INTRODUCTION

"The calculus," wrote John von Neumann (1903–1957), "was the first achievement of modern mathematics, and it is difficult to overestimate its importance" [1].

Today, more than three centuries after its appearance, calculus continues to warrant such praise. It is the bridge that carries students from the basics of elementary mathematics to the challenges of higher mathematics and, as such, provides a dazzling transition from the finite to the infinite, from the discrete to the continuous, from the superficial to the profound. So esteemed is calculus that its name is often preceded by "the," as in von Neumann's observation above. This gives "*the* calculus" a status akin to "*the* law"—that is, a subject vast, self-contained, and awesome.

Like any great intellectual pursuit, the calculus has a rich history and a rich *pre*history. Archimedes of Syracuse (ca. 287–212 BCE) found certain areas, volumes, and surfaces with a technique we now recognize as proto-integration. Much later, Pierre de Fermat (1601–1665) determined slopes of tangents and areas under curves in a remarkably modern fashion. These and many other illustrious predecessors brought calculus to the threshold of existence.

Nevertheless, this book is not about forerunners. It goes without saying that calculus owes much to those who came before, just as modern art owes much to the artists of the past. But a specialized museum—the Museum of Modern Art, for instance—need not devote room after room to premodern influences. Such an institution can, so to speak, start in the middle. And so, I think, can I.

Thus I shall begin with the two seventeenth-century scholars, Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716), who gave birth to the calculus. The latter was first to publish his work in a 1684 paper whose title contained the Latin word *calculi* (a system of calculation) that would attach itself to this new branch of mathematics. The first textbook appeared a dozen years later, and the calculus was here to stay.

As the decades passed, others took up the challenge. Prominent among these pioneers were the Bernoulli brothers, Jakob (1654–1705) and Johann (1667–1748), and the incomparable Leonhard Euler (1707–1783), whose research filled many thousands of pages with mathematics of the highest quality. Topics under consideration expanded to include limits, derivatives, integrals, infinite sequences, infinite series, and more. This extended body of material has come to be known under the general rubric of "analysis."

With increased sophistication came troubling questions about the underlying logic. Despite the power and utility of calculus, it rested upon a less-than-certain foundation, and mathematicians recognized the need to recast the subject in a precise, rigorous fashion after the model of Euclid's geometry. Such needs were addressed by nineteenth-century analysts like Augustin-Louis Cauchy (1789–1857), Georg Friedrich Bernhard Riemann (1826–1866), Joseph Liouville (1809–1882), and Karl Weierstrass (1815–1897). These individuals worked with unprecedented care, taking pains to define their terms exactly and to prove results that had hitherto been accepted uncritically.

But, as often happens in science, the resolution of one problem opened the door to others. Over the last half of the nineteenth century, mathematicians employed these logically rigorous tools in concocting a host of strange counterexamples, the understanding of which pushed analysis ever further toward generality and abstraction. This trend was evident in the set theory of Georg Cantor (1845–1918) and in the subsequent achievements of scholars like Vito Volterra (1860–1940), René Baire (1874–1932), and Henri Lebesgue (1875–1941).

By the early twentieth century, analysis had grown into an enormous collection of ideas, definitions, theorems, and examples—and had developed a characteristic manner of thinking—that established it as a mathematical enterprise of the highest rank.

What follows is a sampler from that collection. My goal is to examine the handiwork of those individuals mentioned above and to do so in a manner faithful to the originals yet comprehensible to a modern reader. I shall discuss theorems illustrating the development of calculus over its formative years and the genius of its most illustrious practitioners. The book will be, in short, a "great theorems" approach to this fascinating story.

To this end I have restricted myself to the work of a few representative mathematicians. At the outset I make a full disclosure: my cast of characters was dictated by personal taste. Some whom I have included, like Newton, Cauchy, Weierstrass, would appear in any book with similar objectives. Some, like Liouville, Volterra and Baire, are more idiosyncratic. And others, like Gauss, Bolzano, and Abel, failed to make my cut.

