Preface

1 What Is *The Companion*?

Bertrand Russell, in his book *The Principles of Mathematics*, proposes the following as a definition of pure mathematics.

> Pure Mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of *such that*, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics *uses* a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

The Princeton Companion to Mathematics could be said to be about everything that Russell's definition leaves out.

Russell's book was published in 1903, and many mathematicians at that time were preoccupied with the logical foundations of the subject. Now, just over a century later, it is no longer a new idea that mathematics can be regarded as a formal system of the kind that Russell describes, and today's mathematician is more likely to have other concerns. In particular, in an era where so much mathematics is being published that no individual can understand more than a tiny fraction of it, it is useful to know not just which arrangements of symbols form grammatically correct mathematical statements, but also which of these statements deserve our attention.

Of course, one cannot hope to give a fully objective answer to such a question, and different mathematicians can legitimately disagree about what they find interesting. For that reason, this book is far less formal than Russell's and it has many authors with many different points of view. And rather than trying to give a precise answer to the question, "What makes a mathematical statement interesting?" it simply aims to present for the reader a large and representative sample of the ideas that mathematicians are grappling with at the beginning of the twenty-first century, and to do so in as attractive and accessible a way as possible.

2 The Scope of the Book

The central focus of this book is *modern, pure* mathematics, a decision about which something needs to be said. "Modern" simply means that, as mentioned above, the book aims to give an idea of what mathematicians are now doing: for example, an area that developed rapidly in the middle of the last century but that has now reached a settled form is likely to be discussed less than one that is still developing rapidly. However, mathematics carries its history with it: in order to understand a piece of present-day mathematics, one will usually need to know about many ideas and results that were discovered a long time ago. Moreover, if one wishes to have a proper perspective on today's mathematics, it is essential to have some idea of how it came to be as it is. So there is plenty of history in the book, even if the main reason for our including it is to illuminate the mathematics of today.

The word "pure" is more troublesome. As many have commented, there is no clear dividing line between pure and applied mathematics, and, just as a proper appreciation of modern mathematics requires some knowledge of its history, so a proper appreciation of pure mathematics requires some knowledge of applied mathematics and theoretical physics. Indeed, these areas have provided pure mathematicians with many fundamental ideas, which have given rise to some of the most interesting, important, and currently active branches of pure mathematics. This book is certainly not blind to the impact on pure mathematics of these other disciplines, nor does it ignore the practical and intellectual applications of pure mathematics. Nevertheless, the scope is narrower than it could be. At one stage it was suggested that a more accurate title would be "The Princeton Companion to Pure Mathematics": the only reason for rejecting this title was that it does not sound as good as the actual title.

Another thought behind the decision to concentrate on pure mathematics was that it would leave open the possibility of a similar book, a companion *Companion* so to speak, about applied mathematics and theoretical physics. Until such a book appears, *The Road to Reality*, by Roger Penrose (Knopf, New York, 2005), covers a very wide variety of topics in mathematical physics, written at a level fairly similar to that of this book, and Elsevier has recently brought out a five-volume *Encyclopedia of Mathematical Physics* (Elsevier, Amsterdam, 2006).

3 *The Companion* Is Not an Encyclopedia

The word "companion" is significant. Although this book is certainly intended as a useful work of reference, you should not expect too much of it. If there is a particular mathematical concept that you want to find out about, you will not necessarily be able to find out about it here, even if it is important; though the more important it is, the more likely it is to be included. In this respect, the book is like a human companion, complete with gaps in its knowledge and views on some topics that may not be universally shared. Having said that, we have at least aimed at some sort of balance: many topics are not covered, but those that are covered range very widely (much more so than one could reasonably expect of any single human companion). In order to achieve this kind of balance, we have been guided to some extent by "objective" indicators such as the American Mathematical Society's classification of mathematical topics, or the way that mathematics is divided into sections at the four-yearly International Congress of Mathematicians. The broad areas, such as number theory, algebra, analysis, geometry, combinatorics, logic, probability, theoretical computer science, and mathematical physics, are all represented, even if not all their subareas are. Inevitably, some of the choices about what to include, and at what length, were not the result of editorial policy, but were based on highly contingent factors such as who agreed to write, who actually submitted after having agreed, whether those who submitted stuck to their word limit, and so on. Consequently, there are some areas that are not as fully represented as we would have liked, but the point came where it was better to publish an imperfect volume than to spend several more years striving for perfect balance. We hope that there will be future editions of *The Companion*: if so, there will be a chance to remedy any defects that there might be in this one.

