INTRODUCTION

I.1 Opening Remarks

I.1.1. Mathematical Modernism

In this book I argue that the period from 1890 to 1930 saw mathematics go through a modernist transformation. Here, modernism is defined as an autonomous body of ideas, having little or no outward reference, placing considerable emphasis on formal aspects of the work and maintaining a complicated—indeed, anxious—rather than a naïve relationship with the day-to-day world, which is the de facto view of a coherent group of people, such as a professional or discipline-based group that has a high sense of the seriousness and value of what it is trying to achieve.

This brisk definition is certainly compatible with what many creators of the artistic modernisms thought they were doing. Consider, for example, these remarks by Guillaume Apollinaire from his *The Beginnings of Cubism*, published in 1912, where, speaking of many young painters, he said:

> These painters, while they still look at nature, no longer imitate it, and carefully avoid any representation of natural scenes which they may have observed, and then reconstructed from preliminary studies. Real resemblance no longer has any importance, since everything is sacrificed by the artist to truth, to the necessities of a higher nature whose existence he assumes, but does not lay bare. The subject has little or no importance any more. . . . Thus we are moving towards an entirely new art which will stand, with respect to painting as envisaged heretofore, as music stands to literature. It will be pure painting, just as music is pure literature. . . . This art of pure painting, if it succeeds in freeing itself from the art of the past, will not necessarily cause the latter to disappear; the development of music has not brought in its train the abandonment of the various genres of literature, nor has the acridity of tobacco replaced the savoriness of food.[1]

[1] In Chipp 1968, 222–223.

Or the American art critic Arthur Dow, writing in 1917, who characterized modernism in these seven points (not all of which met with his approval):

1. Freedom from the constraint of juries, critics or any law making art-body, involving
2. The rejection of most of the traditional ideas of art, even to the denial that beauty is worth seeking. As this seems opposed to the principle of evolution, and is only negative, I do not see how it can be maintained.
3. Interest in the expression of each individual, whether it conforms to a school or not, whether it be agreeable or the reverse.
4. Less attention to subject, more to form. Line, mass and color have pure aesthetic value whether they represent anything or not. Ceasing to make representation a standard but comparing the visual arts with music. Finding a common basis for all the visual arts.
5. Convincing us that there are limitless fields yet unrevealed by art. C. Lewis Hind says that "Matisse flashes upon canvas the unexplored three-fourths of life."
6. New expression by color, not by the colors of things, or color in historic art. Seeking hitherto unexpressed relations of color.
7. Approaching, through non-applied design and in other methods the creation of new types of design, decoration and craft work.[2]

The modernization of mathematics is most apparent at the foundational level, but it had important ramifications throughout the subject. Indeed, its roots are in explicit mathematical practice—problem solving and theory building—and modernism emerged and was successful in mathematics because it connected fruitfully with what mathematicians were doing and with the image they were creating for themselves as an autonomous body of professionals within, or alongside, the disciplines of philosophy and science, more specifically physics.[3] It is important to stress this point, in order to obviate misconceptions about the role of foundational inquiries in mathematics. I argue that the explosion of interest in the foundation of mathematics around 1900 has quite specific historical features that separate it from earlier and later phases. When German mathematicians spoke of a new conceptual mathematics (*begriffliche Mathematik*) they were directing attention not merely to the increasingly abstract nature of their subject—mathematics has always struck some people as rarefied—but to its newly self-contained aspect and the way it was being built up independently of references to the outside world and even the world of science.

This book argues the case for a decisive transformation of mathematical ontology around 1900, not only geometry and the well-known but perhaps misunderstood domain of analysis, where I also draw out, after Epple (1999), the way in which

[2] See Dow 1917, quote on 115–116. Dow goes on to say that not all of these demands were new, and in particular the Japanese artist, Keisai Yeisen, was more expressive in his use of line than Kandinsky. I am indebted to Linda Henderson for bringing this to my attention.

[3] Science on any definition is not synonymous with or reducible to physics. But physics was the paradigmatic science for mathematicians throughout the period covered by this book, the one with the most significant overlap and the greatest use of advanced mathematics, so for present purposes I have sometimes allowed the terms "science" and "physics" to elide, in keeping with the usage of the time.

traditional ideas persisted for a time behind a modern façade of Cauchy-Weierstrass style definition, but in algebra with its shift toward set theory.[4] I give my reasons below (see §2.2) for not finding Cauchy's rigorous real analysis modernist; they derive from my view of Cauchy's traditional mathematical ontology. There was, in addition, a radical transformation in mathematical epistemology. Mathematicians have almost always been concerned with proof, but the autonomous ontology raised new standards and eventually led to a decisive break with previous formulations. Thus, even in analysis, the Weierstrassian impulse led more and more to a reliance on algebra and arithmetic, and an interesting polemic between Karl Weierstrass and Leopold Kronecker, the leaders of the Berlin mathematics department, the largest in the world during the later nineteenth century. The formal language of those in Peano's school in Turin a little later marks a further step away from geometric intuition. Then in geometry itself the idea of relative consistency proofs, and the analysis of geometries in Hilbert's *Grundlagen der Geometrie* (1899), went even further in its analysis of proof. Finally, there was Hilbert's remarkable idea of Proof Theory.

