INTRODUCTION

SYLVIA NASAR

When Freeman Dyson, the physicist, greeted John Forbes Nash, Jr. at the Institute for Advanced Study one day in the early 1990s, he hardly expected a response. A mathematics legend in his twenties, Nash had suffered for decades from a devastating mental illness. A mute, ghost-like figure who scrawled mysterious messages on blackboards and occupied himself with numerological calculations, he was known around Princeton only as "the Phantom."

To Dyson's astonishment, Nash replied. He'd seen Dyson's daughter, an authority on computers, on the news, he said. "It was beautiful," recalled Dyson. "Slowly, he just somehow woke up."

Nash's miraculous emergence from an illness long considered a life sentence was neither the first, nor last, surprise twist in an extraordinary life.

The eccentric West Virginian with the movie star looks and Olympian manner burst onto the mathematical scene in 1948. A one-line

In describing John Nash's contributions to economics and mathematics, I drew from essays by Avinash Dixit, John Milnor, Roger Myerson, and Ariel Rubinstein as well as from my biography, *A Beautiful Mind*. Avinash Dixit and Harold Kuhn kindly commented on my draft. Any errors are, of course, mine alone.

letter of recommendation—"This man is a genius."—introduced the twenty-year-old to Princeton's elite math department. A little more than a year later, Nash had written the twenty-seven-page thesis that would one day win him a Nobel.

Over the next decade, his stunning achievements and flamboyant behavior made Nash a celebrity in the mathematics world. Donald Newman, a mathematician who knew him in the early 1950s, called him "a bad boy, but a great one." Lloyd Shapley, a fellow graduate student at Princeton, said of Nash, "What redeemed him was a clear, logical, beautiful mind."

Obsessed with originality, disdainful of authority, supremely self-confident, Nash rushed in where more conventional minds refused to tread. "Everyone else would climb a peak by looking for a path somewhere on the mountain," recalled Newman. "Nash would climb another mountain altogether and from that distant peak would shine a searchlight back on the first peak."

By his thirtieth birthday, Nash seemed to have it all: he was married to a gorgeous young physicist and was about to be promoted to full professor at MIT; *Fortune* magazine had just named him one of the brightest stars of the younger generation of "new" mathematicians.

Less than a year later, however, the brilliant career was shattered. Diagnosed with paranoid schizophrenia, Nash abruptly resigned from MIT and fled to Paris on a quixotic quest to become a world citizen. For the next decade, he was in and out of mental hospitals. By forty, he'd lost everything: friends, family, profession. Only the compassion of his wife, Alicia, saved him from homelessness. Sheltered by Alicia and protected by a handful of loyal former colleagues, Nash haunted the Princeton campus, in the thrall of a delusion that he was "a religious figure of great, but secret importance."

While Nash was lost in his dreams, his name surfaced more and more often in journals and textbooks in fields as far-flung as economics and biology, mathematics and political science: "Nash equilibrium," "Nash bargaining solution," "Nash program," "De Georgi-Nash," "Nash embedding," "Nash-Moser theorem," "Nash blowing up."

Outside Princeton, scholars who built on his work often assumed he was dead. But his ideas were very much alive, becoming more influential even as their author sank deeper into obscurity. Nash's contributions to pure mathematics—embedding of Riemannian manifolds, existence of solutions of parabolic and elliptic partial differential equations—paved the way for important new developments. By the 1980s, his early work in game theory had permeated economics and helped create new fields within the discipline, including experimental economics. Philosophers, biologists, and political scientists adopted his insights.

The growing impact of his ideas was not limited to the groves of academe. Advised by game theorists, governments around the world began to auction "public" goods from oil drilling rights to radio spectra, reorganize markets for electricity, and devise systems for matching doctors and hospitals. In business schools, game theory was becoming a staple of management training.

The contrast between the influential ideas and the bleak reality of Nash's existence was extreme. The usual honors passed him by. He wasn't affiliated with a university. He had virtually no income. A small band of contemporaries had always recognized the importance of his work. By the late 1980s, their ranks were swelled by younger scholars who launched a fight to get Nash long-overdue recognition. They succeeded spectacularly: in 1994, after an explosive behind-the-scenes debate and a narrow vote, the Swedish Academy of Sciences awarded Nash a Nobel prize in economics for his early work on non-cooperative games. The prize, which he shared with Reinhard Selten and John Harsanyi, was more than an intellectual triumph; it was a victory for those who believed that mental illness shouldn't be a bar to the ne plus ultra of scientific honors.

