INTRODUCTION

Turning on the Light

A FEW YEARS AGO the PBS program *Nova* featured an interview with Andrew Wiles. Wiles is the Princeton mathematician who gave the final resolution to what was perhaps the most famous mathematical problem of all time—the Fermat conjecture. The solution to Fermat was Wiles's life ambition. "When he revealed a proof in that summer of 1993, it came at the end of seven years of dedicated work on the problem, a degree of focus and determination that is hard to imagine."[1] He said of this period in his life, "I carried this thought in my head basically the whole time. I would wake up with it first thing in the morning, I would be thinking about it all day, and I would be thinking about it when I went to sleep. Without distraction I would have the same thing going round and round in my mind."[2] In the *Nova* interview, Wiles reflects on the process of doing mathematical research:

> Perhaps I can best describe my experience of doing mathematics in terms of a journey through a dark unexplored mansion. You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture, but gradually you learn where each piece of furniture is. Finally after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated. You can see exactly where you were. Then you move into the next room and spend another six months in the dark. So each of these breakthroughs, while sometimes they're momentary, sometimes over a period of a day or two, they are the culmination of—and couldn't exist without—the many months of stumbling around in the dark that precede them.

This is the way it is! This is what it means to do mathematics at the highest level, yet when people talk about mathematics, the elements that make up Wiles's description are missing. What is missing is the creativity of mathematics—the essential dimension without which there is no mathematics. Ask people about mathematics and they will talk about arithmetic, geometry, or statistics, about mathematical techniques or theorems they have learned. They may talk about the logical structure of mathematics, the nature of mathematical arguments. They may be impressed with the precision of mathematics, the way in which things in mathematics are either right or wrong. They may even feel that mathematics captures "the truth," a truth that goes beyond individual bias or superstition, that is the same for all people at all times. Rarely, however, do most people mention the "doing" of mathematics when they talk about mathematics.

Unfortunately, many people talk about and use mathematics despite the fact that the light switch has never been turned on *for them*. They are in a position of knowing where the furniture is, to use Wiles's metaphor, but they are still in the dark. Most books about mathematics are written with the aim of showing the reader where the furniture is located. They are written from the point of view of someone for whom the light switch has been turned on, but they rarely acknowledge that without turning on the switch the reader will forever remain in the dark. It is indeed possible to know where the furniture is located without the light switch having ever been turned on. "Locating the furniture" is a relatively straightforward, mechanical task, but "turning on the light" is of another order entirely. One can prepare for it, can set the stage, so to speak, but one can neither predict nor program the magical moment when things "click into place." This book is written in the conviction that we need to talk about mathematics in a way that has a place for the darkness as well as the light and, especially, a place for the mysterious process whereby the light switch gets turned on.

Almost everyone uses mathematics of some kind almost every day, and yet, for most people, their experience of mathematics is the experience of driving a car—you know that if you press on the gas the car will go forward, but you don't have any idea why. Thus, most people are in the dark with respect to the mathematics that they use. This group includes untold numbers of students in mathematics classrooms in elementary schools, high schools, and universities around the world. It even includes intelligent people who use fairly sophisticated mathematics in their personal or professional lives. Their position, with respect to the mathematics they use every day, is like that of the person in the dark room. They may know where certain pieces of furniture are located, but the light switch has not been turned on. They may not even know about the existence of light switches. Turning on the light switch, the "aha!" experience, is not something that is restricted to the creative research mathematician. Every act of understanding involves the turning on of a light switch. Conversely, if the light has not gone on, then one can be pretty certain that there is no understanding.

If we wish to talk about mathematics in a way that includes acts of creativity and understanding, then we must be prepared to adopt a different point of view from the one in most books about mathematics and science. This new point of view will examine the processes through which new mathematics is created and old mathematics is understood. When mathematics is identified with its content, it appears to be timeless. The new viewpoint will emphasize the dynamic character of mathematics—how it is created and how it evolves over time. In order to arrive at this viewpoint, it will be necessary to reexamine the role of logic and rigor in mathematics. This is because it is the formal dimension of mathematics that gives it its timeless quality. The other dimension—the developmental—will emerge from an examination of situations that have spawned the great creative advances in mathematics. What are the mechanisms that underlie these advances? Do they arise out of the formal structure or is there some other factor at work?