In this context there was a profound reawakening of interest in the philosophy of mathematics, which was matched by many coming from a philosophical background (notably but not exclusively Bertrand Russell). The overlap of logic, mathematics, and language was particularly stimulating at this time, and is worked through here in a more inclusive way than hitherto; much still remains to be done. Inasmuch as this activity also overlaps with contemporary debates in psychology, such an investigation has been greatly eased by the work of Martin Kusch (1997). A significant methodological feature of the present work is its systematic attempt to reopen doors to history of science and the philosophy of mathematics.

I.1.1.1 WHAT IS IN THIS BOOK

Francis Bacon compared the activity of the historian to raising a gigantic wreck from the ocean floor.[5] All one can hope for is that fragments can be brought to the surface, that they are not too damaged in the process, and that their relative positions are not too disturbed. I have tried to bring more to the surface than has been attempted before, and in so doing I have been assisted by the work of many active historians of mathematics. I believe the main novelties of this book are these:

- It gives a new picture of the shift into modern mathematics, one that sees it as a characteristically modernist development.
- It uncovers a rich interconnected web of mathematical ideas joining the foundations of mathematics with questions at the frontiers of research.
- It does this across every branch of mathematics.
- It shows that these questions, patently internal as they are, forged links in the minds of contemporary mathematicians between mathematics, logic, philosophy, and language.

[4] As discussed in Avigad 2006, Corry 1996, Ferreirós 1999, Gray 1992a, and other writers.

[5] Bacon, *The Advancement of learning*, Book II; the comparison is made in the Everyman edition, ed. G. W. Kitchin, 73. The origins of the metaphor are eloquently discussed by Anthony Grafton in the *New York Review of Books*, October 5, 2006.

Accordingly, it brings to light aspects of the life of mathematics not previously discussed: notably, interactions with ideas of international and artificial languages, and with issues in psychology.

It integrates a number of issues too often treated separately: philosophy of mathematics in the hands of Husserl and Frege, Russell and Peirce; interactions between mathematics and physics; theories of measurement.

It opens a door to the history of the philosophy of mathematics.

In particular, it brings in the important figure of Wilhelm Wundt, who is almost always omitted.

It also reminds the historian of mathematics of a number of forgotten figures: Josiah Royce, W. B. Kempe (for once, not for his fallacious proof of the four-color theorem).

It brings in a number of philosophers hitherto marginalized in the history of mathematics (Herbart, Fries, Erdmann, and Lotze) or even completely forgotten, such as W. Tobias, G. F. Lipps, and S. Santerre.

It documents how the very names of Leibniz and Kant were profitably taken as marking major, rival, and evolving positions in logic and the philosophy of mathematics.

It takes the popularization of mathematics around 1900 seriously, as indicative of the fundamental changes mathematics was undergoing, and being seen to undergo.

It looks at the importance of the history of mathematics, which had a resurgence in the period, and considers the ways in which it was written.[6]

It raises the issue of modernism in theology, which coexisted with the situation in mathematics but incurred outright opposition and repression.

The book shows that major changes occurred almost everywhere where mathematics was done at a high level. It therefore argues that taken together all the changes in mathematics here described and the connections to other intellectual disciplines that were then animated constitute a development that cannot be described adequately as progress in this or that branch of mathematics (logic and philosophy) but must be seen as a single cultural shift, which I call mathematical modernism. Such a cultural shift was enabled by the growing autonomy of the mathematicians within the academic profession. The suggestion that the changes in mathematics around 1900 constitute a modernism, much as contemporary changes in art, literature, or music do, is intended to be provocative, because the literature on the artistic modernisms is vast and full of disagreements. I give my grounds for establishing the case on a core definition of modernism, and I believe that mathematical modernism provides a handle with which to grasp otherwise sprawling developments, as well as a store of analogies. Readers familiar with the literature on cultural modernisms are invited to bring its various approaches to the history of mathematics, and see which prove fruitful. One analogy that is pursued here is anxiety, a well-established theme in writing about modernism; I find that in mathematics, too, anxiety was a growing presence. Another theme I have adopted is that of the deep modernist interest in the history of its subject, which was often used as a way of legitimizing the new style, at least in the eyes of its adherents.

[6] Discussed briefly in Gray 1998.

I.1.1.2 THE SPREAD OF MATHEMATICAL MODERNISM

The arrival of modernism in mathematics cannot be understood as a simple event with a definite arrival time and a clear before and after. As with any modernism, there are a few important figures who are early enough to disturb the chronology: two notable examples are the German mathematician Bernhard Riemann and the American philosopher and polymath Charles Sanders Peirce. They illustrate how questions of its success and its spread, especially of a causal kind, have both an institutional and a national aspect. Riemann was centrally placed in Göttingen and managed with no more than a dozen papers to shape the work of two generations of mathematicians, first in Germany and then further afield. Peirce marginalized himself in the already mathematically marginal American scene, and as a result his impact has been greater on historians than on his contemporaries.

