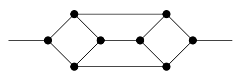
Chapter One

Introduction

This monograph offers a derivation of all classical and exceptional semisimple Lie algebras through a classification of "primitive invariants." Using somewhat unconventional notation inspired by the Feynman diagrams of quantum field theory, the invariant tensors are represented by diagrams; severe limits on what simple groups could possibly exist are deduced by requiring that irreducible representations be of integer dimension. The method provides the full Killing-Cartan list of all possible simple Lie algebras, but fails to prove the existence of F_4, E_6, E_7 and E_8.

One simple quantum field theory question started this project; what is the group-theoretic factor for the following Quantum Chromodynamics gluon self-energy diagram

 $= ?$ (1.1)

I first computed the answer for $SU(n)$. There was a hard way of doing it, using Gell-Mann f_{ijk} and d_{ijk} coefficients. There was also an easy way, where one could doodle oneself to the answer in a few lines. This is the "birdtracks" method that will be developed here. It works nicely for $SO(n)$ and $Sp(n)$ as well. Out of curiosity, I wanted the answer for the remaining five exceptional groups. This engendered further thought, and that which I learned can be better understood as the answer to a different question. Suppose someone came into your office and asked, "On planet Z, mesons consist of quarks and antiquarks, but baryons contain three quarks in a symmetric color combination. What is the color group?" The answer is neither trivial nor without some beauty (planet Z quarks can come in 27 colors, and the color group can be E_6).

Once you know how to answer such group-theoretical questions, you can answer many others. This monograph tells you how. Like the brain, it is divided into two halves: the plodding half and the interesting half.

The plodding half describes how group-theoretic calculations are carried out for unitary, orthogonal, and symplectic groups (chapters 3–15). Except for the "negative dimensions" of chapter 13 and the "spinsters" of chapter 14, none of that is new, but the methods are helpful in carrying out daily chores, such as evaluating Quantum Chromodynamics group-theoretic weights, evaluating lattice gauge theory group integrals, computing $1/N$ corrections, evaluating spinor traces, evaluating casimirs, implementing evaluation algorithms on computers, and so on.

The interesting half, chapters 16–21, describes the "exceptional magic" (a new construction of exceptional Lie algebras), the "negative dimensions" (relations between bosonic and fermionic dimensions). Open problems, links to literature, software and other resources, and personal confessions are relegated to the epilogue, monograph's Web page birdtracks.eu. The methods used are applicable to field-theoretic model building. Regardless of their potential applications, the results are sufficiently intriguing to justify this entire undertaking. In what follows we shall forget about quarks and quantum field theory, and offer instead a somewhat unorthodox introduction to the theory of Lie algebras. If the style is not Bourbaki [29], it is not so by accident.

There are two complementary approaches to group theory. In the *canonical* approach one chooses the basis, or the Clebsch-Gordan coefficients, as simply as possible. This is the method which Killing [189] and Cartan [43] used to obtain the complete classification of semisimple Lie algebras, and which has been brought to perfection by Coxeter [67] and Dynkin [105]. There exist many excellent reviews of applications of Dynkin diagram methods to physics, such as refs. [312, 126].

In the *tensorial* approach pursued here, the bases are arbitrary, and every statement is invariant under change of basis. Tensor calculus deals directly with the invariant blocks of the theory and gives the explicit forms of the invariants, Clebsch-Gordan series, evaluation algorithms for group-theoretic weights, *etc.*

The canonical approach is often impractical for computational purposes, as a choice of basis requires a specific coordinatization of the representation space. Usually, nothing that we want to compute depends on such a coordinatization; physical predictions are pure scalar numbers ("color singlets"), with all tensorial indices summed over. However, the canonical approach can be very useful in determining chains of subgroup embeddings. We refer the reader to refs. [312, 126] for such applications. Here we shall concentrate on tensorial methods, borrowing from Cartan and Dynkin only the nomenclature for identifying irreducible representations. Extensive listings of these are given by McKay and Patera [234] and Slansky [312].

To appreciate the sense in which canonical methods are impractical, let us consider using them to evaluate the group-theoretic factor associated with diagram (1.1) for the exceptional group E_8. This would involve summations over 8 structure constants. The Cartan-Dynkin construction enables us to construct them explicitly; an E_8 structure constant has about $248^3/6$ elements, and the direct evaluation of the group-theoretic factor for diagram (1.1) is tedious even on a computer. An evaluation in terms of a canonical basis would be equally tedious for $SU(16)$; however, the tensorial approach illustrated by the example of section 2.2 yields the answer for all $SU(n)$ in a few steps.