This new point of view will turn our attention away from the content of the great mathematical theories and toward questions that are unresolved, that are in flux and problematic. The problematic is the context within which mathematical creativity is born. People are so motivated to find answers that they sometimes neglect the boundaries of the known, where matters have not settled down, where questions are more meaningful than answers. This book turns matters around; that is, the problematic is regarded as the essence of what is going on. The consequence is that much of the traditional way of looking at mathematics is radically changed. Actual mathematical content does not change, but the point of view that is developed with respect to that content changes a great deal. And the implications of this change of viewpoint will be enormous for mathematics, for science, and for all the cultural projects that get their worldview, directly or indirectly, from mathematics.

Now, not everyone thinks that such a change in viewpoint is necessary or even desirable. There are eminent spokespeople for an opposing point of view, one that maintains that "The ultimate goal of mathematics is to eliminate all need for intelligent thought."[3] This viewpoint, one that is very influential in the artificial intelligence community, is that progress is achieved by turning creative thought into algorithms that make further creativity unnecessary. What is an algorithm? Webster's New Collegiate Dictionary defines it to be "a procedure for solving a mathematical problem in a finite number of steps that frequently involves the repetition of an operation." So an algorithm breaks down some complex mathematical task into a sequence of more elementary tasks, some of which may be repeated many times, and applies these more elementary operations in a step-by-step manner. We are all familiar with the simple mathematical algorithms for addition or multiplication that we learned in elementary school. But algorithms are basic to all of computer programming, from Google's search procedures to Amazon's customer recommendations.

Today the creation of algorithms to solve problems is extremely popular in fields as diverse as finance, communications, and molecular biology. Thus the people I quoted in the above paragraph believe that the essence of what is going on in mathematics is the creation of algorithms that make it *unnecessary* to turn on the light switch. There is no question that some of the greatest advances in our culture involve the creation of algorithms that make calculations into mechanical acts. Because the notion of an algorithm underlies all of computer programming, algorithms are given a physical presence in computers and other computational devices. The evident successes of the computer revolution have moved most people's attention from the creative breakthroughs of the computer scientists and others who create the algorithms to the results of these breakthroughs. We lose track of the "how" and the "why" of information technology because we are so entranced with what these new technologies can do for us. We lose track of the human dimension of these accomplishments and imagine that they have a life independent of human creativity and understanding.

The point of view taken in what follows is that the experience Wiles describes is the essence of mathematics. It is of the utmost importance for mathematics, for science, and beyond that for our understanding of human beings, to develop a way of talking about mathematics that contains the entire mathematical experience, not just some formalized version of the results of that experience. It is not possible to do justice to mathematics, or to explain its importance in human culture, by separating the content of mathematical theory from the process through which that theory is developed and understood.

Different Ways of Using the Mind

Mathematics has something to teach us, all of us, whether or not we like mathematics or use it very much. This lesson has to do with thinking, the way we use our minds to draw conclusions about the world around us. When most people think about mathematics they think about the logic of mathematics. They think that mathematics is characterized by a certain mode of using the mind, a mode I shall henceforth refer to as "algorithmic." By this I mean a step-by-step, rule-based procedure for going from old truths to new ones through a process of logical reasoning. But is this really the only way that we think in mathematics? Is this the way that new mathematical truths are brought into being? Most people are not aware that there are, in fact, other ways of using the mind that are at play in mathematics. After all, where do the new ideas come from? Do they come from logic or from algorithmic processes? In mathematical research, logic is used in a most complex way, as a constraint on what is possible, as a goad to creativity, or as a kind of verification device, a way of checking whether some conjecture is valid. Nevertheless, the creativity of mathematics—the turning on of the light switch—cannot be reduced to its logical structure.