Mathematical modernism was strongest in settings where there already was a professional separation between mathematics and physics, and where there was an existing commitment to research for its own sake. This was the case in Germany, especially in Göttingen and Berlin, the dominant places for mathematics in the world in the late nineteenth century. It was also true in the much thinner atmosphere of the United States, whose fledgling mathematics departments took their lead from Germany where many of the staff had received their training. In France, modernism arrived with what I shall argue, following Gispert (1991), was the emergence of the first coherent group of researchers, as opposed to gifted individuals, and at a time when the cherished link between mathematics and the sciences in France was at its most attenuated. In Italy, too, it was the school around Giuseppe Peano that most thoroughly identified with modernism. In Britain, however, the movement first needed to establish a recognizable group of pure mathematicians, something that had been long delayed by the successes of applied mathematics as a research subject in Cambridge and its visible technological triumphs.

If there is a single exemplary work that ushered in modernism, it is perhaps Hilbert's *Grundlagen der Geometrie* (or, *The Foundations of Geometry*), although Dedekind's *Was sind und was sollen die Zahlen* (or, *What Are Numbers and What Are They For*) would be another strong candidate.[7] But to say this is to make clear that it is not individual works that change the world, but the messages they convey. Those messages go from people to people and, to succeed, must articulate genuine concerns that are also expressed elsewhere. The intellectual concerns therefore must be those of people able to advance them, and so those of significant groups of people with the right opportunities. The professional situation of mathematicians, in particular their relative autonomy from scientists, did not cause modernism to happen, but it enabled it and it promoted it. The social and institutional settings are therefore also necessarily considered in this account.

It is important to my argument to describe an international change in mathematics. One way I have tried to manage the large amount of material this generates is to put mathematicians in the lead. I have been most interested in tracing how mathematicians saw their subject, and the images of it that they tried to present.

[7] It is interesting to note that this was one of the books Einstein and his friends read in 1902 in their Olympia Academy, along with Poincaré's *La science et l'hypothèse*, and various other works. See Howard 2005, 36. For Dedekind's influence on Hilbert and Bernays, see Sieg and Schlimm 2005.

There are other books that could profitably reverse the focus; I hope they will be written and find this one useful. But it has been a deliberate decision to give the mathematicians more space, and if that seems to you, the reader, to distort matters unduly, then I look forward to your reply. Distortion, inevitably, there has been; not everything could be brought to the surface and choices had to be made.

Almost everyone who writes about mathematics for an audience larger than the professional mathematicians apologizes for the difficulty of the subject. I do not. Rather, I think that the issues the mathematicians confronted have a directness and an importance which can be brought out with a minimum of technicalities and a judicious use of analogy. There is no reason why mathematics should be more alarming than many a topic in the history and philosophy of science, which covers alchemy, ancient cosmology, and everything down to zoology with a wealth of detail, many implicit references to the history not only of Europe but all the world, draws in many languages, and so forth. I have tried to show that the mathematicians' questions were generally reasonable, asked for good reasons, and answered in ways that make a certain amount of sense, and because I am concerned to restore voices to the original debates, I have often let the mathematicians speak at length. I invite the reader to meet them halfway. A few passages where I judged that some readers will be helped by explicit mathematics, while others will not, are set off from the rest with an asterisk and the use of an italic font. There is also a glossary of mathematical terms at the end of the book.

I.1.1.3 A FIRST OVERVIEW

The first chapter of this book is an introductory one. It sketches out the themes of modernism in mathematics, then indicates the relation of mathematics to logic, philosophy, and language at some length, and concludes with some remarks about mathematicians and their audiences. This sets out the principal themes of the book. In the chapters that follow, I describe three broad phases in the construction of modernist mathematics: a period before modernism; the piecemeal arrival of modernism in various disciplines; and then modernism's open avowal as a new, and improved, way of doing mathematics. Within each of these chapters, the order of mathematical topics is broadly the same: geometry, mathematical analysis, algebra, philosophy, and logic. This is so that readers may navigate the material in a number of ways and be somewhat selective if they wish. I then widen the picture and look at the changing relationship between mathematics and physics, at the theory of measurement, and at mathematics as a subject fit for energetic popularization. I then widen the picture still further and consider the connections between mathematics and languages both natural and artificial, and the vexed subject of the psychology of mathematics. A strong theme of the period was investigations of how we can know mathematics, and many answers were given.

The final chapter looks at the way some of these themes worked out after the First World War. The war changed the intellectual landscape in many ways, and would require a book of its own to describe, but I feel strongly that it would be wrong to stop the stories I have to tell in 1914 or 1918. I have not attempted a full account, but I hope to have seen the debates of the earlier chapters safely home. At the end I raise, but by no means entirely answer, the question of what establishing that there was a modernist movement in mathematics enables us to say. I discuss the extent to which modernism succeeded in mathematics, and how platonism became the default philosophy of most mathematicians.