Most Nobel laureates, while celebrated within their disciplines, remain invisible to the public at large. And a Nobel rarely changes winners' lives profoundly. Nash is an exception. "We helped lift him into daylight," said Assar Lindbeck, chairman of the Nobel prize committee. "We resurrected him in a way."

Recognition of his ideas has not only redeemed the man—bringing him back to society and mathematics—but has turned Nash into something of a cultural hero. Since winning the Nobel, the mathematician who spent his life "thinking, always thinking" has inspired a *New York Times* profile; a biography, *A Beautiful Mind*; a *Vanity Fair* article; a

Broadway play, *Proof*; and, now, a Hollywood movie, directed by Ron Howard and starring Russell Crowe as Nash.

The ongoing celebration of Nash's inspiring life and unique achievements has generated new interest in the seminal papers he published during his twenties. *The Essential John Nash* was conceived to make these articles accessible to a wide audience. This volume reflects the full range of Nash's diverse contributions. For the first time, readers will have the opportunity to see for themselves why Nash, so nearly forgotten, has been called "the most remarkable mathematician of the second half of the century."*

Nash arrived in Princeton on the first day of Truman's 1948 reelection campaign and found himself suddenly at the center of the mathematical universe. The demigods of twentieth century science were in residence: Einstein, Gödel, Oppenheimer, and John von Neumann. "The air is full of mathematical ideas and formulae," one of Einstein's assistants marveled. It was a heady time. "The notion was that the human mind could accomplish anything with mathematical ideas," one of Nash's fellow graduate students recalled.

The ten or so first-year students were a cocky bunch, but Nash was even cockier. He loved sparring in the common room. He avoided classes. He was rarely seen cracking a book. Pacing endlessly, whistling Bach, he worked inside his own head. John Milnor, the topologist, who was a freshman that year, said, "It was as if he wanted to rediscover, for himself, three hundred years of mathematics." Always on the lookout for a shortcut to fame, Nash would corner visiting lecturers, clipboard and writing pad in hand. "He was very much aware of unsolved problems," said Milnor. "He really cross-examined people."

He was bursting with ideas. Norman Steenrod, Nash's faculty adviser, recalled:

> During his first year of graduate work, he presented me with a characterization of a simple closed curve in the plane. This was essentially the same one given by Wilder in 1932. Some time later

* Mikhail Gromov, 1997.

> he devised a system of axioms for topology based on the primitive concept of connectedness. I was able to refer him to papers by Wallace. During his second year, he showed me a definition of a new kind of homology group which proved to be the same as the Reidemeister group based on homotopy chains.

Nash's first mathematical coup, appropriately enough, involved a game of his own invention. One afternoon von Neumann strolled into the common room to see two students hunched over an unfamiliar game board. Oh, by the way, what was it that they were playing? he later asked a colleague. "Nash," came the answer, "Nash."

Parker Bros. later called Nash's nifty game, which was invented independently by the Danish mathematician Piet Hein, Hex. Nash's playful foray into mathematical games foreshadowed a far more serious involvement in a novel branch of mathematics (see chapter 3, this volume).

Today, the language of game theory permeates the social sciences. In 1948, game theory was brand-new and very much in the air at Princeton's Fine Hall. The notion that games could be used to analyze strategic thinking has a long history. Such games as Kriegspiel, a form of blind chess, were used to train Prussian officers. And mathematicians like Emile Borel, Ernst Zermelo, and Hugo Steinhaus studied parlor games to derive novel mathematical insights. The first formal attempt to create a theory of games was von Neumann's 1928 article, "Zur Theorie der Gesellschaftsspiele," in which he developed the concept of strategic interdependence. But game theory as a basic paradigm for studying decision making in situations where one actor's best options depend on what others do did not come into its own until World War II when the British navy used the theory to improve its hit rate in the campaign against German submarines. Social scientists discovered it in 1944 when von Neumann and the Princeton economist Oskar Morgenstern published their masterpiece, *Theory of Games and Economic Behavior,* in which the authors predicted that game theory would eventually do for the study of markets what calculus had done for physics in Newton's day.

The pure mathematicians around the university and the Institute were inclined to view game theory as "just the latest fad" and "déclassé" because it was applied, not pure mathematics. But in the eyes of Nash and his fellow graduate students, von Neumann's interest in the field lent it instant glamor.

