Preface

The goal of a science of politics is to identify general patterns among abstract concepts. That is, political scientists think about politics and invent abstract concepts that help us describe the political world. But a science of politics requires more than inventive description. It involves the specification of expected causal relationships among the concepts and, ultimately, the determination of whether those expected relationships comport with evidence.

Mathematics is an abstract discipline. All of its concepts have been invented. It is nothing more than a formal language, a set of definitions and rules, a syntax. However, since we are not studying mathematics for the purpose of contributing to its development, we need not concern ourselves with that. Instead, we are interested in mathematics to help us discipline our conceptualization, our theoretical development, our specification of hypotheses, and our testing of hypotheses. In other words, we are interested in mathematics to help us develop rigorous theories of politics and rigorous tests of the implications of those theories.

This book is designed to provide aid in this endeavor. This is *not* a formal text in mathematics. We rarely define terms formally, and we do not offer formal proofs of theorems. Indeed, at times we sacrifice mathematical rigor for intuition in a way that would make a mathematician cringe.[1] And we certainly fail to include many topics that others may find important. We do this for two reasons. One, this book is intended primarily for political scientists, not mathematicians. We believe that many students of political science have had little prior experience with mathematics outside of high school, and further, may have developed a fairly serious case of math phobia. Before one can overcome these obstacles one needs to develop a strong intuition about what math is, what it can tell us, and how it can be useful. We aim to provide a practical text that provides these intuitions. Two, there is an abundance of other, more traditional alternatives for learning math more rigorously, e.g., all the courses offered by math departments.

The book is broken up into five parts. The first part, “Building Blocks,” covers the preliminaries and reviews what should have been learned in most high school curricula, albeit at a higher level. Those with a strong math background can safely skim this section, though we would advise most not to skip it entirely: topics such as relations, proofs, utility representations, and the use of comput-

[1] Though footnotes and other asides will sometimes provide relevant formalism.

ers to aid in performing calculations might be new to many readers. Those whose background is more shaky should read this section carefully, preferably before beginning a math class or "camp," as this material provides the language on which the rest of the book rests. Chapter 1 (re)introduces variables, sets, operators, relations, notation, and proofs. Chapter 2 provides a review of basic algebra, including solving equations, and briefly discusses computational aids. Chapter 3 deals with functions, talks more about relations, and introduces utility representations. Chapter 4 discusses limits and continuity, as well as sequences and series and some related properties of sets not covered in Chapter 1.

The second part covers calculus in one dimension, including optimization. Those with a prior background in calculus can safely skim this section, or skip it entirely if the background is recent. calculus is used in the discussion of continuous distributions, and forms the bedrock for the final section on multivariate calculus and optimization. Chapter 5 introduces calculus and the derivative for functions of one variable. Chapter 6 offers rules of differentiation and provides derivatives of both common and special functions. Chapter 7 introduces the indefinite and definite integral, provides techniques of integration, and discusses the fundamental theorem of calculus. Chapter 8 defines a function's extrema, discusses higher-order derivatives and concave and convex functions, and illustrates techniques for unconstrained optimization in one dimension, including first- and second-order conditions.

The third part tackles probability, from its basics to discrete and continuous distributions. A brief tie-in to statistical inference is provided here, though the focus remains on probability and not on statistics. As this section is perhaps the most essential to the work of political scientists, we would advise a careful reading by all. Chapter 9 presents the basics of probability theory, illustrates how to calculate what are known as "simple" probabilities, and discusses conceptually the utility of probability theory in both statistical inference and formal theory. Chapter 10 introduces the fundamental concepts of a distribution, a random variable and associated probability distribution, and an expectation, and offers examples of relevant discrete probability distributions. Neither of these two chapters requires the material in Part II. Chapter 11 brings calculus back in, thereby permitting us to discuss continuous distributions and expectations of variables over these distributions; it also offers examples of relevant continuous probability distributions and elaborates slightly on statistical inference.

The fourth part is a primer on linear algebra. Linear algebra aids in everything from solving systems of equations to statistical modeling to understanding stochastic processes, yet most students are unlikely to require understanding of all these topics at once. We recommend Chapter 12 to all readers, as it covers the definitions of vectors and matrices, as well as how to perform nuts-and-bolts calculations with them. This is the material that students in statistics courses are most likely to require. For those seeking a somewhat deeper introduction to linear algebra, Chapter 13 delves into vector spaces and norms, spanning vectors, linear independence, matrix rank, and quadratic forms. As a concrete payoff to learning these concepts, it also includes an application of them to the solution of systems of linear equations. Neither of these chapters requires any of the material in Parts II or III of the course, save for the proof that the method of ordinary least squares (OLS) is the best linear unbiased estimator (BLUE) in Chapter 13. Chapter 14 covers, albeit briefly, three more advanced topics: eigenvalues and eigenvectors, matrix decomposition, and Markov processes. These are all useful in statistical analysis, particularly Bayesian statistics, and the last is growing in use in formal theory as well. We strongly recommend the study of probability distributions, discussed in Part III, prior to engaging with the material in Chapter 14.