I.1.1.4 MODERNISMS

How, if at all, were the forces promoting modernism in mathematics related to the better-known modernisms of twentieth-century cultural life?[8] It is clear that some artists and writers picked up on these changes in mathematics. Henderson's treatment (1983) of modern painters' ideas of non-Euclidean geometry and the fourth dimension is exemplary. The use of quasi-mathematical forms in the sculptures of Brancusi, Gabo, and Pevsner is another indication worth analyzing. It is also possible to trace literary implications, although much remains to be done. In the substantial case of Musil, who trained as an engineer, there are many mathematical allusions in *Der Mann ohne Eigenschaften* (*The Man without Qualities*). Considering that Hermann Hesse left school with little formal education, his brief remarks about mathematics in his novel *Das Glasperlenspiel* (*The Glass Bead Game*) are strikingly accurate.

Reference to the best-established cultural modernisms (art and literature, to which one must add music) lead to the ultimately insoluble question of how the modernism in mathematics relates to them. There are certainly occasional overlaps, in the person of Hausdorff, for example (see Mehrtens's book *Moderne Sprache Mathematik*—Modern Language Mathematics). But there are significant departures, most notably in the political dispositions of the actors. Forman's and Ringer's analyses of the decline of the German mandarinate after World War I are the very opposite of an account of cultural modernizers turning in hostility on the bourgeoisie.[9] That said, the political dispositions of musicians and authors was by no means simply, or uniformly, left wing; painters seem to have taken up the most provocative political positions.

There seems to have been little direct influence of the broad cultural shifts into modernism on the practice of mathematics. It is indeed hard to see how a mathematician, drawing whatever inspiration from a cubist painting or James Joyce's *Ulysses*, could do different mathematics, although Hausdorff seems to have been open to such influences. If anything, the reverse seems to have been the case. It therefore seems more profitable to see what historians' discussions of modernism in other spheres can helpfully bring to the mathematical case.

A proper comparative historiographic analysis would be dauntingly long (although instructive), and steps have been taken in this direction by a number of writers. A few tentative, and cautionary, points can be made. The paradigm case of literature offers the same sense of a collective change adopted by a professional group, away from naturalism toward an autonomous practice. It offers the same qualitative mixture of continuities and changes, shifts in critical attitudes and in popular opinion. There is a central focus (for literature, in France; for mathematics, Germany) and the same absence of a leadership. Perhaps even more interestingly, literature offers the same range of awkward cases that obstruct tidy definitions and snappy chronologies. For Hermann Melville, read C. S. Peirce.

[8] In this connection, see Everdell's richly informative cultural history, *The First Moderns* (1997), which covers innovation across many of the arts and sciences and gives mathematics pride of place.

[9] See Forman 1971 and Ringer 1990.

Much the same can be said of painting and sculpture, and also of architecture, as Geoffrey Scott demonstrated in his *The Architecture of Humanism* (1914), but the case of music is more challenging. Here it can be argued that there was not a modernist shift at all. The standard argument that there was one is, one might say, Viennese. It has as its central figures such composers as Brahms, Bruckner, Mahler, then shifts us into Schoenberg, Berg, Webern. It is not at all obvious that the French passage centered on Debussy or that the eastern Europeans Bartok and Stravinsky fit into a modernist transition. The absence of an agreed alternative to the classical model is worrying, all the more so when the twelve-tone movement itself seems to have run out of energy. Careful writers on musical modernism take one of two positions: either they stay close to Vienna, or, as a few do, they argue that indeed modernism can be consistently defined in music.

The awkward case of music can be used to make the analysis of modernism in mathematics more precise, and has indeed been to some extent taken on board in formulating the present thesis. It should certainly counsel against any easy argument to the effect that there was a widespread shift into modernism and mathematics joined in. Rather, it reminds us that modernism is a twofold category. At one level it belongs to the actors themselves. On my definition of the terms the original protagonists must adopt recognizably modernist positions and do recognizably modernist things in a more-or-less coherent way, or my whole analysis collapses. At another level, however, the ascription of the term is the historian's. Historians tidy up the past, they identify trends that were at least partly hidden, and when they do so they have to argue for the legitimacy and the cogency of their actions. The music example is therefore a spur to an analysis of the extent to which the present account of modernism in mathematics conforms to the paradigm cases of literature and art, and how much it is problematic.

I shall argue that the mathematical case conforms rather strongly. In view of the high degree of independence of mathematics from the cases of literature and art, and the close agreement of chronology, this looks like a case of what biologists call convergent evolution. Some features of bats and birds, or ichthyosaurs and dolphins, are alike because the requirements of efficient flying or swimming promote them, but one species does not inherit them from the other. The common features in the present case are hard to discern, but the sheer size of society, its extensive diversification, the existence of cultural activities remote from immediate practical needs, and their high degree of cultural hegemony are certainly present in each.