Simplicity of such calculations is one motivation for formulating a tensorial approach to exceptional groups. The other is the desire to understand their geometrical significance. The Killing-Cartan classification is based on a mapping of Lie algebras onto a Diophantine problem on the Cartan root lattice. This yields an exhaustive classification of simple Lie algebras, but gives no insight into the associated geometries. In the 19th century, the geometries or the invariant theory were the central question, and Cartan, in his 1894 thesis, made an attempt to identify the primitive invariants. Most of the entries in his classification were the classical groups $SU(n)$, $SO(n)$, and $Sp(n)$. Of the five exceptional algebras, Cartan [44] identified G_2 as the group of octonion isomorphisms and noted already in his thesis that E_7 has a skew-symmetric quadratic and a symmetric quartic invariant. Dickson characterized E_6 as a 27-dimensional group with a cubic invariant. The fact that the orthogonal, unitary and symplectic groups were invariance groups of real, complex, and quaternion norms suggested that the exceptional groups were associated with octonions, but it took more than 50 years to establish this connection. The remaining four exceptional Lie algebras emerged as rather complicated constructions from octonions and Jordan algebras, known as the *Freudenthal-Tits construction.* A mathematician's history of this subject is given in a delightful review by Freudenthal [130]. The problem has been taken up by physicists twice, first by Jordan, von Neumann, and Wigner [173], and then in the 1970s by Gürsey and collaborators [149, 151, 152]. Jordan *et al.*'s effort was a failed attempt at formulating a new quantum mechanics that would explain the neutron, discovered in 1932. However, it gave rise to the Jordan algebras, which became a mathematics field in itself. Gürsey *et al.* took up the subject again in the hope of formulating a quantum mechanics of quark confinement; however, the main applications so far have been in building models of grand unification.

Although beautiful, the Freudenthal-Tits construction is still not practical for the evaluation of group-theoretic weights. The reason is this: the construction involves $[3 \times 3]$ octonionic matrices with octonion coefficients, and the 248-dimensional defining space of E_8 is written as a direct sum of various subspaces. This is convenient for studying subgroup embeddings [291], but awkward for group-theoretical computations.

The inspiration for the primitive invariants construction came from the axiomatic approach of Springer [314, 315] and Brown [34]: one treats the defining representation as a single vector space, and characterizes the primitive invariants by algebraic identities. This approach solves the problem of formulating efficient tensorial algorithms for evaluating group-theoretic weights, and it yields some intuition about the geometrical significance of the exceptional Lie groups. Such intuition might be of use to quark-model builders. For example, because $SU(3)$ has a cubic invariant $\epsilon^{abc} q_a q_b q_c$, Quantum Chromodynamics, based on this color group, can accommodate 3-quark baryons. Are there any other groups that could accommodate 3-quark singlets? As we shall see, G_2, F_4, and E_6 are some of the groups whose defining representations possess such invariants.

Beyond its utility as a computational technique, the primitive invariants construction of exceptional groups yields several unexpected results. First, it generates in a somewhat magical fashion a triangular array of Lie algebras, depicted in figure 1.1. This is a classification of Lie algebras different from Cartan's classification; in this new classification, all exceptional Lie groups appear in the same series (the bottom line of figure 1.1). The second unexpected result is that many groups and group representations are mutually related by interchanges of symmetrizations and antisymmetrizations and replacement of the dimension parameter n by $-n$. I call this phenomenon "negative dimensions."

For me, the greatest surprise of all is that in spite of all the magic and the strange diagrammatic notation, the resulting manuscript is in essence not very different from Wigner's [345] 1931 classic. Regardless of whether one is doing atomic, nuclear, or particle physics, all physical predictions ("spectroscopic levels") are expressed in terms of Wigner's $3n$-j coefficients, which can be evaluated by means of recursive or combinatorial algorithms.

Parenthetically, this book is *not* a book about diagrammatic methods in group theory. If you master a traditional notation that covers all topics in this book in a uniform way, more elegantly than birdtracks, more power to you. I would love to learn it.

						0 0	3 $\mathbf{A}_1$ 2
					0 0	1 $\mathbf{U}(1)$ 1	8 $\mathbf{A}_2$ 3
				0 0	0 1	3 $\mathbf{A}_1$ 3	14 $\mathbf{G}_2$ 7
			0 0	0 1	2 $2\mathbf{U}(1)$ 2	9 $\mathbf{3A}_1$ 4	28 $\mathbf{D}_4$ 8
		0 0	0 2	3 $\mathbf{A}_1$ 5	8 $\mathbf{A}_2$ 8	21 $\mathbf{C}_3$ 14	52 $\mathbf{F}_4$ 26
	0 0	0 1	2 $2\mathbf{U}(1)$ 3	8 $\mathbf{A}_2$ 6	16 $\mathbf{2A}_2$ 9	35 $\mathbf{A}_5$ 15	78 $\mathbf{E}_6$ 27
0 0	1 $\mathbf{U}(1)$ 2	3 $\mathbf{A}_1$ 4	9 $\mathbf{3A}_1$ 8	21 $\mathbf{C}_3$ 14	35 $\mathbf{A}_5$ 20	66 $\mathbf{D}_6$ 32	133 $\mathbf{E}_7$ 56
3 $\mathbf{A}_1$ 3	8 $\mathbf{A}_2$ 8	14 $\mathbf{G}_2$ 14	28 $\mathbf{D}_4$ 28	52 $\mathbf{F}_4$ 52	78 $\mathbf{E}_6$ 78	133 $\mathbf{E}_7$ 133	248 $\mathbf{E}_8$ 248

Figure 1.1 The “Magic Triangle” for Lie algebras. The “Magic Square” is framed by the double line. For a discussion, consult chapter 21.