The fifth and final part is the most complex in the book. It introduces selected topics in multivariate calculus, including constrained and unconstrained optimization and implicit differentiation. These topics generally prove useful in more advanced classes in statistics and game theory, and the reader, particularly the reader who has not previously been exposed to calculus, may desire to put most of these chapters off until a later date. We expect that many math classes, and most math camps, will not find time to cover all of these topics, and that this section may serve as a later reference as much as a guide to present instruction. However, the first chapter in this part offers more basic and essential information about multivariate functions and the partial derivative, and should be read on a first pass through the book. This material, covered in Chapter 15, briefly provides the multivariate analogues to the material in Chapters 3–7. It covers functions of several variables, multivariate calculus, and concavity and convexity in more than one dimension. Chapter 16 is technically more complex, and completes the extension of our coverage of calculus to more than one dimension by introducing multidimensional optimization. It then goes further, providing techniques to perform constrained optimization: optimization under constraints on the values the variables might take. For this topic we deviate from our habit of providing intuition into the method and concentrate on elucidating the procedure to perform constrained optimization. Finally, Chapter 17 discusses the concept of comparative statics and describes a tool for accomplishing these, the implicit function theorem. Again, we focus in this chapter on the method, leaving the intuition to more advanced courses.

There is no one right order in which to read the material covered in this book. We chose the order of the parts and chapters to match what we felt was a natural progression: each part from II through V delivers material matched to a different undergraduate math course, and someone reading straight through covers one-dimensional topics fully before moving to the complexity of multiple dimensions in Parts IV and V. Yet Part IV, which covers linear algebra, requires neither calculus nor probability theory, and thus might have been put right after Part I. In fact, we could have moved Chapters 12 and 13, covering vectors, matrices, and vector spaces, to follow Chapter 3, as neither Chapter 12 nor 13 makes use of limits. That ordering could be part of a larger progression that accommodates a department that gets very quickly to quantitative analysis but doesn't require calculus until later. That is, an instructor could begin with Chapters 1–3, move to 12 and 13, pick up limits in Chapter 4, then introduce the basics of probability and distributions in Chapters 9 and 10. Skipped chapters could be included according to taste.

We could imagine other combinations as well, tailored more closely to the needs of a particular group of students. For example, in a department with a course in probability, Part III could be skipped; this could be done in the book's progression or the alternative progression offered in the previous paragraph. The only resulting change that would need to be made would be to skip a couple of the topics in Part IV: the proof that OLS is BLUE in Chapter 13, and Markov processes in Chapter 14. Third, a brief math camp might cover only Chapters 1–8, 12, 13, and 15. We have tried to make the book modular to accommodate selective use, and have noted above the prior requirements of each new chapter. This modularity should also improve the book's utility as a reference, and we would be most pleased if people found themselves returning to it well after completing their coursework.

Throughout each chapter we attempt to provide *reasons* why a student should master particular areas of mathematics. These reasons typically focus on two broad areas of quantitative analysis commonly used in political science—statistics and formal theory. Statistics, loosely speaking, deals with the analysis of data, and is largely, though not entirely, used in the quantitative description of politics and to *test* hypotheses about politics via statistical inference. In political science, statistics is often subsumed under the rubric of "quantitative methods." As we noted above, we do not cover statistical inference in this book. However, as the chapters progress, we work through some of the background behind OLS, which is a method of minimizing the (squared) deviations between one's data and a best-fit line. OLS appears early in most students' statistical training in political science. Formal theory, of which game theory is by far the most commonly used, is the use of mathematical analysis to derive *theoretical* propositions about how the world might work. It is used in political science primarily to derive internally consistent theories and to highlight important interactions between actors. We do not cover game theory in this book, but as the chapters progress we introduce **utility** and **expected utility**, and discuss its **maximization**. This maximization is at the heart of game theory, as it allows game theorists to say what rational actors *should* do in a given situation. These actions form part of **equilibrium** behavior, meaning that each player is satisfied with her optimal action given everyone else's optimal actions, and has no incentive to change it.

Each chapter concludes with a set of exercises designed to develop mastery via practice. We also include worked examples throughout to highlight important concepts or techniques, aimed at fostering a practical knowledge of mathematics that may be brought to coursework and research. In support of this, we have made a special effort to reduce the typical level of bravado present in math texts. We eschew words like "clearly" and "obviously" and instead show the supposedly clear and obvious steps often left out. Our hope is that this reduces some of the unnecessary intimidation often associated with math texts. Further, we frequently reference online resources, as sometimes another perspective on a topic may be all that the stuck student needs to advance.

Finally, we offer a brief note about how we came to write this text. We began with the opinion that math texts generally seek to communicate their content with an emphasis on concise, elegant presentation. Moore, who has had considerably less mathematical training than Siegel, has often found it difficult to follow the presentation in math texts, and wanted to write a text that erred on the side of walking the reader through the material at a level of detail that would leave few, if any, lost. He wrote the first draft for the first twelve chapters.[2] Siegel then drafted the remaining five and revised (and in several cases substantially rewrote) Moore's drafts. Beginning in 2006, various versions of the text have been used in the semester-long course for PhD students in the Political Science Department at Florida State, and those students have been very helpful with their feedback, as has Chris Reenock, who has frequently taught this course. G. Jason Smith contributed to some chapters' exercises, and Xiaoli Guo provided considerable assistance with the figures and other aspects of manuscript preparation. We thank two anonymous reviewers (and their anonymous classes) for additional helpful feedback. Chuck Myers at Princeton University Press offered the project considerable support, and we very much appreciate the assistance of Kathleen Cioffi, Eric Henney, Steven Peter, Rob Tiempo, and others at the press who helped us bring this book to publication.

[2] Joseph K. Young kindly provided the first draft of Chapter 2, Section 1, and Andreas Beger revised the first draft of Chapter 12.