If the case for modernism in mathematics is established, it is possible to examine a number of related historiographic issues. I have argued (in Gray 1998) that mathematicians have written the history of non-Euclidean geometry in ways that reflect the modernist perspective, and indeed to some extent distort the history accordingly. I shall also argue here that views on mathematics, and especially on the proper place of pure mathematics, affected the creation of the Greek canon. In each case, the simultaneous emergence of the modernist movement with the first modern historians of ancient mathematics invites analysis. A case treated at length by Peckhaus (1997) is particularly worth considering: the Leibniz revival of the turn of the century.[10]

[10] Here is the place to mention another book that deserves to be better known, Volkert's *Die Krise der Anschauung* (The Crisis of Intuition), 1986. It has a Lakatosian interest in monsters, and it covers a longer

I.1.2 Mehrtens's *Moderne Sprache Mathematik*

The first historian of mathematics to address the theme of modernism specifically was Herbert Mehrtens, in his book *Moderne Sprache Mathematik* of 1990. This book not only drew together a number of investigations into the history of recent mathematics, it generated several subsequent studies that included a discussion of the "modern" in their accounts of the history of the mathematics of the twentieth century. My book is close to his, and it seems appropriate to discuss it explicitly now.

Mehrtens came to his book after writing a number of social-historical accounts of mathematics, and, shamefully, when he wrote it, sources that would have enabled him to write a book on mathematics in the Nazi time were kept closed to him.[11] Only in the last two or three years has a younger generation of German historians begun to gain access to the relevant archives.[12]

The idea that mathematics underwent a profound transformation around 1900 is not new. It was the view of many of the protagonists at the time, and it was argued for intelligently by Ernst Cassirer (1910). But the first historian of mathematics to associate this view with modernism and to articulate it in a way that brought fresh insights was Mehrtens, and he did this partly through the richness of his scholarship and partly through the way he situated the transformation of mathematical practice in a wider social context. Mehrtens posited a conflict between moderns (*die Moderne*) and countermoderns (*die Gegenmoderne*) that began in the nineteenth century and which he pursued to its end in the 1960s. This is not only a characterization of two opposing tendencies that Mehrtens perceived to have been at work, it reflects an ambiguity he took to be at the heart of the very idea of the modern and which he saw in particular as essential to any adequate account of the Nazi time. He innovated further in the history of mathematics by following Foucault into discourse analysis and stressing the difference between the language of mathematics (its written texts) and the words of the mathematicians. "Modern" mathematicians claimed to work with words, not with (idealized) objects or transcendentally defined objects, and "countermoderns" objected that the very fundament of mathematics had been removed. But, since the moderns won, it is their language that dominates the discourse about mathematics, and so it was only possible to write this history, said Mehrtens, either from within the modern perspective or impossibly and ahistorically from without. This position is mandated, for Mehrtens, by his sense that modernism in mathematics is inseparable from the arrival of modernity, the social condition of the modern world, which he discusses at various points throughout his book.

The conflict opened, or rather, Mehrtens's account of it opens, with Cantor's paradise of set theory and transfinite numbers. Dedekind and Peano are discussed for their radical reformulation of the number concept. Once these dramatic novelties

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chronological sweep than this book, but where the periods overlap so too do many of the examples. It does not specifically address the topic of modernism, but by reaching to the middle of the twentieth century it shows more of the philosophical consequences of what I argue was a modernist transformation of mathematics.

[11] This also explains why Mehrtens, who speaks and writes excellent English, wrote the book in German. It remains a minor scandal that no one has translated the book, which it would still be worthwhile to do.

[12] See, among several of Volker Remmert's essays, Remmert 1999 and Remmert 2004.

have been explained, Mehrtens turned to the more vexed question of space and introduces his second modern, Bernhard Riemann, and the first of his countermoderns, Felix Klein. More precisely, Riemann opened up a separation in mathematics that grew into the distinction between the modern and the countermodern. Klein, with his deep attachment to meanings in mathematics and to the uses of mathematics in science, went one way. His insistence on the importance of intuition made him the opposite of Riemann in certain respects and, later, of David Hilbert in others, for Hilbert, of course, is Mehrtens's general director of the modernism project. The conflict grows ever more heated: what on one view brings monstrous functions and loss of control over the concepts and the purpose of the mathematical enterprise is seen on the other as scope for freedom and creativity.

Hilbert in particular has a major role, and his work is presented ironclad as a program to make all of mathematics abstract, axiomatic, and internally self-consistent. He took up the cause of Georg Cantor, and had the support of Zermelo and, albeit on his own more radical terms, Felix Hausdorff. The opposition was first represented by Klein, then Henri Poincaré, and after his death by Hermann Weyl and above all by Luitzen Brouwer. And so the scene is set for the *Grundlagen Krise* (Foundational crisis) of the 1920s, portrayed as a clash not over the foundations of mathematics, but as an upheaval in the ideas of truth, sense, object, and existence in mathematics. An upheaval, moreover, with an intense social and political context. Victory, Mehrtens suggests, went to the formalists and created in Hilbertian proof theory and Turing's analysis of calculation a paradise for machines. In the next generation the victors were exemplified by Bourbaki, and they strove for rationalization, objectivity, effectiveness, and economy of thought. The losers made a fetish of intuition, turned it into a racial category, and some, led by Ludwig Bieberbach and Oswald Teichmüller, became Nazis.