Nash wrote his first major paper—his now-classic article on bargaining—while attending Albert Tucker's weekly game theory seminar during his first year at Princeton, where he met von Neumann and Morgenstern. But he had come up with the basic idea as an undergraduate at Carnegie Tech—in the only economics course (international trade) he ever took.

Bargaining is an old problem in economics. Despite the rise of the marketplace with millions of buyers and sellers who never interact directly, one-on-one deals—between individuals, corporations, governments, or unions—still loom large in everyday economic life. Yet, before Nash, economists assumed that the outcome of a two-way bargaining was determined by psychology and was therefore outside the realm of economics. They had no formal framework for thinking about how parties to a bargain would interact or how they would split the pie.

Obviously, each participant in a negotiation expects to benefit more by cooperating than by acting alone. Equally obviously, the terms of the deal depend on the bargaining power of each. Beyond this, economists had little to add. No one had discovered principles by which to winnow unique predictions from a large number of potential outcomes. Little if any progress had been made since Edgeworth conceded, in 1881, "The general answer is . . . contract without competition is indeterminate."

In their opus, von Neumann and Morgenstern had suggested that "a real understanding" of bargaining lay in defining bilateral exchange as a "game of strategy." But they, too, had come up empty. It is easy to see why: real-life negotiators have an overwhelming number of potential strategies to choose from—what offers to make, when to make them, what information, threats, or promises to communicate, and so on.

Nash took a novel tack: he simply finessed the process. He visualized a deal as the outcome of either a process of negotiation or else independent strategizing by individuals each pursuing his own interest. Instead of defining a solution directly, he asked what reasonable conditions any division of gains from a bargain would have to satisfy. He then posited four conditions and, using an ingenious mathematical argument, showed that, if the axioms held, a unique solution existed that maximized the product of the participants' utilities. Essentially, he reasoned, how gains are divided reflects how much the deal is worth to each party and what other alternatives each has.

By formulating the bargaining problem simply and precisely, Nash showed that a unique solution exists for a large class of such problems. His approach has become the standard way of modeling the outcomes of negotiations in a huge theoretical literature spanning many fields, including labor-management bargaining and international trade agreements.

Since 1950, the Nash equilibrium—Nash's Nobel-prize-winning idea—has become "*the* analytical structure for studying all situations of conflict and cooperation."*

Nash made his breakthrough at the start of his second year at Princeton, describing it to fellow graduate student David Gale. The latter immediately insisted Nash "plant a flag" by submitting the result as a note to the *Proceedings of the National Academy of Sciences*. In the note, "Equilibrium Points in *n*-Person Games," Nash gives the general definition of equilibrium for a large class of games and provides a proof using the Kakutani fixed point theorem to establish that equilibria in randomized strategies must exist for any finite normal form game (see chapter 5).

After wrangling for months with Tucker, his thesis adviser, Nash provided an elegantly concise doctoral dissertation which contained another proof, using the Brouwer fixed point theorem (see chapter 6). In his thesis, "Non-Cooperative Games," Nash drew the all-important distinction between non-cooperative and cooperative games, namely between games where players act on their own "without collaboration or communication with any of the others," and ones where players have opportunities to share information, make deals, and join coalitions. Nash's theory of games—especially his notion of equilibrium for such games (now known as Nash equilibrium)—significantly extended the boundaries of economics as a discipline.

* Roger Myerson 1999.

All social, political, and economic theory is about interaction among individuals, each of whom pursues his own objectives (whether altruistic or selfish). Before Nash, economics had only one way of formally describing how economic agents interact, namely, the impersonal market. Classical economists like Adam Smith assumed that each participant regarded the market price beyond his control and simply decided how much to buy or sell. By some means—i.e., Smith's famous Invisible Hand—a price emerged that brought overall supply and demand into balance.

Even in economics, the market paradigm sheds little light on less impersonal forms of interaction between individuals with greater ability to influence outcomes. For example, even in markets with vast numbers of buyers and sellers, individuals have information that others do not, and decide how much to reveal or conceal and how to interpret information revealed by others. And in sociology, anthropology, and political science, the market as explanatory mechanism was even more inadequate. A new paradigm was needed to analyze a wide array of strategic interactions and to predict their results.

