Introduction

This book is the full detailed version of the paper [KU1].

In [G1], Griffiths defined and studied the classifying space D of polarized Hodge structures of fixed weight w and fixed Hodge numbers $(h^{p,q})$. In [G5], Griffiths presented a dream of adding points at infinity to D. This book is an attempt to realize his dream.

In the special case

$$w = 1, \quad h^{1,0} = h^{0,1} = g, \quad \text{other } h^{p,q} = 0, \tag{1}$$

the classifying space D coincides with Siegel's upper half space $\mathfrak{h}_g$ of degree g. If $g = 1$, $\mathfrak{h}_g$ is the Poincaré upper half plane $\mathfrak{h} = \{x + iy \mid x, y \in \mathbf{R}, y > 0\}$. For a congruence subgroup Γ of $\mathrm{SL}(2, \mathbf{Z})$, that is, for a subgroup of $\mathrm{SL}(2, \mathbf{Z})$ which contains the kernel of $\mathrm{SL}(2, \mathbf{Z}) \to \mathrm{SL}(2, \mathbf{Z}/n\mathbf{Z})$ for some integer $n \geq 1$, the quotient $\Gamma\backslash\mathfrak{h}$ is a modular curve without cusps. We obtain the compactification $\Gamma\backslash(\mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q}))$ of the modular curve $\Gamma\backslash\mathfrak{h}$ by adding points at infinity called cusps (i.e., the elements of the finite set $\Gamma\backslash\mathbf{P}^1(\mathbf{Q})$). In the case (1) with g general, for a congruence subgroup Γ of $\mathrm{Sp}(g, \mathbf{Z})$, by adding points at infinity, we have toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$ [AMRT] and the Satake-Baily-Borel compactification of $\Gamma\backslash\mathfrak{h}_g$ [Sa1], [BB]. All these compactifications coincide when $g = 1$, but, when $g > 1$, there are many toroidal compactifications and the Satake-Baily-Borel compactification is different from them. The theory of these compactifications is included in a general theory of compactifications of quotients of symmetric Hermitian domains by the actions of discrete arithmetic groups. The points at infinity are often more important than the usual points. For example, the Taylor expansion of a modular form at the standard cusp (i.e., the class of $\infty \in \mathbf{P}^1(\mathbf{Q})$ modulo Γ) of the compactified modular curve $\Gamma\backslash(\mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q}))$ is called the q-expansion and is very important in the theory of modular forms.

However, the classifying space D in general is rarely a symmetric Hermitian domain, and we cannot use the general theory of symmetric Hermitian domains when we try to add points at infinity to D. In this book, we overcome this difficulty. We discuss two subjects in this book.

SUBJECT I. TOROIDAL PARTIAL COMPACTIFICATIONS AND MODULI OF POLARIZED LOGARITHMIC HODGE STRUCTURES

A toroidal compactification of $\Gamma\backslash\mathfrak{h}_g$ is defined depending on the choice of a certain fan (cone decomposition). If the fan is not sufficiently big, we have a toroidal partial compactification of $\Gamma\backslash\mathfrak{h}_g$, which need not be compact and which is locally isomorphic to an open set of a toroidal compactification.

In this book, for general D, we construct a kind of toroidal partial compactification $\Gamma\backslash D_\Sigma$ of $\Gamma\backslash D$ associated with a fan Σ and a discrete subgroup Γ of $\mathrm{Aut}(D)$ satisfying a certain compatibility with Σ.

In the case (1), the classes of polarized Hodge structures in $\Gamma\backslash\mathfrak{h}_g$ converge to a point at infinity of $\Gamma\backslash\mathfrak{h}_g$ when the polarized Hodge structures become degenerate. As in [Sc], nilpotent orbits appear when polarized Hodge structures become degenerate. In our definition of D_Σ for general D, a nilpotent orbit itself is viewed as a point at infinity.

As is discussed in detail in this book, the theory of nilpotent orbits is regarded as a local aspect of the theory of polarized logarithmic Hodge structures. The "polarized logarithmic Hodge structure" (PLH) is formulated by using the theory of logarithmic structures introduced by Fontaine and Illusie and developed in [Kk1],[KkNc], and it is something like the logarithmic degeneration of the PH (polarized Hodge structure).