Mehrtens did not depict the clash of modern and countermodern as a simple battle between two sides. Certainly he saw it as a deep division between two views of mathematics, and one that was of a piece with the other ways in which the modern world was created. But it is only if anyone descends into Nazi ideology that Mehrtens labels them "antimodern"; modern and countermodern are yoked together. Mehrtens's critique was written in a post-Marxist spirit, influenced by such writers as Foucault. Modernization, for him, is not progress, it is also part of the catastrophe of Nazism. If he is less clear that the search for meaning and a place in the world has its good side, it is only because he sees more clearly the ways in which late capitalism is antithetical to all of that.[13]

It will be clear that I believe Mehrtens identified all the major German players in the modernization of mathematics, and a number of its leading opponents. They were mostly known to have been in place before—only the discussion of Hausdorff was strikingly new in 1990—but they are richly described and held together in a particularly interesting matrix. The polarity of modern and countermodern is a genuine dialectic: both sides need each other for their full expression. They express themselves in a way that reflects the wider cultural and social changes of modernization, and which, moreover, Mehrtens sees as thoroughly ambiguous. Given his natural wish to see modernization as part of the story of the Nazi catastrophe, this is

[13] Two further avowedly speculative chapters close the book, which go further into cultural criticism than I need to follow here.

only reasonable. Nonetheless, my disagreements with his treatment are most noticeable at this, as it might be called, outermost level.

Where Mehrtens succeeds most impressively is in his chapter on the modernization of mathematics. Here his focus on the German community and in particular on Göttingen pays off with a subtlety of analysis that is necessitated by the awkward fact that, while it is Hilbert and Göttingen that are the hub of modernity, it is Göttingen and Klein who are the very center of a powerful and diverse identification of mathematics. Klein pushed with extraordinary effect for the institutionalization of mathematics within school teaching, technology, engineering, and science, as well as a subject in its own right. A victory then for the moderns, or the countermoderns? The answer Mehrtens gives is "both." The moderns and countermoderns are inseparable parts of the mathematics profession: they offer views that are both contradictory and complementary. They came to cohabit in Göttingen and in Germany generally to their mutual advantage because the institutional base of mathematics was broad and secure and because the moderns exercised a hegemonic role within the profession.

That said, explanations of major social movements do not sit comfortably with individuals and their actions, and while Mehrtens does well to give his central figures the autonomy they have, ultimately they are small parts of a machine that seems to have a logic of its own, giving his whole account an oddly Hegelian ring. I am not sure there is more than a resonance that links mathematical modernism to the arrival of modernity, modern capitalism, and its horrific opponent, the Nazi state. We are far from knowing much about the societies we inhabit. We do not really know, I believe, which economic and social features will prove to be essential to any dynamic account of a culture, country or community, and even if we did, why should we prefer generalities of wide application to a series of more localized but richer accounts? The new capitalisms of China and India will offer endless material for studies comparing them with the American, European, and Japanese versions, but we may yet manage better without grand overarching accounts (whether or not they also serve immediate political agendas).

It is unfortunate for Mehrtens that the Nazis shared the widespread dislike of mathematics and found mathematicians irrelevant to their purposes, just as liberals, conservatives, and communists did. They tore apart the great mathematical community in Göttingen and later in all of Germany because of their hatred of Jews, but finding no racial component to mathematics had no real interest in trashing the subject itself.[14] It so happens that Bieberbach, who had few scruples about striking poses that would advance his career, chose when the time came to be a Nazi, that Teichmüller was a passionate Nazi, and that some other figures, such as Helmut Hasse, were right-wing conservatives who had no great problems with the Nazis. It so happens that Emil Artin, one of the leading modern mathematicians and very much part of Emmy Noether's group, was a cultural modernist in many other ways, just as it does that Weyl, deeply rooted as he was in German philosophy, emigrated when the Nazis came to power. But Hasse also worked closely with Emmy Noether, as did the Dutchman Bartel van der Waerden, who chose to stay in Germany from 1933 right through the war, and Oscar Perron, who also stayed but was a courageous

[14] See Segal 2003 and also the review by Remmert, http://www.h-net.org/reviews/showpdf.cgi?path=220081064463863, H-Net Reviews in Humanities and the Social Sciences.

anti-Nazi throughout, was strongly opposed to abstract modern mathematics. It ceases to be obvious that the dance of modern and countermodern lines up very well with the grander cultural clash of modern and traditional. And indeed, with the turn back to applications in mathematics in the last twenty years, which was scarcely visible in 1990, it is less clear that objects, sense, and meaning have been so thoroughly driven out of mathematics. For all these reasons, the book's tight focus on Germany is unfortunate. Its core story is at the mercy, to some extent, of the people one looks at, and its outermost level, the account of modernity, for all its sources in contemporary writing, is to an extent a priori.

If we eschew such lofty considerations, Mehrtens's book can, however, be seen to point the way to a number of interesting questions. What, for example, was the contemporary situation outside Germany? Were many mathematicians caught up in the transition to modernism, or was it the business of only a few major figures? As some of its critics have pointed out, the book says very little about the presumably changing relation between mathematics and physics at the time. It says almost nothing about relationships with philosophers and philosophy, and despite its interests in *Sprache* (language) where discourses about mathematics are concerned, it says almost nothing about the mathematical language used in research or about the contemporary situation in linguistics. All of these topics are investigated here.