Nash's solution concept for games with many players provided that alternative. Economists usually assume that each individual will act to maximize his or her own objective. The concept of the Nash equilibrium, as Roger Myerson has pointed out, is essentially the most general formulation of that assumption. Nash formally defined equilibrium of a non-cooperative game to be "a configuration of strategies, such that no player acting on his own can change his strategy to achieve a better outcome for himself." The outcome of such a game must be a Nash equilibrium if it is to conform to the assumption of rational individual behavior. That is, if the predicted behavior doesn't satisfy the condition for Nash equilibrium, then there must be at least one individual who could achieve a better outcome if she were simply made aware of her own best interests.

In one sense, Nash made game theory relevant to economics by freeing it from the constraints of von Neumann and Morgenstern's two-person, zero-sum theory. By the time he was writing his thesis, even the strategists at RAND had come to doubt that nuclear warfare, much less post-war reconstruction, could usefully be modeled as a game in which the enemy's loss was a pure gain for the other side. Nash had the critical insight that most social interactions involve neither pure competition nor pure cooperation but rather a mix of both.

From a perspective of half a century later, Nash did much more than that. After Nash, the calculus of rational choice could be applied to situations beyond the market itself to analyze the system of incentives created by any social institution. Myerson's eloquent assessment of Nash's influence on economics is worth quoting at length:

> Before Nash, price theory was the one general methodology available to economics. The power of price theory enabled economists to serve as highly valued guides in practical policy making to a degree that was not approached by scholars in any other social science. But even within the traditional scope of economics, price theory has serious limits. Bargaining situations where individuals have different information . . . the internal organization of a firm . . . the defects of a command economy . . . crime and corruption that undermine property rights. . . .
>
> The broader analytical perspective of non-cooperative game theory has liberated practical economic analysis from these methodological restrictions. Methodological limitations no longer deter us from considering market and non-market systems on an equal footing, and from recognizing the essential interconnections between economic, social, and political institutions in economic development. . . .
>
> By accepting non-cooperative game theory as a core analytical methodology alongside price theory, economic analysis has returned to the breadth of vision that characterized the ancient Greek social philosophers who gave economics its name.*

Von Neumann, however, didn't think much of Nash's breakthrough. When Nash met with him, the Hungarian polymath dismissed the younger man's result as "trivial." The 1953 edition of his and Morgenstern's *Theory of Games and Economic Behavior* included only a perfunctory mention of "non-cooperative games" in the Preface.

* Myerson 1999.

His doctorate in his pocket, Nash headed off to RAND, the ultra-secret cold war think tank, in the summer of 1950. He would be part of "the Air Force's big-brain-buying venture"—whose stars would eventually serve as models for Dr. Strangelove—for the next four years, spending every other summer in Santa Monica.

Game theory was considered RAND's secret weapon in a nuclear war of wits against the Soviet Union. "We hope [the theory of games] will work, just as we hoped in 1942 that the atomic bomb would work," a Pentagon official told *Fortune* at the time. Nash got an excited reception. Researchers like Kenneth Arrow, who won a Nobel for his social choice theory, were already chafing at RAND's "preoccupation with the two-person zero-sum game." As weapons became ever more destructive, all-out war could not be seen as a situation of pure conflict in which opponents shared no common interests. Nash's model thus seemed more promising than von Neumann's.

Probably the single most important work Nash did at RAND involved an experiment. Designed with a team that included Milnor and published as "Some Experimental *n*-Person Games," it anticipated by several decades the now-thriving field of experimental economics. At the time the experiment was regarded as a failure, Alvin Roth has pointed out, casting doubt on the predictive power of game theory. But it later became a model because it drew attention to two aspects of interaction. First, it highlighted the importance of information possessed by participants. Second, it revealed that players' decisions were, more often than not, motivated by concerns about fairness. Despite the experiment's simplicity, it showed that watching how people actually play a game drew researchers' attention to elements of interaction—such as signaling and implied threats—that weren't part of the original model.

Nash, whose own interests were rapidly shifting away from game theory to pure mathematics, became fascinated with computers at RAND. Of the dozen or so working papers he wrote during his summers in Santa Monica, none is more visionary than one, written in his last summer at the think tank, called "Parallel Control" (see chapter 9).

