Chapter One

Introduction

Besides the introduction, front matter, back matter, and Appendix (Chapter 15), the book consists of two parts. The first part comprises Chapters 2–7. Here, fundamental properties and constructions of linear algebra are explored in the context of quaternions, such as matrix decompositions, numerical ranges, Jordan and Kronecker canonical forms, canonical forms under congruence, determinants, invariant subspaces, etc. The exposition in the first part is on the level of an upper undergraduate or graduate textbook. The second part comprises Chapters 8–14. Here, the emphasis is on canonical forms of quaternion matrix pencils with symmetries or, what is the same, pairs of matrices with symmetries, and the exposition approaches that of a research monograph. Applications are given to systems of linear differential equations with symmetries, and matrix equations.

The mathematical tools used in the book are easily accessible to undergraduates with a background in linear algebra and rudiments of complex analysis and, on occasion, multivariable calculus. The exposition is largely based on tools of matrix analysis. The author strived to make the book self-contained and inclusive of complete proofs as much as possible, at the same time keeping the size of the book within reasonable limits. However, some compromises were inevitable here. Thus, proofs are often omitted for many linear algebra results that are standard for real and complex matrices, are often presented in textbooks, and are valid for quaternion matrices as well with essentially the same proofs.

The book can be used in a variety of ways. More than 200 exercises are provided, on various levels of difficulty, ranging from routine verification of facts and numerical examples designed to illustrate the results to open-ended questions. The exercises and detailed exposition make the book suitable in teaching as supplementary material for undergraduate courses in linear algebra, as well as for students' independent study or reading courses. For students' benefit, several appendices are included that contain background material used in the main text. The book can serve as a basis for a graduate course named advanced linear algebra, topics in linear algebra, or (for those who want to keep the narrower focus) quaternion linear algebra. For example, one can build a graduate course based on Chapters 2–8 and selections from later chapters.

Open problems presented in the book provide an opportunity to do original research. The open problems are on various levels: open-ended problems that may serve as subject for research by mathematicians and concrete, more-specific problems that are perhaps more suited for undergraduate research work under faculty supervision, honors theses, and the like.

For working mathematicians in both theoretical and applied areas, the book may serve as a reference source. Such areas include, in particular, vector calculus, ordinary and partial differential equations, and boundary value problems (see, e.g., Gürlebeck and Sprössig [60]), and numerical analysis (Bunse-Gerstner et al. [22]). The accessibility and importance of the mathematics should make this book a widely useful work not only for mathematicians, but also for scientists and engineers.

Quaternions have become increasingly useful for practitioners in research, both in theory and applications. For example, a significant number of research papers on quaternions, perhaps even most of them, appear regularly in mathematical physics journals, and quantum mechanics based on quaternion analysis is mainstream physics. In engineering, quaternions are often used in control systems, and in computer science they play a role in computer graphics. Quaternion formalism is also used in studies of molecular symmetry. For practitioners in these areas, the book can serve as a valuable reference tool.

New, previously unpublished results presented in the book with complete proofs will hopefully be useful for experts in linear algebra and matrix analysis. Much of the material appears in a book for the first time; this is true for Chapters 5–14, most of Chapter 4, and a substantial part of Chapter 3.

As far as the author is aware, this is the first book dedicated to systematic exposition of quaternion linear algebra. So far, there are only a few expository papers and chapters in books on the subject (for example, Chapter 1 in Gürlebeck and Sprössig [60], Brieskorn [20], Zhang [164], or Farenick and Pidkowich [38]) as well as algebraic treatises on skew fields (e.g., Cohn [29] or Wan [156]).

It is inevitable that many parts of quaternion linear algebra are not reflected in the book, most notably those parts pertaining to numerical analysis (Bunse-Gerstner et al. [22] and Faßbender et al. [40]). Also, the important classes of orthogonal, unitary, and symplectic quaternion matrices are given only brief exposure.

We now describe briefly the contents of the book chapter by chapter.

Chapter 2 concerns (scalar) quaternions and the basic properties of quaternion algebra, with emphasis on solution of equations such as $axb = c$ and $ax - xb = c$. Description of all automorphisms and antiautomoprhisms of quaternions is given, and representations of quaternions in terms of 2×2 complex matrices and 4×4 real matrices are introduced. These representations will play an important role throughout the book.