We give here a rough illustration of the idea of the PLH. Let X be a complex manifold endowed with a divisor Y with normal crossings, and let $U = X - Y$. Let H be a PLH on X with respect to the "logarithmic structure of X associated with Y." Then the restriction $H|_U$ of H to U is a family of usual PH parametrized by U. At $x \in U$, the fiber $H(x)$ of H is a usual PH. At Y, this family can become degenerate in the classical sense. At each point $x \in Y$, the fiber of H at x corresponds to a nilpotent orbit (in the classical theory) toward $x \in Y$. Schematically, we have the following:

$$\begin{tikzcd}
(H(x) : \text{a PH}) \arrow[r, equal] & (H(x) : \text{a PLH}) \\
(H|_U : \text{a family of PH on } U) \arrow[u, "\text{a fiber over a point } x \in U"'] \arrow[r, "\text{extension}"] \arrow[d, "\text{degeneration toward } x \in Y"'] & (H : \text{PLH on } X) \arrow[u, "\text{a fiber over a point } x \in U"'] \arrow[d, "\text{a fiber over a point } x \in Y"'] \\
(\text{a nilpotent orbit toward } x) \arrow[r, equal] & (H(x) : \text{a PLH}).
\end{tikzcd} \tag{2}$$

(See 0.2.20, 0.4.25.)

Our main theorem concerning Subject I is stated roughly as follows (for the precise statement, see Theorem 0.4.27 below).

Theorem. $\Gamma\backslash D_\Sigma$ *is the fine moduli space of "polarized logarithmic Hodge structures" with a "Γ-level structure" whose "local monodromies are in the directions in Σ."*

Roughly speaking,

$$\begin{aligned}
\Gamma\backslash D &= (\text{polarized Hodge structures with a ``}\Gamma\text{-level structure''}) \\
&\cap \\
\Gamma\backslash D_\Sigma &= \Gamma\backslash\{\sigma\text{-nilpotent orbit} \mid \sigma \in \Sigma\} \\
&= \left(\begin{array}{l}\text{``polarized logarithmic Hodge structures'' with a ``}\Gamma\text{-level structure''} \\ \text{whose ``local monodromies are in the directions in } \Sigma\text{.''}\end{array}\right)
\end{aligned}$$

Here, a σ-nilpotent orbit is a nilpotent orbit in the direction of the cone σ. For $\sigma = \{0\}$, a σ-nilpotent orbit is nothing but a point of D; hence we can regard $D \subset D_\Sigma$.

In the classical case (1), $\Gamma\backslash D_\Sigma$ for a congruence subgroup Γ and for a sufficiently big Σ is a toroidal compactification of $\Gamma\backslash D$. Already, in this classical case, this theorem gives moduli-theoretic interpretations of the toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$.

The space $\Gamma\backslash D_\Sigma$ has a kind of complex structure, but a delicate point is that, in general, this space can have locally the shape of a "complex analytic space with a slit" (for example, $\mathbf{C}^2$ minus $\{(0, z) \mid z \in \mathbf{C}, z \neq 0\}$), and hence it is often not locally compact. However, it is very close to a complex analytic manifold. $\Gamma\backslash D_\Sigma$ is a logarithmic manifold in the sense of 0.4.17 below. Infinitesimal calculus can be performed nicely on $\Gamma\backslash D_\Sigma$. These phenomena were first examined in the easiest nontrivial case in [U2].

One motivation of Griffiths for adding points at infinity to D was the hope that the period map $\Delta^* \to \Gamma\backslash D$ ($\Delta^* = \{q \in \mathbf{C} \mid 0 < |q| < 1\}$) associated with a variation of polarized Hodge structure on Δ^* could be extended to $\Delta \to \Gamma\backslash(D \cup$ (points at infinity)) ($\Delta = \{q \in \mathbf{C} \mid |q| < 1\}$). By using the above main theorem and the nilpotent orbit theorem of Schmid, we can actually extend the period map to $\Delta \to \Gamma\backslash D_\Sigma$ for some suitable Σ (see 0.4.30 and 4.3.1, where a more general result is given).

SUBJECT II. THE EIGHT ENLARGEMENTS OF *D* AND THE FUNDAMENTAL DIAGRAM

In the classical case (1) above, there is another compactification $\Gamma\backslash D_{\mathrm{BS}}$ of $\Gamma\backslash D$ (Γ is a congruence subgroup of $\mathrm{Sp}(g, \mathbf{Z})$) called the Borel-Serre compactification, where D_{BS} is the Borel-Serre space denoted by $\bar{D}$ in [BS], which is a real manifold with corners containing D as a dense open set.