Where *does* mathematical creativity come from? This book will point toward a certain kind of situation that produces creative insights. This situation, which I call "ambiguity," also provides a mechanism for acts of creativity. The "ambiguous" could be contrasted to the "deductive," yet the two are not mutually exclusive. Strictly speaking, the "logical" should be contrasted to the "intuitive." The ambiguous situation may contain elements of the logical and the intuitive, but it is not restricted to such elements. An ambiguous situation may even involve the contradictory, but it would be wrong to say that the ambiguous is necessarily illogical.

Of course, it is not my intention to produce some sort of recipe for creativity. On the contrary, my argument is precisely that such a recipe cannot exist. This book directs our attention toward the problematic and the ambiguous because these situations so often form the contexts that produce creative insights.

Normally, the development of mathematics is reconstructed as a rational flow from assumptions to conclusions. In this reconstruction, the problematic is avoided, deleted, or at best minimized. What is radical about the approach in this book is the assertion that creativity and understanding arise out of the problematic, out of situations I am calling "ambiguous." Logic abhors the ambiguous, the paradoxical, and especially the contradictory, but the creative mathematician welcomes such problematic situations because they raise the question, "What is going on here?" Thus the problematic signals a situation that is worth investigating. The problematic is a potential source of new mathematics. How a person responds to the problematic tells you a great deal about them. Does the problematic pose a challenge or is it a threat to be avoided? It is the answer to this question, not raw intelligence, that determines who will become the successful researcher or, for that matter, the successful student.

The Importance of Talking about Mathematics

In preparing to write this introduction, I went back to reread the introductory remarks in that wonderful and influential book, *The Mathematical Experience*. I was struck by the following paragraph:

> I started to talk to other mathematicians about proof, knowledge, and reality in mathematics and I found that my situation of confused uncertainty was typical. But I also found a remarkable thirst for conversation and discussion about our private experiences and inner beliefs.

I've had the same experience. People want to talk about mathematics but they don't. They don't know how. Perhaps they don't have the language, perhaps there are other reasons. Many mathematicians usually don't talk about mathematics because talking is not their thing—their thing is the "doing" of mathematics. Educators talk about teaching mathematics but rarely about mathematics itself. Some educators, like scientists, engineers, and many other professionals who use mathematics, don't talk about mathematics because they feel that they don't possess the expertise that would be required to speak intelligently about mathematics. Thus, there is very little discussion about mathematics going on. Yet, as I shall argue below, there is a great need to think about the nature of mathematics.

What is the audience for a book that unifies the content with the "doing" of mathematics? Is it restricted to a few interested mathematicians and philosophers of science? This book is written in the conviction that what is going on in mathematics is important to a much larger group of people, in fact to everyone who is touched one way or another by the "mathematization" of modern culture. Mathematics is one of the primary ways in which modern technologically based culture understands itself and the world around it. One need only point to the digital revolution and the advent of the computer. Not only are these new technologies reshaping the world, but they are also reshaping the way in which we understand the world. And all these new technologies stand on a mathematical foundation.

Of course the "mathematization" of culture has been going on for thousands of years, at least from the times of the ancient Greeks. Mathematization involves more than just the practical uses of arithmetic, geometry, statistics, and so on. It involves what can only be called a culture, a way of looking at the world. Mathematics has had a major influence on what is meant by "truth," for example, or on the question, "What is thought?" Mathematics provides a good part of the cultural context for the worlds of science and technology. Much of that context lies not only in the explicit mathematics that is used, but also in the assumptions and worldview that mathematics brings along with it.

The importance of finding a way of talking about mathematics that is not obscured by the technical difficulty of the subject is perhaps best explained by an analogy with a similar discussion for physics and biology. Why should nonphysicists know something about quantum mechanics? The obvious reason is that this theory stands behind so much modern technology. However, there is another reason: quantum mechanics contains an implicit view of reality that is so strange, so at variance with the classical notions that have molded our intuition, that it forces us to reexamine our preconceptions. It forces us to look at the world with new eyes, so to speak, and that is always important. As we shall see, the way in which quantum mechanics makes us look at the world—a phenomenon called "complementarity"—has a great deal in common with the view of mathematics that is being proposed in these pages.