Chapter 3 covers basics on the vector space of columns with quaternion components, matrix algebra, and various matrix decomposition. The real and complex representations of quaternions are extended to vectors and matrices. Various matrix decompositions are studied; in particular, Cholesky factorization is proved for matrices that are hermitian with respect to involutions other than the conjugation. A large part of this chapter is devoted to numerical ranges of quaternion matrices with respect to conjugation as well as with respect to other involutions. Finally, a brief exposition is given for the set of quaternion subspaces, understood as a metric space with respect to the metric induced by the gap function.

In a short Chapter 4 we develop diagonal canonical forms and prove inertia theorems for hermitian and skewhermitian matrices with respect to involutions (including the conjugation). We also identify dimensions of subspaces that are neutral or semidefinite relative to a given hermitian matrix and are maximal with respect to this property. The material in Chapters 3 and 4 does not depend on the more involved constructions such as the Jordan form and its proof.

Chapter 5 is a key chapter in the book. Root subspaces of quaternion matrices are introduced and studied. The Jordan form of a quaternion matrix is presented in full detail, including a complete proof. The complex matrix representation plays a crucial role here. Although the standard definition of a determinant is not very useful when applied to quaternion matrices, nevertheless several notions of determinant-like functions for matrices over quaternions have been defined and used in the literature; a few of these are explored in this chapter as well. Several applications of the Jordan form are treated. These include matrix equations of the form $AX - XB = C$, functions of matrices, and stability of systems of differential equations of the form

$$A_\ell x^{(\ell)}(t) + A_{\ell-1}x^{(\ell-1)}(t) + \cdots + A_1 x'(t) + A_0 x(t) = 0, \qquad t \in \mathsf{R},$$

with constant quaternion matrix coefficients $A_\ell, \ldots, A_0$. Stability of an analogous system of difference equations is studied as well.

The main theme of Chapter 6 concerns subspaces that are simultaneously invariant for one matrix and semidefinite (or neutral) with respect to another. Such subspaces show up in many applications, some of them presented later in Chapter 13. For a given invertible plus-matrix A, the main result here asserts that any subspace which is A-invariant and at the same time nonnegative with respect to the underlying indefinite inner product can be extended to an A-invariant subspace which is maximal nonnegative. Analogous results are proved for related classes of matrices, such as unitary and dissipative, as well as in the context of indefinite inn er products induced by involutions other than the conjugation.

Chapter 7 treats matrix polynomials with quaternion coefficients. A diagonal form (known as the Smith form) is proved for such polynomials. In contrast to matrix polynomials with real or complex coefficients, a Smith form is generally not unique. For matrix polynomials of first degree, a Kronecker form—the canonical form under strict equivalence—is available, which is presented with a complete proof. Furthermore, a comparison is given for the Kronecker forms of complex or real matrix polynomials with the Kronecker forms of such matrix polynomials under strict equivalence using quaternion matrices.

In Chapters 8, 9, and 10 we develop canonical forms of quaternion matrix pencils $A + tB$ in which the matrices A and B are either hermitian or skewhermitian and their applications. Chapter 8 is concerned with the case when both matrices A and B are hermitian. Full and detailed proofs of the canonical forms under strict equivalence and simultaneous congruence are provided, based on the Kronecker form of the pencil $A + tB$. Several variations of the canonical forms are included as well. Among applications here: criteria for existence of a nontrivial positive semidefinite real linear combination and sufficient conditions for simultaneous diagonalizability of two hermitian matrices under simultaneous congruence. A comparison is made with pencils of real symmetric or complex hermitian matrices. It turns out that two pencils of real symmetric matrices are simultaneously congruent over the reals if and only if they are simultaneously congruent over the quaternions. An analogous statement holds true for two pencils of complex hermitian matrices.