For general D, by adding to D points at infinity of different kinds, we obtain eight enlargements of D with maps among them which form the following fundamental diagram (3) (see 5.0.1).

Fundamental Diagram

$$\begin{array}{ccccccc} & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} \\ & & & & \downarrow & & \downarrow \\ D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \\ \downarrow & & \downarrow & & & & \\ D_\Sigma & \leftarrow & D^\sharp_\Sigma & & & & \end{array} \tag{3}$$

Note that the space D_Σ that appeared in Subject I sits at the left lower end of this diagram. The left-hand side of this diagram has Hodge-theoretic nature, and the right-hand side has the nature of the theory of algebraic groups. These are related by the middle map $D^\sharp_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$, which is a geometric interpretation of the SL(2)-orbit theorem of Cattani-Kaplan-Schmid [CKS].

In the case (1) with $g = 1$, the largest Σ exists. For this Σ, $D_\Sigma = \mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q})$, and the above diagram becomes

$$\begin{array}{ccccccc} & & & & \mathfrak{h}_{\mathrm{BS}} & = & \mathfrak{h}_{\mathrm{BS}} \\ & & & & \| & & \| \\ \mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q}) & \leftarrow & \mathfrak{h}_{\mathrm{BS}} & = & \mathfrak{h}_{\mathrm{BS}} & = & \mathfrak{h}_{\mathrm{BS}} \\ \| & & \| & & & & \\ \mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q}) & \leftarrow & \mathfrak{h}_{\mathrm{BS}} & & & & \end{array} \tag{4}$$

The space $\mathfrak{h}_{\mathrm{BS}}$ is described as follows. It is the union of open subsets $\mathfrak{h}_{\mathrm{BS}}(a)$ for $a \in \mathbf{P}^1(\mathbf{Q})$. $\mathfrak{h}_{\mathrm{BS}}(\infty) = \{x + iy \mid x \in \mathbf{R}, 0 < y \leq \infty\} \supset \mathfrak{h} = \{x + iy \mid x \in \mathbf{R}, y > 0\}$. The action of $\mathrm{SL}(2, \mathbf{Q})$ on $\mathfrak{h}$ extends to a continuous action of $\mathrm{SL}(2, \mathbf{Q})$ on $\mathfrak{h}_{\mathrm{BS}}$, and we have $g(\mathfrak{h}_{\mathrm{BS}}(a)) = \mathfrak{h}_{\mathrm{BS}}(ga)$ for $g \in \mathrm{SL}(2, \mathbf{Q})$ and $a \in \mathbf{P}^1(\mathbf{Q})$. In particular, all $\mathfrak{h}_{\mathrm{BS}}(a)$ are homeomorphic to each other. The map $\mathfrak{h}_{\mathrm{BS}} \to \mathfrak{h}_\Sigma = \mathfrak{h} \cup \mathbf{P}^1(\mathbf{Q})$ for the biggest Σ is the identity map on $\mathfrak{h}$ and sends elements of $\mathfrak{h}_{\mathrm{BS}}(a) - \mathfrak{h}$ to a for $a \in \mathbf{P}^1(\mathbf{Q})$.

In the case (1) for general g, the fundamental diagram becomes

$$\begin{array}{ccccccc} & & & & D_{\mathrm{SL}(2),\mathrm{val}} & = & D_{\mathrm{BS},\mathrm{val}} \\ & & & & \downarrow & & \downarrow \\ D_{\Sigma,\mathrm{val}} & \leftarrow & D^\sharp_{\Sigma,\mathrm{val}} & \to & D_{\mathrm{SL}(2)} & = & D_{\mathrm{BS}} \\ \downarrow & & \downarrow & & & & \\ D_\Sigma & \leftarrow & D^\sharp_\Sigma & & & & \end{array} \tag{5}$$