Similarly, it behooves the educated person to attempt to understand a little of modern genetics not only because it provides the basis for the biotechnology that is transforming the world, but also because it is based on a certain way of looking at human nature. This could be summarized by the phrase, "You are your DNA" or, more explicitly, "DNA is nothing less than a blueprint—or, more accurately, an algorithm or instruction manual—for building a living, breathing, thinking human being."[4] Molecular biology carries with it huge implications for our understanding of human nature. To what extent are human beings biological machines that carry their own genetic blueprints? It is vital that thoughtful people, scientists and nonscientists alike, find a way to address the metascientific questions that are implicit in these new scientific and technological advances. Otherwise society risks being carried mindlessly along on the accelerating tide of technological innovations. The question about whether a human being is mechanically determined by their blueprint of DNA has much in common with the question raised by our approach to mathematics, namely, "Is mathematical thought algorithmic?" or "Can a computer do mathematics?"

The same argument that can be made for the necessity to closely examine the assumptions of physics and molecular biology can be made for mathematics. Mathematics has given us the notion of "proof" and "algorithm." These abstract ideas have, in our age, been given a concrete technological embodiment in the form of the computer and the wave of information technology that is inundating our society today. These technological devices are having a significant impact on society at all levels. As in the case of quantum mechanics or molecular biology, it is not just the direct impact of information technology that is at issue, but also the impact of this technological revolution on our conception of human nature. How are we to think about consciousness, about creativity, about thought? Are we all biological computers with the brain as hardware and the "mind" defined to be software? Reflecting on the nature of mathematics will have a great deal to contribute to this crucial discussion.

The three areas of modern science that have been referred to above all raise questions that are interrelated. These questions involve, in one way or another, the intellectual models—metaphors if you will—that are implicit in the culture of modern science. These metaphors are at work today molding human beings' conceptions of certain fundamental human attributes. It is important to bring to conscious awareness the metascientific assumptions that are built into these models, so that people can make a reasonable assessment of these assumptions. Is a machine, even a sophisticated machine like a computer, a reasonable model for thinking about human beings? Most intelligent people hesitate even to consider these questions because they feel that the barrier of scientific expertise is too high. Thus, the argument is left to the "experts," but the fact is that the "experts" do not often stop to consider such questions for two reasons: first, they are too busy keeping up with the accelerating rate of scientific development in their field to consider "philosophical" questions; second, they are "insiders" to their fields and so have little inclination to look at their fields from the outside. In order to have a reasonable discussion about the worldview implicit in certain scientific disciplines, it would therefore be necessary to carry a dual perspective; to be inside and outside simultaneously. In the case of mathematics, this would involve assuming a perspective that arises from mathematical practice—from the actual "doing" of mathematics—as well as looking at mathematics as a whole as opposed to discussing some specific mathematical theory.

INTRODUCTION

The "Light of Reason" or the "Light of Ambiguity"?

What is it that makes mathematics mathematics? What are the precise characteristics that make mathematics into a discipline that is so central to every advanced civilization, especially our own? Many explanations have been attempted. One of these sees mathematics as the ultimate in rational expression; in fact, the expression "the light of reason" could be used to refer to mathematics. From this point of view, the distinguishing aspect of mathematics would be the precision of its ideas and its systematic use of the most stringent logical criteria. In this view, mathematics offers a vision of a purely logical world. One way of expressing this view is by saying that the natural world obeys the rules of logic and, since mathematics is the most perfectly logical of disciplines, it is not surprising that mathematics provides such a faithful description of reality. This view, that the deepest truth of mathematics is encoded in its formal, deductive structure, is definitely *not* the point of view that this book assumes. On the contrary, the book takes the position that the logical structure, while important, is insufficient even to begin to account for what is really going on in mathematical practice, much less to account for the enormously successful applications of mathematics to almost all fields of human thought.