The subject matter of Chapter 9 is concerned mainly with matrix pencils of the form $A + tB$, where one of the matrices A or B is skewhermitian and the other may be hermitian or skewhermitian. Canonical forms of such matrix pencils are given under strict equivalence and under simultaneous congruence, with full detailed proofs, again based on the Kronecker forms. Comparisons with real and complex matrix pencils are presented. In contrast to hermitian matrix pencils, two complex skewhermitian matrix pencils that are simultaneously congruent under quaternions need not be simultaneously congruent under the complex field, although an analogous property is valid for pencils of real skewsymmetric matrices. Similar results hold for real or complex matrix pencils $A+tB$, where A is real symmetric or complex hermitian and B is real skewsymmetric or complex skewhermitian. In each case, we sort out the relationships of simultaneous congruence over the complex field of complex matrix pencils where one matrix is hermitian and the other is skewhermitian versus simultaneous congruence over the skew field of quaternions for such pencils. As an applications we obtain a canonical form for quaternion matrices under (quaternion) congruence.

In Chapter 10 we study matrices (or linear transformations) that are selfadjoint or skewadjoint with respect to a nondegenerate hermitian or skewhermitian inner product. As an application of the canonical forms obtained in Chapters 8 and 9, canonical forms for such matrices are derived. Matrices that are skewadjoint with respect to skewhermitian inner products are known as Hamiltonian matrices; they play a key role in many applications such as linear control systems (see Chapter 14). The canonical forms allow us to study invariant Lagrangian subspaces; in particular, they give criteria for existence of such subspaces. Another application involves boundedness and stable boundedness of linear systems of differential equations with constant coefficients under suitable symmetry requirements.

The development of material in Chapters 11, 12, and 13 is largely parallel to that in Chapters 8, 9, and 10, but with respect to an involution of the quaternions other than the conjugation and with respect to indefinite inner products induced by matrices that are hermitian or skewhermitian with respect to such involutions. Thus, letting ϕ be a fixed involution of the quaternions which is different from the conjugation, the canonical forms (under both strict equivalence and simultaneous ϕ-congruence) of quaternion matrix pencils $A+tB$, where each of A and B is either ϕ-hermitian or ϕ-skewhermitian, are given in Chapters 11 and 12. As before, full and detailed proofs are supplied.

Applications are made in Chapter 14 to various types of matrix equations over quaternions, such as

$$Z^m + \sum_{j=0}^{m-1} A_j Z^j = 0,$$

where $A_0, \ldots, A_{m-1}$ are given $n \times n$ quaternion matrices,

$$ZBZ + ZA - DZ - C = 0,$$

where A, B, C, and D are given quaternion matrices of suitable sizes, and the symmetric version of the latter equation,

$$ZDZ + ZA + A^*Z - C = 0, \tag{1.0.1}$$

where D and C are assumed to be hermitian. The theory of invariant subspaces of quaternion matrices—and for equation (1.0.1) also of subspaces that are simultaneously invariant and semidefinite—plays a crucial role in study of these matrix equations. Equation (1.0.1) and its solutions, especially hermitian solutions, are important in linear control systems. A brief description of such systems and their relation to equations of the type of (1.0.1) is also provided.

For the readers' benefit, in Chapter 15 we bring several well-known canonical forms for real and complex matrices that are used extensively in the text. No proofs are given; instead we supply references that contain full proofs and further bibliographical information.