Nash, however, was bent on proving himself a pure mathematician. Even before completing his thesis on game theory, he turned his attention to the trendy topic of geometric objects called manifolds. Manifolds play a role in many physical problems, including cosmology. Right off the bat, he made what he called "a nice discovery relating to manifolds and real algebraic varieties." Hoping for an appointment at Princeton or another prestigious math department, he returned to Princeton for a post-doctoral year and devoted himself to working out the details of the difficult proof.

Many breakthroughs in mathematics come from seeing unsuspected connections between objects that appear intractable and ones that mathematicians have already got their arms around. Dismissing conventional wisdom, Nash argued that manifolds were closely related to a simpler class of objects called algebraic varieties. Loosely speaking, Nash asserted that for any manifold it was possible to find an algebraic variety one of whose parts corresponded in some essential way to the original object. To do this, he showed, one has to go to higher dimensions.

Nash's theorem was initially greeted with skepticism. Experts found the notion that every manifold could be described by a system of polynomial equations implausible. "I didn't think he would get anywhere," said his Princeton adviser.

Nash completed "Real Algebraic Manifolds," his favorite paper and the only one he concedes is nearly perfect, in the fall of 1951 (see chapter 10). Its significance was instantly recognized. "Just to conceive the theorem was remarkable," said Michael Artin, a mathematician at MIT. Artin and Barry Mazur, who was a student of Nash's at MIT, later used Nash's result to resolve a basic problem in dynamics, the estimation of periodic points. Artin and Mazur proved that any smooth map from a compact manifold to itself could be approximated by a smooth map such that the number of periodic points of period p grows at most exponentially with p. The proof relied on Nash's work by translating the dynamic problem into an algebraic one of counting solutions to polynomial equations.

Nonetheless, Nash's hoped-for appointment at Princeton did not materialize. Instead, he got an offer at MIT, then still the nation's leading engineering school but not the great research university that it was to become.

In 1955, Nash unveiled a stunning result to a disbelieving audience at the University of Chicago. "I did this because of a bet," he announced.

One of his colleagues at MIT had, two years earlier, challenged him. "If you're so good, why don't you solve the embedding problem . . . ?" When Nash took up the challenge and announced that "he had solved it, modulo details," the consensus around Cambridge was that "he is getting nowhere."

The precise question that Nash was posing—"Is it possible to embed any Riemannian manifold in a Euclidian space?"—was a challenge that had frustrated the efforts of eminent mathematicians for three-quarters of a century.

By the early 1950s, interest had shifted to geometric objects in higher dimensions, partly because of the large role played by distorted-time and space relationships in Einstein's theory of relativity. Embedding means presenting a given geometric object as a subset of a space of possibly higher dimension, while preserving its essential topological properties. Take, for instance, the surface of a balloon, which is two-dimensional. You cannot put it on a blackboard, which is two-dimensional, but you can make it a subset of a space of three or more dimensions.

John Conway, the Princeton mathematician who discovered surreal numbers, calls Nash's result "one of the most important pieces of mathematical analysis in this century." Nash's theorem stated that any kind of surface that embodied a special notion of smoothness could actually be embedded in a Euclidean space. He showed, essentially, that you could fold a manifold like a handkerchief without distorting it. Nobody would have expected Nash's theorem to be true. In fact, most people who heard the result for the first time couldn't believe it. "It took enormous courage to attack these problems," said Paul Cohen, a mathematician who knew Nash at MIT.

After the publication of "The Imbedding Problem for Riemannian Manifolds" in the *Annals of Mathematics* (see chapter 11), the earlier perspective on partial differential equations was completely altered. "Many of us have the power to develop existing ideas," said Mikhail Gromov, a geometer whose work was influenced by Nash. "We follow paths prepared by others. But most of us could never produce anything comparable to what Nash produced. It's like lightening striking . . . there has been some tendency in recent decades to move from harmony to chaos. Nash said that chaos was just around the corner."

Nominally attached to the Institute for Advanced Study during a leave from MIT in the academic year 1956–57, Nash instead gravitated to the Courant Institute at New York University, "the national capital of applied mathematical analysis."

At Courant, then housed in a former hat factory off Washington Square in Greenwich Village, a group of young mathematicians was responsible for the rapid progress stimulated by World War II in the field of partial differential equations. Such equations were useful in modeling a wide variety of physical phenomena, from air passing under the wings of a jet to heat passing through metal. By the mid-1950s, mathematicians knew simple routines for solving ordinary differential equations using computers. But straightforward methods for solving most nonlinear partial differential equations—the kind potentially useful for describing large or abrupt changes—did not exist. Stanislaw Ulam complained that such systems of equations were "baffling analytically," noting that they defied "even qualitative insights by present methods."