Another respect in which this book differs from an encyclopedia is that it is arranged thematically rather than alphabetically. The advantage of this is that, although the articles can be enjoyed individually, they can also be regarded as part of a coherent whole. Indeed, the structure of the book is such that it would not be ridiculous to read it from cover to cover, though it would certainly be time-consuming.

4 The Structure of *The Companion*

What does it mean to say that *The Companion* is "arranged thematically"? The answer is that it is divided into eight parts, each with a different general theme and a different purpose. Part I consists of introductory material, which gives a broad overview of mathematics and explains, for the benefit of readers with less of a background in mathematics, some of the basic concepts of the subject. A rough rule of thumb is that a topic belongs in part I if it is part of the necessary background of all mathematicians rather than belonging to one specific area. GROUPS [I.3 §2.1] and VECTOR SPACES [I.3 §2.3] belong in this category, to take two obvious examples.

Part II is a collection of essays of a historical nature. Its aim is to explain how the distinctive style of modern mathematics came into being. What, broadly speaking, are the main differences between the way mathematicians think about their subject now and the way they thought about it 200 years ago (or more)? One is that there is a universally accepted standard for what counts as a proof. Closely related to this is the fact that mathematical analysis (calculus and its later extensions and developments) has been put on a rigorous footing. Other notable features are the extension of the concept of number, the abstract nature of algebra, and the fact that most modern geometers study *non-Euclidean* geometry rather than the more familiar triangles, circles, parallel lines, and the like.

Part III consists of fairly short articles, each one dealing with an important mathematical concept that has not appeared in part I. The intention is that this part of the book will be a very good place to look if there is a concept you do not know about but have often heard mentioned. If another mathematician, perhaps a colloquium speaker, assumes that you are familiar with a definition—for example, that of a SYMPLECTIC FORM [III.88], or the INCOMPRESSIBLE EULER EQUATION [III.23], or a SOBOLEV SPACE [III.29 §2.4], or the IDEAL CLASS GROUP [IV.1 §7]—and if you are too embarrassed to admit that in fact you are not, then you now have the alternative of looking these concepts up in *The Companion*.

The articles in part III would not be much use if all they gave was formal definitions: to understand a concept one wants to know what it means intuitively, why it is important, and why it was first introduced. Above all, if it is a fairly general concept, then one wants to know some good examples—ones that are not too simple and not too complicated. Indeed, it may well be that providing and discussing a well-chosen example is all that such an article needs to do, since a good example is much easier to understand than a general definition, and more experienced readers will be able to work out a general definition by abstracting the important properties from the example.

Another use of part III is to provide backup for part IV, which is the heart of the book. Part IV consists of twenty-six articles, considerably longer than those of part III, about different areas of mathematics. A typical part IV article explains some of the central ideas and important results of the area it treats, and does so as informally as possible, subject to the constraint that it should not be too vague to be informative. The original hope was for these articles to be "bedtime reading," that is, clear and elementary enough that one could read and understand them without continually stopping to think. For that reason, the authors were chosen with two priorities in mind, of equal importance: expertise and expository skill. But mathematics is not an easy subject, and in the end we had to regard the complete accessibility we originally hoped for as an ideal that we would strive toward, even if it was not achieved in every last subsection of every article. But even when the articles are tough going, they discuss what they discuss in a clearer and less formal way than a typical textbook, often with remarkable success. As with part III, several authors have achieved this by looking at illuminating examples, which they sometimes follow with more general theory and sometimes leave to speak for themselves.

Many part IV articles contain excellent descriptions of mathematical concepts that would otherwise have had articles devoted to them in part III. We originally planned to avoid duplication completely, and instead to include cross-references to these descriptions in part III. However, this risked irritating the reader, so we decided on the following compromise. Where a concept is adequately explained elsewhere, part III does not have a full article, but it does have a short description together with a cross-reference. This way, if you want to look a concept up quickly, you can use part III, and only if you need more detail will you be forced to follow the cross-reference to another part of the book.