1.1 NOTATION AND CONVENTIONS

Numbers, sets, spaces

$A := B$—the expression or item A is defined by the expression or item B
R—the real field
$\lfloor x \rfloor$—the greatest integer not exceeding $x \in \mathsf{R}$
C—the complex field
$\mathfrak{I}(z) = (z - \overline{z})/(2\mathsf{i}) \in \mathsf{R}$—the imaginary part of a complex number z
C_+—closed upper complex half-plane
$D_\epsilon(\lambda) := \{z \in \mathsf{C}_+ \, : \, |z - \lambda| < \epsilon\}$—part of the open circular disk centered at λ with radius ϵ that lies in C_+
$\mathsf{C}_{+,0}$—open upper complex half-plane
H—the skew field of the quaternions
$\mathsf{i}, \mathsf{j}, \mathsf{k}$—the standard quaternion imaginary units
$\mathfrak{R}(x) = x_0$ and $\mathfrak{V}(x) = x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k}$—the real and the vector part of x, respectively, for $x = x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k} \in \mathsf{H}$, where $x_0, x_1, x_2, x_3 \in \mathsf{R}$
$|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$—the length of $x \in \mathsf{H}$
$\mathrm{Inv}\,(\phi) := \{x \in \mathsf{H} \, : \, \phi(x) = x\}$—the set (real vector space) of quaternions invariant under an involution ϕ of H
$\beta(\phi) \in \mathsf{H}$—quaternion with the properties that $\phi(\beta(\phi)) = -\beta(\phi)$ and $|\beta(\phi)| = 1$, where ϕ is an involution of H that is different from the quaternion conjugation; for a given ϕ, the quaternion $\beta(\phi) \in \mathsf{H}$ is unique up to negation
$\mathrm{Con}\,(\alpha) = \{y^*\alpha y \, : \, y \in \mathsf{H} \setminus \{0\}\}$—the congruence orbit of $\alpha \in \mathsf{H}$
$\mathrm{Sim}\,(\alpha) = \{y^{-1}\alpha y \, : \, y \in \mathsf{H} \setminus \{0\}\}$—the similarity orbit of $\alpha \in \mathsf{H}$
$\mathsf{F}^{n\times 1}$—the vector space of n-components columns with components in F, where $\mathsf{F} = \mathsf{R}$, $\mathsf{F} = \mathsf{C}$, or $\mathsf{F} = \mathsf{H}$; $\mathsf{H}^{n\times 1}$ is understood as a right quaternion vector space
$e_j \in \mathsf{H}^{n\times 1}$—the vector with 1 in the jth component and zero elsewhere; n is understood from context
$\langle x, y\rangle := y^*x,\ x, y \in \mathsf{H}^{n\times 1}$—the standard inner product defined on $\mathsf{H}^{n\times 1}$
$\|x\|_{\mathsf{H}} = \sqrt{\langle x, x\rangle}$ or $\|x\|$—the norm of $x \in \mathsf{H}^{n\times 1}$
$\mathsf{F}^{m\times n}$—the vector space of $m\times n$ matrices with entries in F, where $\mathsf{F} = \mathsf{R}$, $\mathsf{F} = \mathsf{C}$, or $\mathsf{F} = \mathsf{H}$, and $\mathsf{H}^{m\times n}$ is understood as a left quaternion vector space; if $m = n$, then $\mathsf{C}^{n\times n}$ is a complex algebra, whereas $\mathsf{R}^{n\times n}$ and $\mathsf{H}^{n\times n}$ are real algebras

Subspaces

$P_{\mathcal{M}} \in \mathsf{F}^{n\times n}$—the orthogonal projection onto the subspace $\mathcal{M} \subseteq \mathsf{F}^{n\times 1}$; here $\mathsf{F} = \mathsf{R}$, $\mathsf{F} = \mathsf{C}$, or $\mathsf{F} = \mathsf{H}$
$S_{\mathcal{M}} := \{x \in \mathcal{M}\,;\, \|x\| = 1\}$—the unit sphere of a nonzero subspace $\mathcal{M}$
$A|_{\mathcal{R}}$—restriction of a square-size matrix A to its invariant subspace $\mathcal{R}$ (we represent $A|_{\mathcal{R}}$ as a matrix with respect to some basis in $\mathcal{R}$)
$\mathcal{M} \dot{+} \mathcal{N}$—direct sum of subspaces $\mathcal{M}$ and $\mathcal{N}$
$\mathrm{Span}\,\{x_1, \ldots, x_p\}$ or $\mathrm{Span}_{\mathsf{H}}\,\{x_1, \ldots, x_p\}$—the quaternion subspace spanned by vectors $x_1, \ldots, x_p \in \mathsf{H}^{n\times 1}$
$\mathrm{Span}_{\mathsf{R}}\,\{x_1, \ldots, x_p\}$—the real subspace spanned by vectors $x_1, \ldots, x_p \in \mathsf{H}^{n\times 1}$
$\dim_{\mathsf{H}} \mathcal{M}$ or $\dim \mathcal{M}$—the (quaternion) dimension of a quaternion vector space $\mathcal{M}$

Grass_n—the set of all (quaternion) subspaces in $\mathsf{H}^{n\times 1}$
$\theta(\mathcal{M},\mathcal{N}) = \|P_{\mathcal{M}} - P_{\mathcal{N}}\|$—the gap between subspaces $\mathcal{M}$ and $\mathcal{N}$

