Preface

This monograph contains the material taught during a series of lectures at Zhejiang University in Hangzhou. The lectures were given during several visits spread out over a period of almost three years which started during the summer of 2010. I also presented large portions of the material in courses at the Johns Hopkins University during the spring semesters of 2010 and 2012. This endeavor was partially supported by the Chinese Ministry of Education, the NSF and the Simons Foundation.

Since there were naturally people at Zhejiang University who could only attend, say, lectures for one of the three years, I attempted to split the material into three parts, each corresponding to one of those years. I also tried to organize the material and present it so that if somebody joined the course during one of the last two years it would be possible to follow the lectures needing minimal background from the earlier material, such as the statement of some of the missed theorems and not the proofs.

The first part of the course, which corresponds to the first three chapters of the monograph, was mainly devoted to the proof of the sharp Weyl formula. The proof presented uses the Hadamard parametrix and much of this part of the course was devoted to its construction using elementary properties of the geodesic flow. In this part of the course I also showed that no improvements of the sharp Weyl formula are possible for the sphere equipped with the standard round metric due to the fact that, in this case, the eigenvalues repeat with the highest possible multiplicity. In the other direction, at the end of this part of the course, I showed that for manifolds with nonpositive curvature and especially for the n-torus one can make significant improvements for bounds for the remainder term in the Weyl law. An explanation is that in addition to having eigenvalues repeating with the highest possible multiplicity, the sphere has the largest possible collection of periodic geodesics in the sense that every geodesic on S^n is periodic. In both cases, the torus and manifolds with nonpositive curvature have exactly the opposite type of behavior. I devoted the remaining part of the course to better understand this type of phenomena.

To do so, I had to first present the material in the fourth chapter of the monograph, which corresponds to the second part of the course. This is a rapid (and admittedly sketchy) introduction to microlocal analysis and the theory of pseudodifferential operators. For the applications in this third part of the course, which I hope was its highlight, I needed to cover basics from the calculus of pseudodifferential operators and, more importantly, special cases of Egorov's theorem and Hörmander's theorem on propagation of singularities. My attempt in this part of the course (as well as in the first part) was to present the minimal amount of material that would be needed at the end.

In the final part of the course I presented two main results that hearken back to the material presented at the end of the first part. I also chose to present them since both have proofs that naturally bring together focal points of earlier parts of the course.

The first of these, which is presented in the fifth chapter, is the Duistermaat-Guillemin theorem which says that if the set of periodic geodesics in a given compact Riemannian manifold M is of measure zero then the remainder term in the Weyl law for M satisfies better bounds than the one for the sphere. The proof that I present uses an additional idea of Ivrii showing how one can use Hörmander's theorem about propagation of singularities and a generalized version of the earlier Weyl law involving pseudodifferential operators to reduce everything to a calculation that only requires information about the wave kernel near the time $t = 0$. Thus, the punchline of the proof uses the Hadamard parametrix, just as in the proof of the sharp Weyl formula from the first part of the course. The proof of the "generalized Weyl formula" that is presented is just a small variation on this earlier result as well.

The Duistermaat-Guillemin theorem says that the favorable (but generic) assumption that the set of periodic geodesics is of measure zero leads to more favorable estimates for the Weyl law. The last main theorem that I presented in the course is of a similar nature. Roughly speaking, it says that if the long-term geodesic flow is uniformly distributed than (most) eigenfunctions exhibit a similar behavior in the sense that their L^2-mass becomes equidistributed as their energies go to infinity. More specifically, if the geodesic flow is ergodic than there always exists a subsequence $\{\lambda_{j_k}\}$ of full density so that, for the corresponding subsequence of L^2-normalized eigenfunctions $\{e_{j_k}\}$, the associated unit probability measures $|e_{\lambda_{j_k}}|^2\, dV_g$ tend in the weak* topology to the uniform probability measure $dV_g/\mathrm{Vol}_g(M)$. A stronger theorem is valid as well, which is the main result in the sixth chapter, which says that the eigenfunctions $\{e_{\lambda_{j_k}}\}$ become uniformly distributed in phase space as well. This result is due in varying degrees of generality to Snirelman, Zelditch and Colin de Verdiére. Like the Duistermaat-Guillemin theorem, the proof brings together highlights from earlier parts of the course—in this case Egorov's theorem, the generalized Weyl formula and von Neumann's mean ergodic theorem.

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