Part V is a complement to part III. Again, it consists of short articles on important mathematical topics, but now these topics are the theorems and open problems of mathematics rather than the basic objects and tools of study. As with the book as a whole, the choice of entries in part V is necessarily far from comprehensive, and has been made with a number of criteria in mind. The most obvious one is mathematical importance, but some entries were chosen because it is possible to discuss them in an entertaining and accessible way, others because they have some unusual feature (an example is the FOUR-COLOR THEOREM [V.12], though this might well have been included anyway), some because the authors of closely related part IV articles felt that certain theorems should be discussed separately, and some because authors of several other articles wanted to assume them as background knowledge. As with part III, some of the entries in part V are not full articles but short accounts with cross-references to other articles.

Part VI is another historical section, about famous mathematicians. It consists of short articles, and the aim of each article is to give very basic biographical information (such as nationality and date of birth), together with an explanation of why the mathematician in question is famous. Initially, we planned to include living mathematicians, but in the end we came to the conclusion that it would be almost impossible to make a satisfactory selection of mathematicians working today, so we decided to restrict ourselves to mathematicians who had died, and moreover to mathematicians who were principally known for work carried out before 1950. Later mathematicians do of course feature in the book, since they are mentioned in other articles. They do not have their own entries, but one can get some idea of their achievements by looking them up in the index.

After six parts mainly about pure mathematics and its history, part VII finally demonstrates the great external impact that mathematics has had, both practically and intellectually. It consists of longer articles, some written by mathematicians with interdisciplinary interests and others by experts from other disciplines who make considerable use of mathematics.

The final part of the book contains general reflections about the nature of mathematics and mathematical life. The articles in this part are on the whole more accessible than the longer articles earlier in the book, so even though part VIII is the final part, some readers may wish to make it one of the first parts they look at.

The order of the articles within the parts is alphabetical in parts III and V and chronological in part VI. The decision to organize the articles about mathematicians in order of their dates of birth was carefully considered, and we made it for several reasons: it would encourage the reader to get a sense of the history of the subject by reading the part right through rather than just looking at individual articles; it would make it much clearer which mathematicians were contemporaries or near contemporaries; and after the slight inconvenience of looking up a mathematician by guessing his or her date of birth relative to those of other mathematicians, the reader would learn something small but valuable.

In the other parts, some attempt has been made to arrange the articles thematically. This applies in particular to part IV, where the ordering attempts to follow two basic principles: first, that articles about closely related branches of mathematics should be close to each other in the book; and second, that if it makes obvious sense to read article A before article B, then article A should come before article B in the book. This is easier said than done, since some branches are hard to classify: for instance, should arithmetic geometry count as algebra, geometry, or number theory? A case could be made for any of the three and it is artificial to decide on just one. So the ordering in part IV should not be taken as a classification scheme, but just as the best linear ordering we could think of.

As for the order of the parts themselves, the aim has been to make it the most natural one from a pedagogical point of view and to give the book some sense of direction. Parts I and II are obviously introductory, in different ways. Part III comes before part IV because in order to understand an area of mathematics one tends to start by grappling with new definitions. But part IV comes before part V because in order to appreciate a theorem it is a good idea to know how it fits into an area of mathematics. Part VI is placed after parts III–V because one can better appreciate the contribution of a famous mathematician after knowing some mathematics. Part VII is near the end for a similar reason: to understand the influence of mathematics, one should understand mathematics first. And the reflections of part VIII are a sort of epilogue, and therefore an appropriate way for the book to sign off.

5 Cross-References

From the start, it was planned that *The Companion* would have a large number of cross-references. One or two have even appeared in this preface, signalled by THIS FONT, together with an indication of where to find the relevant article. For example, the reference to a SYMPLECTIC FORM [III.88] indicated that symplectic forms are discussed in article number 88 of part III, and the reference to the IDEAL CLASS GROUP [IV.1 §7] pointed the reader to section 7 of article number 1 in part IV.