This book offers another vision of mathematics, a vision in which the logical is merely one dimension of a larger picture. This larger picture has room for a number of factors that have traditionally been omitted from a description of mathematics and are translogical—that is, beyond logic—though not illogical. Thus, there is a discussion of things like ambiguity, contradiction, and paradox that, surprisingly, also have an essential role to play in mathematical practice. This flies in the face of conventional wisdom that would see the role of mathematics as eliminating such things as ambiguity from a legitimate description of the worlds of thought and nature. As opposed to the formal structure, what is proposed is to focus on the central ideas of mathematics, to take ideas—instead of axioms, definitions, and proofs—as the basic building blocks of the subject and see what mathematics looks like when viewed from that perspective.

The phenomenon of ambiguity is central to the description of mathematics that is developed in this book. In his description of his own personal development, Alan Lightman says, "Mathematics contrasted strongly with the ambiguities and contradictions in people. The world of people had no certainty or logic."[5] For him, mathematics is the domain of certainty and logic. On the other hand, he is also a novelist who "realized that the ambiguities and complexities of the human mind are what give fiction and perhaps all art its power." This is the usual way that people divide up the arts from the sciences: ambiguity in one, certainty in the other. I suggest that mathematics is also a human, creative activity. As such, ambiguity plays a role in mathematics that is analogous to the role it plays in art—it imbues mathematics with depth and power.

Ambiguity is intrinsically connected to creativity. In order to make this point, I propose a definition of ambiguity that is derived from a study of creativity.[6] The description of mathematics that is to be sketched in this book will be a description that is grounded in mathematical practice—what mathematicians actually do—and, therefore, must include an account of the great creativity of mathematics. We shall see that many creative insights of mathematics arise out of ambiguity, that in a sense the deepest and most revolutionary ideas come out of the most profound ambiguities. Mathematical ideas may even arise out of contradiction and paradox. Thus, eliminating the ambiguous from mathematics by focusing exclusively on its logical structure has the unwanted effect of making it impossible to describe the creative side of mathematics. When the creative, open-ended dimension is lost sight of, and, therefore, mathematics becomes identified with its logical structure, there develops a view of mathematics as rigid, inflexible, and unchanging. The truth of mathematics is mistakenly seen to come exclusively from a rigid, deductive structure. This rigidity is then transferred to the domains to which mathematics is applied and to the way mathematics is taught, with unfortunate consequences for all concerned.

Thus, there are two visions of mathematics that seem to be diametrically opposed to one another. These could be characterized by emphasizing the "light of reason," the primacy of the logical structure, on the one hand, and the light that Wiles spoke of, a creative light that I maintain often emerges out of ambiguity, on the other (this is itself an ambiguity!). My job is to demonstrate how mathematics transcends these two opposing views: to develop a picture of mathematics that includes the logical and the ambiguous, that situates itself equally in the development of vast deductive systems of the most intricate order and in the birth of the extraordinary leaps of creativity that have changed the world and our understanding of the world.

This is a book about mathematics, yet it is not your average mathematics book. Even though the book contains a great deal of mathematics, it does not systematically develop any particular mathematical subject. The subject is mathematics as a whole—its methodology and conclusions, but also its culture. The book puts forward a new vision of what mathematics is all about. It concerns itself not only with the culture of mathematics in its own right, but also with the place of mathematics in the larger scientific and general culture.

The perspective that is being developed here depends on finding the right way to think about mathematical rigor, that is, logical, deductive thought. Why is this way of thinking so attractive? In our response to reason, we are the true descendents of the Greek mathematicians and philosophers. For us, as for them, rational thought stands in contrast to a world that is all too often beset with chaos, confusion, and superstition. The "dream of reason" is the dream of order and predictability and, therefore, of the power to control the natural world. The means through which we attempt to implement that dream are mathematics, science, and technology. The desired end is the emergence of clarity and reason as organizational principles of the entire cosmos, a cosmos that of course includes the human mind. People who subscribe to this view of the world might think that it is the role of mathematics to eliminate ambiguity, contradiction, and paradox as impediments to the success of rationality. Such a view might well equate mathematics with its formal, deductive structure. This viewpoint is incomplete and simplistic. When applied to the world in general, it is mistaken and even dangerous. It is dangerous because it ignores one of the most basic aspects of human nature—in mathematics or elsewhere—our aesthetic dimension, our originality and ability to innovate. In this regard let us take note of what the famous musician, Leonard Bernstein, had to say: "ambiguity . . . is one of art's most potent aesthetic functions. The more ambiguous, the more expressive."[7] His words apply not only to music and art, but surprisingly also to science and mathematics. In mathematics, we could amend his remarks by saying, "the more ambiguous, the more potentially original and creative."