Nash proved basic local existence, uniqueness, and continuity theorems (and also speculated about relations with statistical mechanics, singularities, and turbulence.) He used novel methods of his own invention. He had a theory that deep problems wouldn't yield to a frontal attack. Taking an ingeniously roundabout approach, he first transformed the non-linear equations into linear ones and then attacked them with non-linear means. Today rocket scientists on Wall Street use Nash-inspired methods for solving a particular class of parabolic partial differential equations that arise in finance problems.

When he returned to MIT the following fall, there were still gaps in the proof. "It was as if he was a composer and could hear the music, but he didn't know how to write it down." Nash organized a cadre of mathematicians to help him get the paper ready for publication. "It was like building the atom bomb . . . a kind of factory," said one of them later. The complete proof was published in 1958 in "Continuity of Solutions of Parabolic and Elliptic Equations" (see chapter 12).

As Nash's thirtieth birthday approached, he seemed poised to make more groundbreaking contributions. He told colleagues of "an idea of an idea" about a possible solution to the Riemann hypothesis, the deepest puzzle in all of mathematics. He set out "to revise quantum theory," along lines he had once, as a first-year graduate student, described to Einstein. Writing to Oppenheimer in 1957, Nash had said, "To me one of the best things about the Heisenberg paper is its restriction to observable quantities . . . I want to find a different and more satisfying under-picture of a non-observable reality."

Later, he blamed the onset of his terrible disease on intellectual overreaching. No one can know what he might have accomplished had full-blown schizophrenia not set in. In the event, despite the ravages of his illness, he did go on to publish several more papers. "Le problème de Cauchy pour les équations différentielles d'une fluide générale," which appeared in 1962, is described as "basic and noteworthy" in *The Encyclopedic Dictionary of Mathematics* and inspired a good deal of subsequent work by others. He continued to tackle new subjects. Hironaka eventually wrote up one of his conjectures, dating from 1964, as "Nash Blowing Up." In 1966, he published "Analyticity of Solutions of Implicit Function Problems with Analytic Data," which pursued his ideas about partial differential equations to their natural conclusion. And in 1967 he completed a much-cited draft, "Arc Structure of Singularities," that was eventually published in a 1995 special issue of the *Duke Journal of Mathematics.*

"If you're going to develop exceptional ideas, it requires a type of thinking that is not simply practical thinking," Nash told a reporter recently. When Nash won the Nobel in 1994, he was not invited to deliver the customary hour-long Nobel lecture in Stockholm. He did, however, give a talk in Uppsala just after the Nobel ceremonies about his recent attempt to develop a mathematically correct theory of a non-expanding universe that is consistent with known physical observations. More recently, Nash has been working on game theory again. He has received a grant from the National Science Foundation to develop a new "evolutionary" solution concept for cooperative games. To get your life back is a marvelous thing, he has said. But to be able to create exciting new mathematics is now, as ever, his greatest ambition.

References

Dixit, A., and B. Nalebuff. *Thinking Strategically.* New York: W. W. Norton, 1991.

Kuhn, H. W. "Introduction," to *A Celebration of John F. Nash, Jr.,* ed. H. W. Kuhn, L. Nirenberg, and P. Sarnak, pp. i–v. In *Duke Mathematical Journal,* 81, nos. 1 and 2 (1995).

———. "Foreword." In *Classics in Game Theory,* ed. H. W. Kuhn, pp. ix–x. Princeton: Princeton University Press, 1997.

Milnor, J. "A Nobel Prize for John Nash." *The Mathematical Intelligencer* 17, no. 3 (1995): 11–17.

Myerson, R. B. "Nash Equilibrium and the History of Economic Theory." *Journal of Economic Literature* 37 (1999): 1067–82.

Nasar, S. "The Lost Years of the Nobel Laureate," *New York Times*, November 13, 1994, sec. F, pp. 1, 8.

———. *A Beautiful Mind.* New York: Simon & Schuster, 1998.

Roth, A. "Game Theory as a Tool for Market Design" (1999). Available at http://www.economics.harvard.edu/˜aroth/design.pdf.

Rubinstein, A. "John Nash: The Master of Economic Modeling." *The Scandinavian Journal of Economics* 97, no. 1 (1995): 9–13.