We have tried as hard as possible to produce a book that is a pleasure to read, and the aim is that cross-references should contribute to this pleasure. This may seem a rather strange thing to say, since it can be annoying to interrupt what one is reading in a book in order to spend a few seconds looking something up elsewhere. However, we have also tried to keep the articles as self-contained as is feasible. Thus, if you do not want to pursue the cross-references, then you will usually not have to. The main exception to this is that authors have been allowed to assume some knowledge of the concepts discussed in part I. If you do not know any university mathematics, then you would be well-advised to start by reading part I in full, as this will greatly reduce your need to look things up while reading later articles.

Sometimes a concept is introduced in an article and then explained in that article. The usual convention in mathematical writing is to italicize a term when it is being defined. We have stuck to something like that convention, but in an informal article it is not always clear what constitutes the moment of definition of a new or unfamiliar term. Our rough policy has been to italicize a term the first time it is used if that use is followed by a discussion that gives some kind of explanation of the term. We have also italicized terms that are not subsequently explained: this should be taken as a signal that the reader is not required to understand the term in order to understand the rest of the article in question. In more extreme cases of this kind, quotation marks may be used instead.

Many of the articles end with brief "Further Reading" sections. These are exactly that: suggestions for further reading. They should not be thought of as full-scale bibliographies such as one might find at the end of a survey article. Related to this is the fact that it is not a major concern of *The Companion* to give credit to all the mathematicians who made the discoveries that it discusses or to cite the papers where those discoveries appeared. The reader who is interested in original sources should be able to find them from the books and articles in the further reading sections, or from the Internet.

6 Who Is *The Companion* Aimed At?

The original plan for *The Companion* was that all of it should be accessible to anybody with a good background in high school mathematics (including calculus). However, it soon became apparent that this was an unrealistic aim: there are branches of mathematics that are so much easier to understand when one knows at least some university-level mathematics that it does not make good sense to attempt to explain them at a lower level. On the other hand, there are other parts of the subject that decidedly *can* be explained to readers without this extra experience. So in the end we abandoned the idea that the book should have a uniform level of difficulty.

Accessibility has, however, remained one of our highest priorities, and throughout the book we have tried to discuss mathematical ideas at the lowest level that is practical. In particular, the editors have tried very hard not to allow any material into the book that they do not themselves understand, which has turned out to be a very serious constraint. Some readers will find some articles too hard and other readers will find other articles too easy, but we hope that all readers from advanced high school level onwards will find that they enjoy a substantial proportion of the book.

What can readers of different levels hope to get out of *The Companion*? If you have embarked on a university-level mathematics course, you may find that you are presented with a great deal of difficult and unfamiliar material without having much idea why it is important and where it is all going. Then you can use *The Companion* to provide yourself with some perspective on the subject. (For example, many more people know what a ring is than can give a good reason for caring about rings. But there are very good reasons, which you can read about in RINGS, IDEALS, AND MODULES [III.81] and ALGEBRAIC NUMBERS [IV.1].)

If you are coming to the end of the course, you may be interested in doing research in mathematics. But undergraduate courses typically give you very little idea of what research is actually like. So how do you decide which areas of mathematics truly interest you at the research level? It is not easy, but the decision can make the difference between becoming disillusioned and ultimately not getting a Ph.D., and going on to a successful career in mathematics. This book, especially part IV, tells you what mathematicians of many different kinds are thinking about at the research level, and may help you to make a more informed decision.

If you are already an established research mathematician, then your main use for this book will probably be to understand better what your colleagues are up to. Most nonmathematicians are very surprised to learn how extraordinarily specialized mathematics has become. Nowadays it is not uncommon for a very good mathematician to be completely unable to understand the papers of another mathematician, even from an area that appears to be quite close. This is not a healthy state of affairs: anything that can be done to improve the level of communication among mathematicians is a good idea. The editors of this book have learned a huge amount from reading the articles carefully, and we hope that many others will avail themselves of the same opportunity.

7 What Does *The Companion* Offer That the Internet Does Not Offer?

In some ways the character of *The Companion* is similar to that of a large mathematical Web site such as the mathematical part of Wikipedia or Eric Weisstein's "Mathworld" (http://mathworld.wolfram.com/). In particular, the cross-references have something of the feel of hyperlinks. So is there any need for this book?