Matrix-related notation

I_n or I (with n understood from context)—$n\times n$ identity matrix
$0_{u\times v}$, abbreviated to 0_u, if $u = v$—the $u\times v$ zero matrix, also 0 (with u and v understood from context)
C-eigenvalues of A—for a square-size complex matrix A, defined as the (complex) roots of the characteristic polynomial of A, and $\sigma_{\mathsf{C}}(A)$ is the set of all C-eigenvalues; e.g., for $A = \begin{bmatrix} \mathsf{i} & 1 \\ 0 & 2\mathsf{i} \end{bmatrix}$ we have

$$\sigma_{\mathsf{C}}(A) = \{\mathsf{i}, 2\mathsf{i}\},$$

$$\sigma(A) = \{a\mathsf{i} + b\mathsf{j} + c\mathsf{k} : a,b,c \in \mathsf{R},\ \ a^2 + b^2 + c^2 = 1 \ \text{ or } \ a^2 + b^2 + c^2 = 4\}$$

A^T—transposed matrix
A^*—conjugate transposed matrix
$\overline{A}$—the matrix obtained from $A \in \mathsf{C}^{m\times n}$ or $A \in \mathsf{H}^{m\times n}$ by replacing each entry with its complex or quaternion conjugate
$A_\phi \in \mathsf{H}^{n\times m}$, $A = [a_{i,j}]_{i,j=1}^{m,n} \in \mathsf{H}^{m\times n}$—stands for the matrix $[\phi(a_{j,i})]_{j,i=1}^{n,m}$, where ϕ is an involution of H
$\mathrm{Ran}\, A = \{Ax : x \in \mathsf{F}^{n\times 1}\} \subseteq \mathsf{F}^{m\times 1}$—the image or range of $A \in \mathsf{F}^{m\times n}$; here $\mathsf{F} \in \{\mathsf{R},\mathsf{C},\mathsf{H}\}$ (understood from context)
$\mathrm{Ker}\, A = \{x \in \mathsf{F}^{n\times 1} : Ax = 0\}$—the kernel of $A \in \mathsf{F}^{m\times n}$; here $\mathsf{F} \in \{\mathsf{R},\mathsf{C},\mathsf{H}\}$ (understood from context)
$\|A\|_{\mathsf{H}}$ or $\|A\|$—the norm of $A \in \mathsf{H}^{m\times n}$; it is taken to be the largest singular value of A
$\mathrm{rank}\, A$—the (quaternion) rank of a matrix $A \in \mathsf{H}^{m\times n}$; if A is real or complex, then $\mathrm{rank}\, A$ coincides with the rank of A as a real or complex matrix
$B \geq C$ or $C \leq B$—for hermitian matrices $B, C \in \mathsf{H}^{n\times n}$, indicates that the difference $B - C$ is positive semidefinite
$\mathrm{In}_+(A)$, $\mathrm{In}_-(A)$, $\mathrm{In}_0(A)$—the number of positive, negative, or zero eigenvalues of a quaternion hermitian matrix A, respectively, counted with algebraic multiplicities
$(\mathrm{In}_+(H), \mathrm{In}_-(H), \mathrm{In}_0(H))$—the $\beta(\phi)$-inertia, or the $\beta(\phi)$-signature, of a ϕ-skewhermitian matrix $H \in \mathsf{H}^{n\times n}$; here ϕ is an involution of H different from the quaternion conjugation
$\mathrm{diag}\,(X_1,\ldots,X_k) = \mathrm{diag}\,(X_j)_{j=1}^k = X_1 \oplus X_2 \oplus \cdots \oplus X_k$—block diagonal matrix with the diagonal blocks $X_1, X_2, \ldots, X_k$ (in that order)
$\mathrm{row}_{j=1,2,\ldots,p}\, X_j = \mathrm{row}\,(X_j)_{j=1}^p = [X_1\ X_2\ \cdots\ X_p]$—block row matrix
$\mathrm{col}_{j=1,2,\ldots,p}\, X_j = \mathrm{col}\,(X_j)_{j=1}^p = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}$—block column matrix
$X^{\oplus m}$—$X \in \mathsf{H}^{\delta_1\times\delta_2}$, the $m\delta_1 \times m\delta_2$ matrix $X \oplus \cdots \oplus X$, where X is repeated m times

1.2 STANDARD MATRICES

In this section we collect matrices in standard forms and fixed notation that will be used throughout the book, sometimes without reference. The subscript in notation for a square-size matrix will always denote the size of the matrix.