In this book, I chose to investigate how broadly an account of mathematical modernism could be drawn, going beyond its heartland in Germany, and my conclusion, evidently, is that it can be. This led me to explore the ways mathematics was thought about, and the ways it was used, in neighboring disciplines, and it took me away from the sociological dimension of Mehrtens's work. So I have not used his juxtaposition of modern and countermodern, even though I appreciate the insights it brought him and the complexity it enabled him to handle. The tensions and divides Mehrtens found in Germany and I find broadly repeated elsewhere are complicated and shifting, but they function in his book because they bridge the gap between society and the individual. I see that gap differently, and his terms would be mere labels in my setting. I have also had the advantage of more than fifteen years of research in the history of mathematics and other subjects, years in which postmodernist certainties have swelled, older Marxist certainties have diminished, and now too postmodern approaches seem to be losing what charm they had. The result is a broader picture of mathematics and its connections than Mehrtens attempted, which is perhaps more suggestive of a modernist sweep but is shorn of deliberate political overtones. I might conjecture that his list of modernists and mine agree, but I must admit that his vision of modernism is not mine.

I.1.3 Disclaimers

I.1.3.1 MODERNITY

It remains to discuss the relation of the historical developments described in this book to "modernity." This is the most nebulous of the "m" words. I have attempted to make clear what I mean by modernism; modernization is the process by which something is made modern, but modernity is something like a condition. It might be the condition of a state or society, it might be people's perception of that condition. Authors of books and articles, museum and gallery curators presume on occasion that we inhabit it, or have inhabited it. It is associated with novelty, rapid change, loss of control, social disturbance, the automobile and the plane, the telephone, films, social planning, all manner of aspects of twentieth-century living in Europe and America especially. It is in danger, like the concept of time in the well-known aphorism, of being something we understand until we think about it. It is certainly unclear what aspects of modern life particularly qualify for discussion under the heading of modernity. Unfortunately, this has not stopped a stream of authors writing about it until one can have the impression that it is a well-understood term—or at least that it is well understood by everyone else, and so may be written about without further thought.[15]

I do not doubt that people's sense of themselves and their society matters. There are times in this book when I allow a glimpse of the wider world and its concerns, because that glimpse amplifies the story I wish to tell. But modernity, and Modern with a capital "m," are vast and misty generalizations I have not tried to bring to the page.

I.1.3.2 WHAT THIS BOOK IS NOT

Of the many things this book is not, some are worth mentioning. It is not a whole history, or even a package history, of mathematics in the period around 1900. It presents an argument that mathematics underwent a modernist transformation, and to that end it deals with what I believe are enough good examples to make a persuasive case. Topics only qualified for inclusion insofar as they engage with the case. Among the topics that are missing are a number which would push the case even further, but I judged excessive. For example, and with some regrets, I left out the work of Noether and her school on structural algebra in the 1920s and '30s, although it is impeccably modernist, because it would have placed even more demands on the reader.[16] For the same reason I have not pushed the theme of modern topology, so well argued by Moritz Epple (1999). It has been persuasively argued that there was a revolution in probability theory around 1900, culminating in the work of the Russian mathematician Andrei Kolmogorov, but I chose not to discuss it for the same reasons.[17] And I decided not to say much about responses to Einstein's theories of special and general relativity, not just from a feeling that this was pushing on a (fast-moving?) open door, but because it was a dramatic change in our ideas about physics rather than one in mathematics.[18] The same goes for nineteenth-century studies in the psychology of perception that questioned whether we see Euclidean or non-Euclidean space, investigations of the horopter curve, discussions of whether space or time might be discrete, and such topics.[19]

Nor do I deal in any detail with developments that might imperil a thesis I have not argued for (which would claim that modernism deeply affected every branch of mathematics). There are indeed some branches of mathematics that did

[15] Two good books that can be mentioned here are Berman 1982, and Kern 1983.

[16] For accounts of this, see Corry 1997, 2007, and McLarty 2006 and forthcoming papers.

[17] See Krüger, Daston, and Heidelberger 1989 and on Kolmogorov see Plato 1994 and Hochkirchen 1999.

[18] On the responses by philosophers, one can consult Hentschel 1990. Otherwise, the Einstein literature is huge and growing. Many interesting papers were given at the rolling conferences in the Einstein year 2005 and will doubtless find their ways into print; the series Einstein Studies is always interesting, and as good a place to start as any are Stachel 1995 and, for different reasons, Pais 1982.

[19] On these matters, see Heelan 1981.

not modernize. I give my assessment of the importance of them at the end of the book. I do not believe either that it is necessary to argue that modernism won completely, or that the survival of more traditional forms of mathematics weakens my case. Indeed, there were major mathematicians who were only partly persuaded of the modernist cause, Poincaré and Weyl among them. But I do believe that by far the major part of mathematics was transformed in a modernist way. That is for the reader to judge, and while I am offering a view it cannot be the only view, even if it is found to be convincing—the historical record is far too large for that. Instead, I have tried to make it clear where I stand, and to survey enough of mathematics to establish how sweeping the modernist transformation was. Modernism is many things, and if taking some interpretation of it in, say, the history of music proves a productive spur to asking worthwhile questions in the history of mathematics, then that will be some justification for emphasising it so strongly here. I have argued for the existence of substantial links between mathematics and physics, philosophy, logic, psychology, and even the study of language in the period, in the belief that it will be valuable if these links survive further analysis and generate further research.