In this case, for a subgroup Γ of $\mathrm{Sp}(g, \mathbf{Z})$ of finite index and for a suitable Σ, $\Gamma\backslash D_\Sigma$ is a toroidal compactification [AMRT] of $\Gamma\backslash D$, $\Gamma\backslash D_{\Sigma,\mathrm{val}}$ is obtained from $\Gamma\backslash D$ by blow-ups, and the maps $\Gamma\backslash D^\sharp_\Sigma \to \Gamma\backslash D_\Sigma$ and $\Gamma\backslash D^\sharp_{\Sigma,\mathrm{val}} \to \Gamma\backslash D_{\Sigma,\mathrm{val}}$ are proper surjective maps whose fibers are products of finite copies of $\mathbf{S}^1$. On the other hand, $\Gamma\backslash D_{\mathrm{BS}}$ is the Borel-Serre compactification [BS] of $\Gamma\backslash D = \Gamma\backslash\mathfrak{h}_g$. The spaces D_{BS} and $D^\sharp_\Sigma$ are real manifolds with corners (they are like $\mathbf{R}^m \times \mathbf{R}^n_{\geq 0}$ locally). Already, in this classical case, the fundamental diagram (5) gives a relation between toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$ and the Borel-Serre compactification of $\Gamma\backslash\mathfrak{h}_g$, which were not known before (see 0.5.28 below).

For general D, these eight spaces are defined as

$D_{\Sigma}=$ (the space of nilpotent orbits) (1.3.8),
$D^{\sharp}_{\Sigma}=$ (the space of nilpotent i-orbits) (1.3.8),
$D_{\mathrm{SL}(2)}=$ (the space of SL(2)-orbits) (5.2.6),
$D_{\mathrm{BS}}=$ (the space of Borel-Serre orbits) (5.1.5),
$D_{\Sigma,\mathrm{val}}=$ (the space of valuative nilpotent orbits) (5.3.5),
$D^{\sharp}_{\Sigma,\mathrm{val}}=$ (the space of valuative nilpotent i-orbits) (5.3.5),
$D_{\mathrm{SL}(2),\mathrm{val}}=$ (the space of valuative SL(2)-orbits) (5.2.7),
$D_{\mathrm{BS},\mathrm{val}}=$ (the space of valuative Borel-Serre orbits) (5.1.6).

The space D_{BS} was constructed in [KU2] and [BJ] independently, by using the work [BS] of Borel-Serre on Borel-Serre compacifications. The spaces $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, and $D_{\mathrm{BS},\mathrm{val}}$ are defined in [KU2].

Roughly speaking, these eight spaces appear as follows:

$\Gamma\backslash D_{\Sigma}$ is like an analytic manifold with slits,
$D^{\sharp}_{\Sigma}$ and $D_{\mathrm{SL}(2)}$ are like real manifolds with corners and slits,
D_{BS} is a real manifold with corners,
$\Gamma\backslash D_{\Sigma,\mathrm{val}}$ and $D^{\sharp}_{\Sigma,\mathrm{val}}$ are the projective limits of "blow-ups" of $\Gamma\backslash D_{\Sigma}$ and $D^{\sharp}_{\Sigma}$, respectively, associated with rational subdivisions of Σ,
$D_{\mathrm{SL}(2),\mathrm{val}}$ and $D_{\mathrm{BS},\mathrm{val}}$ are the projective limits of certain "blow-ups" of $D_{\mathrm{SL}(2)}$ and D_{BS}, respectively.

The maps $\Gamma\backslash D^{\sharp}_{\Sigma}\to\Gamma\backslash D_{\Sigma}$ and $\Gamma\backslash D^{\sharp}_{\Sigma,\mathrm{val}}\to\Gamma\backslash D_{\Sigma,\mathrm{val}}$ are proper surjective maps whose fibers are products of a finite number of copies of $\mathbf{S}^1$, where this number is varying.

Like nilpotent orbits, SL(2)-orbits also appear in the theory of degenerations of polarized Hodge structures [Sc], [CKS]. The fundamental diagram (3) shows how nilpotent orbits, SL(2)-orbits, and the theory of Borel and Serre are related.

In this book, we study all these eight spaces. To prove the main theorem in Subject I and to prove that $\Gamma\backslash D_{\Sigma}$ has good properties such as the Hausdorff property, nice infinitesimal calculus, etc., we need to consider all other spaces in the diagram (3); we discuss the spaces from the right to the left in the fundamental diagram (3) to deduce the nice properties of $\Gamma\backslash D_{\Sigma}$, starting from the properties of the Borel-Serre compactifications (which were proved in [BS] by using arithmetic theory of algebraic groups).

The organization of this book is as follows. In Chapter 0, we give an overview of the book. In Chapters 1–4, we formulate the main theorem in Subject I. In Chapters 5–8, we prove the main theorem, considering all eight enlargements of D in the fundamental diagram (3), and we also prove various properties of the eight enlargements. In Chapters 9–12, we give complementary results.

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