famous problems in mathematics concerns whether it’s always possible to color the regions of every map with four colors so that neighboring regions are colored differently. This Four Color Problem was the idea of a young British mathematician in the mid-nineteenth century and eventually gained notoriety and interest as this problem became better known. The Four Color Problem, famous not only for the length of time it took to solve but for the controversial method used to

solve it, is discussed in Chapter 11. This led to coloring the vertices of a graph and how this can be used to solve a variety of problems, from scheduling problems to traffic-light phase problems.

Not only is it of interest to consider coloring the vertices of a graph, both from practical and theoretical points of view, it is also of interest to consider coloring its edges. This is the topic of Chapter 12. Here too this can aid us in solving certain types of scheduling problems. This also leads us to consider a class of numbers in graph theory called Ramsey numbers. The chapter concludes with a curious theorem called the Road Coloring Theorem, which tells us that in certain traffic systems consisting only of one-way streets in which the same number of roads leave each location, roads can be colored so that directions can be given to arrive at some destination regardless of the location where the traveler presently resides.

While the main purpose of this book is to illustrate how interesting and intriguing (and sometimes mysterious) just one area of mathematics can be, this book can also be used as a textbook. At the end of the book is an “Exercises” section containing several exercises for all chapters in the book.

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A.B., G.C., P.Z.