However, I claim no expertise in those areas. The relevant parts of this book offer a view, as it were, outward from the history of mathematics. I hope experts in those domains will be enabled to offer views outward from where they stand and toward mathematics, thus bringing about a dialogue between history of mathematics and other parts of the history of science and intellectual history.

Finally, I do not claim that the modernization of mathematics was part of a broader cultural push, animated by concurrent changes in the arts. I do claim that the changes were similar in kind and were helped along by a growing diversity and specialization in all walks of cultural and intellectual life. But the mathematical ones described here and the better-known artistic ones happened independently.

I.1.3.3 PLATO'S GHOST

I surely do not need to say that I am far from believing that I have brought this book to perfection. Yeats's poem, with its evocation of hard work leading to worldly success, culminating in a claim for perfection that Plato's ghost only too plausibly denies, stands at the front of this book because it fits the mathematical scene I have tried to describe here remarkably well. Not only is Plato the presiding ancestor of mathematicians, and Platonism commonly supposed to be their default philosophy, philosophers too work in his shadow. Hard work, for many of those discussed here, did lead to worldly success and to lasting fame. More pertinently, the mathematicians often did believe that they had brought something to perfection, and written, at last, the definitive account of this or that aspect of mathematics, or of its philosophy, or its relation to logic or the natural world. Each time, they were mistaken. They may have penetrated deeper than anyone before them, but only to make deeper mistakes. Who better than the ghost of Plato to ask them if indeed it was all over, if nothing more needed to be said?

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I.2 Some Mathematical Concepts

The nineteenth and early twentieth centuries saw the introduction of a number of significant novel ideas that were to be of major importance in the years after 1880. For convenience, a number of these that will occupy us later in the book are noted here, with an indication of their value. For brevity the list is merely indicative, loosely thematic, and not chronological.

In ***Geometry***

Non-Euclidean geometry. A consistent metrical geometry different from Euclidean geometry, its discovery showed that physical space is not necessarily described by Euclidean geometry.

Projective (non-metrical) geometry. A geometry without metrical concepts (angles, lengths, areas, etc.) only those of incidence; more fundamental than Euclidean or non-Euclidean geometry.

The Kleinian view of geometry (Erlangen Program). A hierarchy of geometries within projective geometry; the first clear association of groups of transformations with a geometry, elucidating the idea of different types of geometry.

Axiomatic geometry. A key place for the implicit definition of mathematical terms that are otherwise meaningless; axiomatization became a creative method for producing novel mathematical structures.

n-dimensional geometry. Geometry (Euclidean, non-Euclidean, projective, or of any other kind, such as differential geometry) in any number of dimensions, not merely two or three.

In ***Analysis***

The natural numbers. Defined mathematically by Dedekind, Peano, and Frege.

The real numbers. Defined mathematically by Dedekind and Cantor (and others) in terms of infinite sets.

Continuous but nowhere differentiable functions. Highly counterintuitive, they show that mathematical objects, and therefore physical processes, need not be analytic or obey the naive theory of the calculus.

Continuous curve. A naive concept that admits different, more precise definitions, and allows for counterintuitive examples, such as curves with no length and curves with finite area. The challenge was to match intuitions with proofs: even the Jordan curve theorem (a closed curve which does not cross itself has an inside and an outside) is surprisingly hard to prove.

Measure theory. A fundamental reformulation of the theory of integration independent of the concept of area.

Topology. A branch of mathematics underlying analysis (especially but not exclusively subsets of the real line), complex function theory, and geometry; the abstract setting for the study of continuity and properties invariant under continuous change.

In *Algebra*

Group theory. A group is a set with a composition law; group theory is an abstract setting for generalizing addition (of numbers), composition (of functions), and successive transformations in geometry. Finite groups were shown to have various distinguishing properties, opening the way to a structural classification of groups.

Algebraic number theory. The study of novel types and classes of integers and their properties, including prime and ideal factors.

Galois theory. A theory of groups and fields that arose from earlier studies of the solution of polynomial equations and became the paradigm structural theory in modern algebra.

In *Set Theory*

Set theory. As a foundation of analysis and algebraic number theory, it was introduced by Dedekind. The discovery that there are both countable and uncountable sets, opening the way to a theory of infinite sets, is due to Cantor.

Ordinal and cardinal numbers. The generalization of familiar concepts to infinite sets of arbitrary sizes. Their properties are described by the theory of *transfinite arithmetic*.

In *Logic*

Intuitionistic logic. A logic characterized by the claim that the law of the excluded middle (a statement is either true or false) does not necessarily apply when infinite sets are under consideration.

Formal languages. Introduced to make logic more precise and to handle distinctions almost inaccessible in natural languages.

Proof theory. A mathematical analysis of mathematical arguments designed to provide a rigorous account of all mathematics.

1st- and 2nd- order logics. A logic is first order if its quantifiers (all, some, none) refer only to elements of a set of objects, but second order if they refer to sets of objects.