If one wishes to understand mathematics and plumb its depths, one must reevaluate one's position toward the ambiguous (as I shall define it in Chapter 1) and even the paradoxical. Understanding ambiguity and its role in mathematics will hint at a new kind of organizational principle for mathematics and science, a principle that includes classical logic but goes beyond it. This new principle will be *generative*—it will allow for the dynamic development of mathematics. As opposed to the static nature of logic with its absolute dichotomies, a generative principle will allow for the existence of mathematical creativity, be it in research or in individual acts of understanding. Thus "ambiguity" will force a reevaluation of the essence of mathematics.

Why is it important to reconsider mathematics? The reasons vary from those that are internal to the discipline itself to those that are external and affect the applications of mathematics to other fields. The internal reasons include developing a description of mathematics, a philosophy of mathematics if you will, that is consistent with mathematical practice and is not merely a set of a priori beliefs. Mathematics is a *human* activity; this is a triumph, not a constraint. As such, it is potentially accessible to just about everyone. Just as most people have the capacity to enjoy music, everyone has some capacity for mathematics appreciation. Yet most people are fearful and intimidated by mathematics. Why is that? Is it the mathematics itself that is so frightening? Or is it rather the way in which mathematics is viewed that is the problem?

Beyond the valid "internal" reasons to reconsider the nature of mathematics, even more compelling are the external reasons—the impact that mathematics has, one way or another, on just about every aspect of the modern world. Since mathematics is such a central discipline for our entire culture, reevaluating what mathematics is all about will have many implications for science and beyond, for example, for our conception of the nature of the human mind itself. Mathematics provided humanity with the ideal of reason and, therefore, a certain model of what thinking is or should be, even what a human being should be. Thus, we shall see that a close investigation of the history and practice of mathematics can tell us a great deal about issues that arise in philosophy, in education, in cognitive science, and in the sciences in general. Though I shall endeavor to remain within the boundaries of mathematics, the larger implications of what is being said will not be ignored.

Mathematics is one of the most profound creations of the human mind. For thousands of years, the content of mathematical theories seemed to tell us something profound about the nature of the natural world—something that could not be expressed in any way other than the mathematical. How many of the greatest minds in history, from Pythagoras to Galileo to Gauss to Einstein, have held that "God is a mathematician." This attitude reveals a reverence for mathematics that is occasioned by the sense that nature has a secret code that reveals her hidden order. The immediate evidence from the natural world may seem to be chaotic and without any inner regularity, but mathematics reveals that under the surface the world of nature has an unexpected simplicity—an extraordinary beauty and order. There is a mystery here that many of the great scientists have appreciated. How does mathematics, a product of the human intellect, manage to correspond so precisely to the intricacies of the natural world? What accounts for the "extraordinary effectiveness of mathematics"?

Beyond the content of mathematics, there is the *fact of mathematics*. What is mathematics? More than anything else, mathematics is a way of approaching the world that is absolutely unique. It cannot be reduced to some other subject that is more elementary in the way that it is claimed that chemistry can be reduced to physics. Mathematics is irreducible. Other subjects may use mathematics, may even be expressed in a totally mathematical form, but mathematics has no other subject that stands in relation to it in the way that it stands in relation to other subjects. Mathematics is a *way of knowing*—a unique way of knowing. When I wrote these words I intended to say "a unique *human* way of knowing." However, it now appears that human beings share a certain propensity for number with various animals.[8] One could make an argument that a tendency to see the world in a mathematical way is built into our developmental structure, hard-wired into our brains, perhaps implicit in elements of the DNA structure of our genes. Thus mathematics is one of the most basic elements of the natural world.