At the moment, the answer is yes. If you have ever tried to use the Internet to find out about a mathematical concept, then you will know that it is a hit-and-miss affair. Sometimes you find a good explanation that gives you the information you were looking for. But often you do not. The Web sites just mentioned are certainly useful, and recommended for material that is not covered here, but at the time of writing most of the online articles are written in a different style from the articles in this book: drier, and more concerned with giving the basic facts in an economical way than with reflecting on those facts. And one does not find long essays of the kind contained in parts I, II, IV, VII, and VIII of this book.

Some people will also find it advantageous to have a large collection of material in book form. As has already been mentioned, this book is organized not as a collection of isolated articles but as a carefully ordered sequence that exploits the linear structure that all books necessarily have and that Web sites do not have. And the physical nature of a book makes browsing through it a completely different experience from browsing a Web site: after reading the list of contents one can get a feel for the entire book, whereas with a large Web site one is somehow conscious only of the page one is looking at. Not everyone will agree with this or find it a significant advantage, but many undoubtedly will and it is for them that the book has been written. For now, therefore, *The Princeton Companion to Mathematics* does not have a serious online competitor: rather than competing with the existing Web sites, it complements them.

8 How *The Companion* Came into Being

The Princeton Companion to Mathematics was first conceived by David Ireland in 2002, who was at the time employed in the Oxford office of Princeton University Press. The most important features of the book—its title, its organization into sections, and the idea that one of these sections should consist of articles about major branches of mathematics—were all part of his original conception. He came to visit me in Cambridge to discuss his proposal, and when the moment came (it was clearly going to) for him to ask whether I would be prepared to edit it, I accepted more or less on the spot.

What induced me to make such a decision? It was partly because he told me that I would not be expected to do all the work on my own: not only would there be other editors involved, but also there would be considerable technical and administrative support. But a more fundamental reason was that the idea for the book was very similar to one that I had had myself in an idle moment as a research student. It would be wonderful, I thought then, if somewhere one could find a collection of well-written essays that presented for you the big themes of mathematical research in different areas of mathematics. Thus a little fantasy had been born, and suddenly I had the chance to turn it into a reality.

We knew from the outset that we wanted the book to contain a certain amount of historical reflection, and soon after this meeting David Ireland asked June Barrow-Green whether she was prepared to be another editor, with particular responsibility for the historical parts. To our delight, she accepted, and with her remarkable range of contacts she gave us access to more or less all the mathematical historians in the world.

There then began several meetings to plan the more detailed content of the volume, ending in a formal proposal to Princeton University Press. They sent it out to a team of expert advisers, and although some made the obvious point that it was a dauntingly huge project, all were enthusiastic about it. This enthusiasm was also evident at the next stage, when we began to find contributors. Many of them were very encouraging and said how glad they were that such a book was being produced, confirming what we already thought: that there was a gap in the market. During this stage, we benefited greatly from the advice and experience of Alison Latham, editor of *The Oxford Companion to Music*.

In the middle of 2003, David Ireland left Princeton University Press, and with it this project. This was a big blow, and we missed his vision and enthusiasm for the book: we hope that what we have finally produced is something like what he originally had in mind. However, there was a positive development at around the same time, when Princeton University Press decided to employ a small company called T&T Productions Ltd. The company was to be responsible for producing a book out of the files submitted by the contributors, as well as for doing a great deal of the day-to-day work such as sending out contracts, reminding contributors that their deadlines were approaching, receiving files, keeping a record of what had been done, and so on. Most of this work was done by Sam Clark, who is extraordinarily good at it and manages to be miraculously good-humored at the same time. In addition, he did a great deal of copy-editing as well, where that did not need too great a knowledge of mathematics (though as a former chemist he knows more than most people). With Sam's help we have not just a carefully edited book but one that is beautifully designed as well. Without him, I do not see how it would have ever been completed.