Likewise, some of the theorems I discuss are known to any mathematically literate reader, although their *original* proofs may come as a surprise to those not conversant with the history of mathematics. Into this category fall Leibniz's barely recognizable derivation of the "Leibniz series" from 1673 and Cantor's first but less-well-known proof of the nondenumerability of the continuum from 1874. Other theorems, although part of the folklore of mathematics, seldom appear in modern textbooks; here I am thinking of a result like Weierstrass's everywhere continuous, nowhere differentiable function that so astounded the mathematical world when it was presented to the Berlin Academy in 1872. And some of my choices, I concede, are downright quirky. Euler's evaluation of $\int_0^1 \frac{\sin(\ln x)}{\ln x}dx$, for example, is included simply as a demonstration of his analytic wizardry.

Each result, from Newton's derivation of the sine series to the appearance of the gamma function to the Baire category theorem, stood at the research frontier of its day. Collectively, they document the evolution of analysis over time, with the attendant changes in style and substance. This evolution is striking, for the difference between a theorem from Lebesgue in 1904 and one from Leibniz in 1690 can be likened to the difference between modern literature and *Beowulf*. Nonetheless—and this is critical—I believe that each theorem reveals an ingenuity worthy of our attention and, even more, of our admiration.

Of course, trying to characterize analysis by examining a few theorems is like trying to characterize a thunderstorm by collecting a few raindrops. The impression conveyed will be hopelessly incomplete. To undertake such a project, an author must adopt some fairly restrictive guidelines.

One of mine was to resist writing a comprehensive history of analysis. That is far too broad a mission, and, in any case, there are many works that describe the development of calculus. Some of my favorites are mentioned explicitly in the text or appear as sources in the notes at the end of the book.

A second decision was to exclude topics from both multivariate calculus and complex analysis. This may be a regrettable choice, but I believe it is a defensible one. It has imposed some manageable boundaries upon the contents of the book and thereby has added coherence to the tale. Simultaneously, this restriction should minimize demands upon the reader's background, for a volume limited to topics from *univariate, real* analysis should be understandable to the widest possible audience.

This raises the issue of prerequisites. The book's objectives dictate that I include much technical detail, so the mathematics necessary to follow these theorems is substantial. Some of the early results require considerable algebraic stamina in chasing formulas across the page. Some of the later ones demand a refined sense of abstraction. All in all, I would not recommend this for the mathematically faint-hearted.

At the same time, in an attempt to favor clarity over conciseness, I have adopted a more conversational style than one would find in a standard text. I intend that the book be accessible to those who have majored or minored in college mathematics and who are not put off by an integral here or an epsilon there. My goal is to keep the prerequisites as modest as the topics permit, but no less so. To do otherwise, to water down the content, would defeat my broader purpose.

So, this is not primarily a biography of mathematicians, nor a history of calculus, nor a textbook. I say this despite the fact that at times I provide biographical information, at times I discuss the history that ties one topic to another, and at times I introduce unfamiliar (or perhaps long forgotten) ideas in a manner reminiscent of a textbook. But my foremost motivation is simple: to share some favorite results from the rich history of analysis.

And this brings me to a final observation.

In most disciplines there is a tradition of studying the major works of illustrious predecessors, the so-called "masters" of the field. Students of literature read Shakespeare; students of music listen to Bach. In mathematics such a tradition is, if not entirely absent, at least fairly uncommon. This book is meant to address that situation. Although it is not intended as a *history* of the calculus, I have come to regard it as a *gallery* of the calculus.

To this end, I have assembled a number of masterpieces, although these are not the paintings of Rembrandt or Van Gogh but the theorems of Euler or Riemann. Such a gallery may be a bit unusual, but its objective is that of all worthy museums: to serve as a repository of excellence.

Like any gallery, this one has gaps in its collection. Like any gallery, there is not space enough to display all that one might wish. These limitations notwithstanding, a visitor should come away enriched by an appreciation of genius. And, in the final analysis, those who stroll among the exhibits should experience the mathematical imagination at its most profound.