From its roots in our biology, human beings have developed mathematics as a vast cultural project that spans the ages and all civilizations. The nature of mathematics gives us a great deal of information, both direct and indirect, on what it means to be human. Considering mathematics in this way means looking not merely at the content of individual mathematical theories, but at mathematics as a whole. What does the nature of mathematics, viewed globally, tell us about human beings, the way they think, and the nature of the cultures they create? Of course, the latter, global point of view can only be seen clearly by virtue of the former. You can only speak about mathematics with reference to actual mathematical topics. Thus, this book contains a fair amount of actual mathematical content, some very elementary and some less so. The reader who finds some topic obscure is advised to skip it and continue reading. Every effort has been made to make this account self-contained, yet this is not a mathematics textbook—there is no systematic development of any large area of mathematics. The mathematics that is discussed is there for two reasons: first, because it is intrinsically interesting, and second, because it contributes to the discussion of the nature of mathematics in general. Thus, a subject may be introduced in one chapter and returned to in subsequent chapters.

It is not always appreciated that the story of mathematics is also a story about what it means to be human—the story of beings blessed (some might say cursed) with self-consciousness and, therefore, with the need to understand the natural world and themselves. Many people feel that such a human perspective on mathematics would demean it in some way, diminish its claim to be revealing absolute, objective truth. To anticipate the discussion in Chapter 8, I shall claim that mathematical truth exists, but is not to be found in the content of any particular theorem or set of theorems. The intuition that mathematics accesses the truth is correct, but not in the manner that it is usually understood. The truth is to be found more in the fact than in the content of mathematics. Thus it is consistent, in my view, to talk simultaneously about the truth of mathematics and about its contingency.

The truth of mathematics is to be found in its human dimension, not by avoiding this dimension. This human story involves people who find a way to transcend their limitations, about people who dare to do what appears to be impossible and *is* impossible by any reasonable standard. The impossible is rendered possible through acts of genius—this is the very definition of an act of genius, and mathematics boasts genius in abundance. In the aftermath of these acts of genius, what was once considered impossible is now so simple and obvious that we teach it to children in school. In this manner, and in many others, mathematics is a window on the human condition. As such, it is not reserved for the initiated, but is accessible to all those who have a fascination with exploring the common human potential.

We do not have to look very far to see the importance of mathematics in practically every aspect of contemporary life. To begin with, mathematics is the language of much of science. This statement has a double meaning. The normal meaning is that the natural world contains patterns or regularities that we call scientific laws and mathematics is a convenient language in which to express these laws. This would give mathematics a descriptive and predictive role. And yet, to many, there seems to be something deeper going on with respect to what has been called "the unreasonable effectiveness of mathematics in the natural sciences."[9] Certain of the basic constructs of science cannot, in principle, be separated from their mathematical formulation. An electron *is* its mathematical description via the Schrödinger equation. In this sense, we cannot see any deeper than the mathematics. This latter view is close to the one that holds that there exists a mathematical, Platonic substratum to the real world. We cannot get closer to reality than mathematics because the mathematical level *is* the deepest level of the real. It is this deeper level that has been alluded to by the brilliant thinkers that were mentioned above. This deeper level was also what I meant by calling mathematics irreducible.

Our contemporary civilization has been built upon a mathematical foundation. Computers, the Internet, CDs, and DVDs are all aspects of a digital revolution that is reshaping the world. All these technologies involve representing the things we see and hear, our knowledge, and the contents of our communications in digital form, that is, reducing these aspects of our lives to a common numerical basis. Medicine, politics, and social policy are all increasingly expressed in the language of the mathematical and statistical sciences. No area of modern life can escape from this mathematization of the world.

If the modern world stands on a mathematical foundation, it behooves every thoughtful, educated person to attempt to gain some familiarity with the world of mathematics. Not only with some particular subject, but with the culture of mathematics, with the manner in which mathematicians think and the manner in which they see this world of their own creation.

Why Am I Writing This Book?