We continued to have regular meetings, to plan the book in more detail and to discuss progress on it. These meetings were now ably organized and chaired by Richard Baggaley, also from the Oxford office of Princeton University Press. He continued to do this until the summer of 2004, when Anne Savarese, Princeton's new reference editor, took over. Richard and

Anne have also been immensely useful, asking the editors the right awkward questions when we have been tempted to forget about the parts of the book that were not quite going to plan, and forcing on us a level of professionalism that, to me at least, does not come naturally.

In early 2004, at what we naively thought was a late stage in the preparation of the book, but which we now understand was actually near the beginning, we realized that, even with June's help, I had far too much to do. One person immediately sprang to mind as an ideal coeditor: Imre Leader, who I knew would understand what the book was trying to achieve and would have ideas about how to achieve it. He agreed, and quickly became an indispensable member of the editorial team, commissioning and editing several articles.

By the second half of 2007, we really were at a late stage, and by that time it had become clear that additional editorial help would make it much easier to complete the tricky tasks that we had been postponing and actually get the book finished. Jordan Ellenberg and Terence Tao agreed to help, and their contribution was invaluable. They edited some of the articles, wrote others, and enabled me to write several short articles on subjects that were outside my area of expertise, safe in the knowledge that they would stop me making serious errors. (I would have made several without their help, but take full responsibility for any that may have slipped through the net.) Articles by the editors have been left unattributed, but a note at the end of the contributor list explains which ones were written by which editor.

9 The Editorial Process

It is not always easy to find mathematicians with the patience and understanding to explain what they are doing to nonexperts or colleagues from other areas: too often they assume you know something that you do not, and it is embarrassing to admit that you are completely lost. However, the editors of this book have tried to help you by taking this burden of embarrassment on themselves. An important feature of the book has been that the editorial process has been a very active one: we have not just commissioned the articles and accepted whatever we have been sent. Some drafts have had to be completely discarded and new articles written in the light of editorial comments. Others have needed substantial changes, which have sometimes been made by the authors and sometimes by the editors. A few articles were accepted with only trivial changes, but these were a very small minority.

The tolerance, even gratitude, with which almost all authors have allowed themselves to be subjected to this treatment has been a very welcome surprise and has helped the editors maintain their morale during the long years of preparation of this volume. We would like to express our gratitude in return, and we hope that they agree that the whole process has been worthwhile. To us it seems inconceivable that this amount of work could go into the articles *without* a substantial payoff. It is not my place to say how successful I think the outcome has been, but, given the number of changes that were made in the interests of accessibility, and given that interventionist editing of this type is rare in mathematics, I do not see how the book can fail to be unusual in a good way.

A sign of just how long everything has taken, and also of the quality of the contributors, is that a significant number of contributors have received major awards and distinctions since being invited to contribute. At least three babies have been born to authors in the middle of preparing articles. Two contributors, Benjamin Yandell and Graham Allan, have sadly not lived to see their articles in print, but we hope that in a small way this book will be a memorial to them.

10 Acknowledgments

An early part of the editorial process was of course planning the book and finding authors. This would have been impossible without the help and advice of several people. Donald Albers, Michael Atiyah, Jordan Ellenberg, Tony Gardiner, Sergiu Klainerman, Barry Mazur, Curt McMullen, Robert O'Malley, Terence Tao, and Avi Wigderson all gave advice that in one way or another had a beneficial effect on the shape of the book. June Barrow-Green has been greatly helped in her task by Jeremy Gray and Reinhard Siegmund-Schultze. In the final weeks, Vicky Neale very kindly agreed to proofread certain sections of the book and help with the index; she was amazing at this, picking up numerous errors that we would never have spotted ourselves and are very pleased to have corrected. And there is a long list of mathematicians and mathematical historians who have patiently answered questions from the editors: we would like to thank them all.

I am grateful to many people for their encouragement, including virtually all the contributors to this volume and many members of my immediate family,

particularly my father, Patrick Gowers: this support has kept me going despite the mountainous appearance of the task ahead. I would also like to thank Julie Barrau for her less direct but equally essential help. During the final months of preparation of the book, she agreed to take on much more than her fair share of our domestic duties. Given that a son was born to us in November 2007, this made a huge difference to my life, as has she.

Timothy Gowers