I_r or I (with r understood from context)—the $r \times r$ identity matrix

$0_{u\times v}$—often abbreviated to 0_u; if $u = v$ or 0 (with u and v understood from context), the $u \times v$ zero matrix

Jordan blocks:

$$J_m(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \in \mathsf{H}^{m\times m}, \quad \lambda \in \mathsf{H} \tag{1.2.1}$$

real Jordan blocks:

$$J_{2m}(a \pm \mathrm{i}b) = \begin{bmatrix} a & b & 1 & 0 & \cdots & 0 & 0 \\ -b & a & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 \\ 0 & 0 & -b & a & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & 0 & \cdots & -b & a \end{bmatrix} \in \mathsf{R}^{2m\times 2m}, \tag{1.2.2}$$

$$a \in \mathsf{R}, \quad b \in \mathsf{R} \setminus \{0\}$$

symmetric matrices:

$$F_m := \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} = F_m^{-1}, \tag{1.2.3}$$

$$G_m := \begin{bmatrix} 0 & \cdots & \cdots & 1 & 0 \\ \vdots & & & 0 & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} F_{m-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.2.4}$$

$$\widetilde{G}_m := F_m G_m F_m = \begin{bmatrix} 0 & 0 \\ 0 & F_{m-1} \end{bmatrix}. \tag{1.2.5}$$

We also define

$$\Xi_m(\alpha) := \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha \\ 0 & 0 & \cdots & -\alpha & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & (-1)^{m-2}\alpha & \cdots & 0 & 0 \\ (-1)^{m-1}\alpha & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathsf{H}^{m\times m}, \tag{1.2.6}$$

where $\alpha \in \mathsf{H}$, and

$$\begin{aligned} \Phi_m(\beta) \quad &:= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \beta \\ 0 & 0 & \cdots & 0 & -\beta & -1 \\ 0 & 0 & \cdots & \beta & -1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ (-1)^{m-1}\beta & -1 & 0 & \cdots & 0 & 0 \end{bmatrix} \\ &= \Xi_m(\beta) - \widetilde{G}_m \in \mathsf{H}^{m\times m}, \end{aligned} \tag{1.2.7}$$

where $\beta \in \mathsf{H}$. Note that

$$\Xi_m(\alpha) = (-1)^{m-1}(\Xi_m(\alpha))^T, \qquad \alpha \in \mathsf{H};$$

in particular $\Xi_m(\alpha) = (-1)^m(\Xi_m(\alpha))^*$ if and only if the real part of α is zero.

$$Y_{2m} = \begin{bmatrix} 0 & & & & 1 & 0 \\ & & & & 0 & -1 \\ & & 1 & 0 & & \\ & & 0 & -1 & & \\ & & \iddots & & & \\ 1 & 0 & & & & \\ 0 & -1 & & & & 0 \end{bmatrix}. \tag{1.2.8}$$

Note that Y_{2m} is real symmetric.

Real matrix pencils—

$$Z_{2m}(t,\,\mu,\,\nu) := (t+\mu)F_{2m} + \nu Y_{2m} + \begin{bmatrix} F_{2m-2} & 0 \\ 0 & 0_2 \end{bmatrix}, \tag{1.2.9}$$

where $\mu \in \mathsf{R}$, $\nu \in \mathsf{R} \setminus \{0\}$.

Singular matrix pencils—

$$L_{\varepsilon\times(\varepsilon+1)}(t) = [0_{\varepsilon\times 1} \;\; I_\varepsilon] + t[I_\varepsilon \;\; 0_{\varepsilon\times 1}] \in \mathsf{H}^{\varepsilon\times(\varepsilon+1)}. \tag{1.2.10}$$

Here ε is a positive integer; $L_{\varepsilon\times(\varepsilon+1)}(t)$ is of size $\varepsilon \times (\varepsilon+1)$.