What is my purpose in writing this book? Where do the ideas come from? Obviously, I think that the ideas are important because the point of view from which they are written is unusual. But putting aside the content of the book for a moment, there is also an important personal reason for me. This book weaves together two of the most important strands in my life. One strand is mathematics: I have spent a good part of the last forty years doing the various things that a university mathematician does—teaching, research, and administration. When I look back at my motivation for going into mathematics, what appealed to me was the clarity and precision of the kind of thinking that doing mathematics called for. However, clarity was not a sufficient condition for doing research. Research required something else—a need to understand. This need to understand often took me into realms of the obscure and the problematic. How, I asked myself, can one find one description of mathematics that unifies the logical clarity of formal mathematics with the sense of obscurity and flux that figures so prominently in the doing and learning of mathematics?

The second strand in my life was and is a strenuous practice of Zen Buddhism. Zen helped me confront aspects of my life that went beyond the logical and the mathematical. Zen has the reputation for being antilogical, but that is not my experience. My experience is that Zen is not confined to logic; it does not see logic as having the final word. Zen demonstrates that there is a way to work with situations of conflict, situations that are problematic from a normal, rational point of view. The rational, for Zen, is just another point of view. Paradox, in Zen, is used constructively as a way to direct the mind to subverbal levels out of which acts of creativity arise.

I don't think that Zen has anything to say about mathematics per se, but Zen contains a viewpoint that is interesting when applied to mathematics. It is a viewpoint that resonates with many interesting things that are happening in our culture. They all involve moving away from an "absolutist" position, whether this means distrust of all ideologies or in rejection of "absolute, objective, and timeless truth." For me, this means that ambiguity, contradiction, and paradox are an essential part of mathematics—they are the things that keep it changing and developing. They are the motor of its endless creativity.

In the end, I found that these two strands in my life—mathematics and Zen—fit together very well indeed. I expect that you, the reader, will find this voyage beyond the boundaries of the rational to be challenging because it requires a change in perspective; but for that very reason I hope that you will find it exciting. Ambiguity opens up a world that is never boring because it is a world of continual change and creativity.

The Structure of the Book

The book is divided into three sections. The first, "The Light of Ambiguity," begins by introducing the central notion of ambiguity. Actually one could look at the entire book as an exploration of the role of ambiguity in mathematics, as an attempt to come to grips with the elusive notion of ambiguity. In order to highlight my contention that the ambiguous always has a component of the problematic about it, I spend a couple of chapters talking about contradiction and paradox in mathematics. These chapters also enable me to build up a certain body of mathematical results so as to enable even readers who are a little out of touch with mathematics to get up to speed, so to say.

The second section is called "The Light as Idea." It discusses the nature of ideas in mathematics—especially those ideas that arise out of situations of ambiguity. Of course the creative process is intimately tied to the birth and the processing of mathematical ideas. Thus thinking about ideas as the fundamental building blocks of mathematics (as opposed to the logical structure, for example) pushes us toward a reevaluation of just what mathematics is all about. This section demonstrates that even something as problematic as a paradox can be the source of a productive idea. Furthermore, I go on to claim that some of the most profound ideas in mathematics arise out of situations that are characterized not by logical harmony but by a form of extreme conflict. I call the ideas that emerge out of these extreme situations "great ideas," and a good deal of the book involves a discussion of such seminal ideas.

The third section, "The Light and the Eye of the Beholder," considers the implications of the point of view that has been built up in the first two sections. One chapter is devoted to a discussion of the nature of mathematical truth. Is mathematics absolutely true in some objective sense? For that matter, what do we mean by "objectivity" in mathematics? Thinkers of every age have attested to the mystery that lies at the heart of the relationship between mathematics and truth. My "ambiguous" approach leads me to look at this mystery from a perspective that is a little unusual. Finally, I spend a concluding chapter discussing the fascinating and essential question of whether the computer is a reasonable model for the kind of mathematical activity that I have discussed in the book. Is mathematical thought algorithmic in nature? Is the mind of the mathematician a kind of software that is implemented on the hardware that we call the brain? Or is mathematical activity built on a fundamental and irreducible human creativity—a creativity that comes from a deep need that we human beings have to understand—to create meaning out of our lives and our environment? This drive for meaning is inevitably accompanied by conflict and struggle, the very ingredients that we shall find in situations of ambiguity.